CISC 2210 – Introduction to Discrete Structures

Solutions to the Midterm Project

1. 36 students take the Discrete Structures class. They had 3 quizzes called \(X\), \(Y\), and \(Z\).

- Surprisingly, only 1 student did not get an \(A\) on any quiz while the rest of the students got an \(A\) (aced) on at least one quiz.
- 21 students aced \(X\), 19 students aced \(Y\), and 17 students aced \(Z\).
- 9 students aced both \(X\) and \(Y\), 11 students aced both \(X\) and \(Z\), and 8 students aced both \(Y\) and \(Z\).

**Answer:** Let \(X\), \(Y\), and \(Z\) be the sets of students who aced quizzes \(X\), \(Y\), and \(Z\) respectively. The problem provides the sizes of the following sets: \(|X| = 21\), \(|Y| = 19\), \(|Z| = 17\), \(|X \cap Y| = 9\), \(|X \cap Z| = 11\), \(|Y \cap Z| = 8\), and \(|X \cup Y \cup Z| = 1\).

(a) How many students aced at least one quiz? \(|X \cup Y \cup Z| = 35\).

\[
|X \cup Y \cup Z| = 36 - |X \cup Y \cup Z| = 36 - 1 = 35
\]

(b) How many students aced all three quizzes? \(|X \cap Y \cap Z| = 6\).

- The principle of inclusion and exclusion implies that

\[
35 = |X \cup Y \cup Z| = |X| + |Y| + |Z| - |X \cap Y| - |X \cap Z| - |Y \cap Z| + |X \cap Y \cap Z|
= 21 + 19 + 17 - 9 - 11 - 8 + |X \cap Y \cap Z|
= 29 + |X \cap Y \cap Z|
\]

- This is equivalent to \(|X \cap Y \cap Z| = 35 - 29 = 6\)

(c) How may students aced both \(X\) and \(Y\) but not \(Z\)? \(|X \cap Y \cap \bar{Z}| = 3\).

\[
|X \cap Y \cap \bar{Z}| = |X \cap Y| - |X \cap Y \cap Z| = 9 - 6 = 3
\]

(d) How may students aced both \(X\) and \(Z\) but not \(Y\)? \(|X \cap \bar{Y} \cap Z| = 5\).

\[
|X \cap \bar{Y} \cap Z| = |X \cap Z| - |X \cap Y \cap Z| = 11 - 6 = 5
\]

(e) How may students aced both \(Y\) and \(Z\) but not \(X\)? \(|\bar{X} \cap Y \cap Z| = 2\).

\[
|\bar{X} \cap Y \cap Z| = |Y \cap Z| - |X \cap Y \cap Z| = 8 - 6 = 2
\]

(f) How may students aced only \(X\)? \(|X \cap \bar{Y} \cap \bar{Z}| = 7\).

\[
|X \cap \bar{Y} \cap \bar{Z}| = |X| - |X \cap Y \cap \bar{Z}| - |X \cap \bar{Y} \cap Z| - |X \cap Y \cap Z| = 21 - 3 - 5 - 6 = 7
\]
(g) How may students aced only $Y$? $|\bar{X} \cap Y \cap \bar{Z}| = 8$. 

$$|\bar{X} \cap Y \cap \bar{Z}| = |Y| - |X \cap Y \cap \bar{Z}| - |\bar{X} \cap Y \cap Z| - |X \cap Y \cap Z| = 19 - 3 - 2 - 6 = 8$$

(h) How may students aced only $Z$? $|\bar{X} \cap \bar{Y} \cap Z| = 4$. 

$$|\bar{X} \cap \bar{Y} \cap Z| = |Z| - |X \cap \bar{Y} \cap Z| - |\bar{X} \cap Y \cap Z| - |X \cap Y \cap Z| = 17 - 5 - 2 - 6 = 4$$

$$36 = 6 + 3 + 2 + 5 + 7 + 8 + 4 + 1$$
2. Prove the following logic distributive law.

\[ x \land (y \lor z \lor w) \equiv (x \land y) \lor (x \land z) \lor (x \land w) \]

**Notations:** Let \( L = x \land (y \lor z \lor w) \) and let \( R = (x \land y) \lor (x \land z) \lor (x \land w) \).

**Proof outline:** Consider the two options for \( x \).

- If \( x = F \) then \( L = F \) and \( R = F \) since each of its three clauses is false.
- If \( x = T \) then both \( L = F \) and \( R = F \) only if \( x = y = z = F \).

