Discrete Structures: Midterm Exam I – Solutions

Part I: Sets

1. (a) For a universal set \( U \) and two sets \( A \subseteq U \) and \( B \subseteq U \), what is \( A \cup B \) if \( A \cap B = \emptyset \)?

   **Answer:** \( A \cup B = U \). That is, \( A \) and \( B \) together contain all the objects of the universal set \( U \).

   **Proof:** By the De Morgan’s laws, \( \overline{A \cap B} = \overline{A} \cup \overline{B} \). Since \( A \cap B = \emptyset \), it follows that also \( \overline{A} \cup \overline{B} = \emptyset \). Thus,

   \[
   A \cup B = \overline{A \cap B} = \overline{\emptyset} = U
   \]

   (b) For a universal set \( U \) and two sets \( A \subseteq U \) and \( B \subseteq U \), what is \( A \cap B \) if \( A \cup B = U \)?

   **Answer:** \( A \cap B = \emptyset \). That is, \( A \) and \( B \) are two disjoint sets.

   **Proof:** By the De Morgan’s laws, \( \overline{A \cup B} = \overline{A} \cap \overline{B} \). Since \( A \cup B = U \), it follows that also \( \overline{A} \cap \overline{B} = \emptyset \). Thus,

   \[
   A \cap B = \overline{A \cup B} = \overline{U} = \emptyset
   \]

2. (a) Suppose that \( A \cup B = A \cup C \) such that \( B \) and \( C \) are two disjoint sets. What could you say about \( B \) and \( C \) related to \( A \)?

   **Answer:** \( B \subseteq A \) and \( C \subseteq A \). That is, both \( B \) and \( C \) are subsets of \( A \).

   **Proof:** Assume to the contrary that \( x \in B \) but \( x \notin A \). Since \( A \cup B = A \cup C \), it follows that \( x \in A \cup C \) and since \( x \notin A \), it follows that \( x \in C \). A contradiction since \( B \) and \( C \) are disjoint sets. Therefore, \( B \subseteq A \). Similar arguments show that \( C \subseteq A \).

   (b) Suppose that \( A \cap B = A \cap C \) such that \( B \) and \( C \) are two disjoint sets. What could you say about \( B \) and \( C \) related to \( A \)?

   **Answer:** \( A \cap B = A \cap C = \emptyset \). That is \( A \) and \( B \) are disjoint sets and also \( A \) and \( C \) are disjoint sets.

   **Proof:** Assume to the contrary that \( x \in A \cap B \). Since \( A \cap B = A \cap C \), it follows that \( x \in A \cap C \) and therefore \( x \in C \). A contradiction since \( B \) and \( C \) are disjoint sets. Therefore, \( A \cap B = \emptyset \). Similar arguments show that \( A \cap C = \emptyset \).
3. Assume a class of $x$ students where $x \in \{33, 36, 39, 42, 45\}$. Some of the $x$ students play basketball, some of them play football, and some of them play soccer.

- 5 students play all of them while 7 students play none of them.
- The same number of students play each one of the sports listed.
- There are no students who play only two of the sports listed but not the third one.

How many students play Basketball?

**Answer:** $\frac{x}{3} + 1$. For $x = 33, 36, 39, 42, 45$ the answer is 12, 13, 14, 15, 16 respectively.

**Proof:** Let $B$, $F$, and $S$ be the sets of students that play basketball, football, and soccer respectively. Since 7 students play none of the sports, it follows that $|B \cup F \cup S| = x - 7$. The principle of inclusion exclusion implies that

$$|B \cup F \cup S| = |B| + |F| + |S| - |B \cap F| - |B \cap S| - |F \cap S| + |B \cap F \cap S| = x - 7$$

By the assumptions, $|B \cap F \cap S| = 5$ and this is also the size of all the pairwise intersections because there are no students who play only two of the sports. Therefore

$$x - 7 = |B| + |F| + |S| - 5 - 5 - 5 + 5 = |B| + |F| + |S| - 10$$

The assumption that $|B| = |F| = |S|$ implies that

$$x - 7 = 3|B| - 10$$

This is equivalent to

$$|B| = \frac{x + 3}{3} = \frac{x}{3} + 1$$

**Bonus:** Observe that all the options for $x$ are multiples of 3. This is because after subtracting 7 and then 5 from $x$, the remaining number, $y = x - 12$, is the total number of students who play exactly one of the sports. Since he same number of students play each one of the sports, it follow that $y$ must be a multiple of 3 and therefore also $x = y + 12$ is a multiple of 3.
1. (a) i. \( x \land (x \lor y \lor z) \equiv x \) regardless of the True/False assignments to \( y \) and \( z \).

**Proof:** If \( x \) is True, then \((x \lor y \lor z)\) is True which implies that the formula is True. If \( x \) is False, then the formula is False regardless of the value of \((x \lor y \lor z)\). In both cases the formula is equivalent to \( x \).

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ii. \( \neg x \land (x \land y \land z) \) is a contradiction, it is always False.

**Proof:** By the associative law the formula is equivalent to

\[(\neg x \land x) \land (y \land z)\]

Since \((\neg x \land x)\) is always False, it follows that the formula is also always False regardless of the value of \((y \land z)\).

\[
\begin{array}{ccc|c|c|c}
 x & y & z & \neg x & x \land y \land z & \neg x \land (x \land y \land z) \\
 T & T & T & F & T & F \\
 T & T & F & F & F & F \\
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\]
(b) i. \( x \lor (\neg x \lor \neg y \lor \neg z) \) is a tautology, it is always True.

**Proof:** By the associative law the formula is equivalent to
\[
(x \lor \neg x) \lor (\neg y \lor \neg z)
\]
Since \((x \lor \neg x)\) is always True, it follows that the formula is also always True regardless of the value of \((y \lor z)\).