The two truth tables below evaluate \( L \) and \( R \) for all the \( 2^4 = 16 \) possible assignments for the variables \( x, y, z, \) and \( w \). The distributive law is correct since the column for \( L = x \land (y \lor z \lor w) \) in the first table is identical to the column of \( R = (x \land y) \lor (x \land z) \lor (x \land w) \) in the second table.

| \( x \) | \( y \) | \( z \) | \( w \) | \( (y \lor z \lor w) \) | \( L = x \land (y \lor z \lor w) \)
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| \( x \) | \( y \) | \( z \) | \( w \) | \( (x \land y) \) | \( (x \land z) \) | \( (x \land w) \) | \( R = (x \land y) \lor (x \land z) \lor (x \land w) \)
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3. Let \( S \subseteq U = \{2, 3, \ldots, 21\} \) be a non-empty set containing integers larger than 1 and smaller than 22. The following are three statements about \( S \):

(A) \( S \) does not contain prime numbers.

(B) \( S \) contains at least two even numbers.

(C) \( S \) contains at most two odd numbers.

In the following eight questions you are asked to find the smallest and the largest possible sets \( S_{\text{min}} \) and \( S_{\text{max}} \) for which some of the statements are TRUE and some of the statements are FALSE. \( S_{\text{min}} \) should contain the minimum possible number of members from \( U \) while \( S_{\text{max}} \) should contain the maximum possible number of members from \( U \).

(a) Find \( S_{\text{min}} \) and \( S_{\text{max}} \) for which all three statements are TRUE.

\[
S_{\text{min}} = \{4, 6\} \quad S_{\text{max}} = \{4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20\}
\]

(b) Find \( S_{\text{min}} \) and \( S_{\text{max}} \) for which statements \( A \) and \( B \) are TRUE but statement \( C \) is FALSE.

\[
S_{\text{min}} = \{4, 6, 9, 15, 21\} \quad S_{\text{max}} = \{4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21\}
\]

(c) Find \( S_{\text{min}} \) and \( S_{\text{max}} \) for which statements \( A \) and \( C \) are TRUE but statement \( B \) is FALSE.

\[
S_{\text{min}} = \{4\} \quad S_{\text{max}} = \{4, 9, 15\}
\]

(d) Find \( S_{\text{min}} \) and \( S_{\text{max}} \) for which statements \( B \) and \( C \) are TRUE but statement \( A \) is FALSE.

\[
S_{\text{min}} = \{2, 4\} \quad S_{\text{max}} = \{2, 3, 4, 5, 6, 8, 10, 12, 14, 16, 18, 20\}
\]

(e) Find \( S_{\text{min}} \) and \( S_{\text{max}} \) for which statement \( A \) is TRUE but statements \( B \) and \( C \) are FALSE.

\[
S_{\text{min}} = \{9, 15, 21\} \quad S_{\text{max}} = \{4, 9, 15, 21\}
\]

(f) Find \( S_{\text{min}} \) and \( S_{\text{max}} \) for which statement \( B \) is TRUE but statements \( A \) and \( C \) are FALSE.

\[
S_{\text{min}} = \{2, 3, 4, 5, 7\} \quad S_{\text{max}} = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21\}
\]

(g) Find \( S_{\text{min}} \) and \( S_{\text{max}} \) for which statement \( C \) is TRUE but statements \( A \) and \( B \) are FALSE.

\[
S_{\text{min}} = \{2\} \quad S_{\text{max}} = \{2, 3, 5\}
\]

(h) Find \( S_{\text{min}} \) and \( S_{\text{max}} \) for which all three statements are FALSE.

\[
S_{\text{min}} = \{3, 5, 7\} \quad S_{\text{max}} = \{2, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21\}
\]
4. The goal is to count the number of passwords with repetitions of length 7 on the digits 
\{0, 1, \ldots , 9\}. Note that unlike numbers, such passwords may start with a 0. 
Recall that there are \( n^k \) lists with repetition of length \( k \) with objects from a size \( n \) set.

(a) How many passwords of length 7 are there?
   - A password is a list with repetition of length 7 from a set of size 10. Therefore, there 
     are \( 10^7 = 10000000 \) passwords.

(b) How many passwords of length 7 are there that contain only odd digits?
   - A password that contains only odd digits is a list with repetition of length 7 from a 
     set of size 5. Therefore, there are \( 5^7 = 78125 \) passwords.

(c) How many passwords of length 7 are there that do not contain the digit 0?
   - A password that does not contain the digit 0 is a list with repetition of length 7 from 
     a set of size 9. Therefore, there are \( 9^7 = 4782969 \) passwords.

(d) How many passwords of length 7 are there that contain the digit 0 at least once?
   - The set of such passwords is the complement of the set from part (c). Therefore there 
     are \( 10^7 - 9^7 = 5217031 \) such passwords.

(e) How many passwords of length 7 are there that contain the digit 0 exactly once?
   - There are 7 options for the location of the digit 0 in the password. For each option, 
     the rest of the password is a list with repetition of length 6 from a set of size 9. 
     Therefore, there are \( 7 \cdot 9^6 = 3720087 \) such passwords.