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ii. \( \neg x \land (\neg x \lor \neg y \lor \neg z) \equiv \neg x \) regardless of the True/False assignments to \(y\) and \(z\).

**Proof:** If \(x\) is True, then the formula is False regardless of the value of \((\neg x \lor \neg y \lor \neg z)\). If \(x\) is False, then \((\neg x \lor \neg y \lor \neg z)\) is True which implies that the formula is True. In both cases the formula is equivalent to \(\neg x\).

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2. (a) How many students failed the exam if only one of the following statements is True?
   - At least one student failed.
   - Fewer than three students failed.
   - At least three students failed.

   **Answer:** Zero students failed.

   **Proof I:** The 2nd statement and the 3rd statement contradict each other. As a result, one of them is True and one of them is False. Since only one of the statements is True, it follows that the 1st statement must be False. That is, fewer than one student failed which implies that no one failed.

   **Proof II:** If three or more students failed, then both the 1st and the 3rd statements are True. A contradiction, because only one statement can be True. If one or two students failed, then both the 1st and the 2nd statements are True. A contradiction, because only one statement can be True. The remaining case is when no one failed which implies that only the 2nd statement is True.

   **Proof III:** Consider the following three options depending on which of the three statements is True.
   - The 1st statement is True: Then the second statement is True if one or two students failed and the third statement is True if three or more students failed. In either case the number of True statements would be more than one. A contradiction.
   - The 2nd statement is True: Then the only way the first statement is False is when no one failed.
   - The 3rd statement is True: Then the first statement is also True. A contradiction.

   Only the 2nd option does not lead to a contradiction. Thus, no one failed the test.

(b) Either Alice, or Bob, or Charlie failed the test, the other two got an A in the test.
   - Alice says that Bob failed.
   - Bob says that Alice is lying.
   - Charlie says that he did not fail.

   Only one of them is telling the truth. Who failed the test?

   **Answer:** Charlie failed the test.

   **Proof I:** If Bob is telling the truth then Alice is lying and if Bob is lying then Alice is telling the truth. In either case, one of them is telling the truth and since only one person is telling the truth, it follows that Charlie is lying. If Charlie is lying then he failed the test and Bob is the only one who is telling the truth.

   **Proof II:** Consider the following three options depending on who is telling the truth.
   - If Alice is telling the truth, then Bob failed the test and therefore Charlie also is telling the truth. A contradiction since only one of them is telling the truth.
   - If Bob is telling the truth, then both Alice and Charlie are lying and hence Charlie failed the test.
   - If Charlie is telling the truth, then Bob is lying which implies that Alice is telling the truth. A contradiction since only one of them is telling the truth.

   Only the 2nd option does not lead to a contradiction. Thus, Charlie failed the test.

   **Proof III:** Consider the following three options depending on who is the only one who failed the test.
   - If Alice failed the test, then Alice is lying and both Bob and Charlie are telling the truth. A contradiction since only one of them is telling the truth.
   - If Bob failed the test, then Alice is telling the truth and so is Charlie. A contradiction since only one of them is telling the truth.
   - If Charlie failed the test, then both Alice and Charlie are lying and Bob is telling the truth.

   Only the 3rd option does not lead to a contradiction. Thus, Charlie failed the test.
3. (a) Find all True/False assignments for the 3 variables (out of the $2^3 = 8$ possible assignments) that satisfy the formula 

$$(x \lor x_1) \land (x \lor x_2)$$

**Answer:** If $x = T$, then the formula is satisfied regardless of the True/False assignments to $x_1$ and $x_2$. If $x = F$, then the formula is satisfied only if both $x_1 = T$ and $x_2 = T$. Therefore, only the following five True/False assignments to $(x, x_1, x_2)$ satisfy the formula:

$$(T, T, T) \quad (T, T, F) \quad (T, F, T) \quad (T, F, F) \quad (F, T, T)$$

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**Bonus:** For an integer $n \geq 1$, determine the number of True/False assignments for the $n+1$ variables (out of the $2^{n+1}$ possible assignments) that satisfy the formula 

$$(x \lor x_1) \land (x \lor x_2) \land \cdots \land (x \lor x_n)$$

**Answer:** If $x = T$, then the formula is satisfied regardless of the True/False assignments to $x_1, x_2, \ldots, x_n$. There are $2^n$ such assignments. If $x = F$, then the formula is satisfied only if $x_1 = x_2 = \cdots = x_n = T$. There is only one such assignment. In total, there are $2^n + 1$ True/False assignments that satisfy the formula.
(b) Find all True/False assignments for the 3 variables (out of the $2^3 = 8$ possible assignments) that satisfy the formula

$$(y \land y_1) \lor (y \land y_2)$$

**Answer:** If $y = F$, then the formula cannot be satisfied regardless of the True/False assignments to $y_1$ and $y_2$. If $y = T$, then the formula is satisfied if either $y_1 = T$ or $y_2 = T$. Therefore, only the following three True/False assignments to $(y, y_1, y_2)$ satisfy the formula:

$(T, T, T)$  $(T, T, F)$  $(T, F, T)$

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**Bonus:** For an integer $n \geq 1$, determine the number of True/False assignments for the $n+1$ variables (out of the $2^{n+1}$ possible assignments) that satisfy the formula:

$$(y \land y_1) \lor (y \land y_2) \lor \cdots \lor (y \land y_n)$$

**Answer:** If $y = F$, then the formula cannot be satisfied regardless of the True/False assignments to $y_1, y_2, \ldots, y_n$. If $y = T$, then the formula is satisfied if either one of $y_1, y_2, \ldots, y_n$ is $T$. There are $2^n - 1$ such assignments.