(f) How many passwords of length 7 are there that contain the digit 0 at most once?
   - Such a password either does not contain the digit 0 or contains it exactly once. By 
     parts (c) and and (e), the answer is \( 9^7 + 7 \cdot 9^6 = 16 \cdot 9^6 = 8503056 \).

(g) How many passwords of length 7 are there that contain the digit 0 exactly twice?
   - There are \( \binom{7}{2} = \frac{7 \cdot 6}{2} = 21 \) options for the two locations of the digit 0 in the password. 
     For each option, the rest of the password is a list with repetition of length 5 from a 
     set of size 9. Therefore there are \( 21 \cdot 9^5 = 1240029 \) such passwords.

(h) How many passwords of length 7 are there that contain the digit 0 exactly once and the 
    digit 1 exactly once?
   - There are \( 7 \cdot 6 = 42 \) options for the two locations of the digits 0 and 1 in the password. 
     For each option, the rest of the password is a list with repetition of length 5 from a 
     set of size 8. Therefore there are \( 42 \cdot 8^5 = 1376256 \) such passwords.
5. Using the identity
\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]
simplify the following expression as a fraction with as few as possible terms
\[
\binom{m}{3} + \binom{m+1}{3} + \binom{m+2}{3}
\]

**Answer:** Observe that since \(\frac{n!}{(n-3)!} = n(n-1)(n-2)\) it follows that
\[
\binom{n}{3} = \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{6}
\]
The simplification follows by replacing \(n\) with \(m\), \((m+1)\), and \((m+2)\) and applying algebra rules.

\[
\binom{m}{3} + \binom{m+1}{3} + \binom{m+2}{3} = \frac{m(m-1)(m-2)}{6} + \frac{(m+1)m(m-1)}{6} + \frac{(m+2)(m+1)m}{6}
\]
\[
= \frac{m(m-1)(m-2) + (m+1)m(m-1) + (m+2)(m+1)m}{6}
\]
\[
= \frac{(m^3 - 3m^2 + 2m) + (m^3 - m) + (m^3 + 3m^2 + 2m)}{6}
\]
\[
= \frac{3m^3 + 3m}{6}
\]
\[
= \frac{m^3 + m}{2}
\]
\[
= \frac{m(m^2 + 1)}{2}
\]
6. Prove the following identity by induction on \( n \geq 1 \) or by any other method.

\[
\sum_{i=1}^{n} (6i + 3) = 3n(n + 2)
\]

**Proof by induction:**

- **Notations.**

\[
L(n) = (6 \cdot 1 + 3) + (6 \cdot 2 + 3) + \cdots + (6n + 3) \\
R(n) = 3n(n + 2)
\]

- **Induction base.** Prove that \( L(1) = R(1) \):

\[
L(1) = 6 \cdot 1 + 3 = 9 \\
R(1) = 3 \cdot 1(1 + 2) = 9
\]

- **Induction hypothesis.** Assume that \( L(n - 1) = R(n - 1) \) for \( n > 1 \):

\[
L(n - 1) = (6 \cdot 1 + 3) + (6 \cdot 2 + 3) + \cdots + (6(n - 1) + 3) \\
R(n - 1) = 3(n - 1)((n - 1) + 2) \\
\quad = 3(n - 1)(n + 1) \\
\quad = 3(n^2 - 1) \\
\quad = 3n^2 - 3
\]

- **Inductive step.** Prove that \( L(n) = R(n) \) for \( n > 1 \):

\[
L(n) = (6 \cdot 1 + 3) + (6 \cdot 2 + 3) + \cdots + (6(n - 1) + 3) + (6n + 3) \quad \text{(* definition of } L(n) \text{ *)} \\
= L(n - 1) + (6n + 3) \quad \text{(* definition of } L(n - 1) \text{ *)} \\
= R(n - 1) + (6n + 3) \quad \text{(* induction hypothesis *)} \\
= (3n^2 - 3) + (6n + 3) \quad \text{(* evaluation of } R(n - 1) \text{ *)} \\
\quad = 3n^2 + 6n \quad \text{(* algebra *)} \\
\quad = 3n(n + 2) \quad \text{(* algebra *)} \\
\quad = R(n) \quad \text{(* definition of } R(n) \text{ *)}
\]
Second proof: Recall that

\[ \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \]

This identity implies the following,

\[
L(n) = \sum_{i=1}^{n} (6i + 3) \quad (* \text{definition of } L(n) *)
= \sum_{i=1}^{n} (6i) + \sum_{i=1}^{n} 3 \quad (* \text{algebra} *)
= 6 \sum_{i=1}^{n} i + 3n \quad (* \text{algebra} *)
= 6 \frac{n(n+1)}{2} + 3n \quad (* \text{applying above identity} *)
= 3n(n+1) + 3n \quad (* \text{algebra} *)
= 3n^2 + 3n + 3n \quad (* \text{algebra} *)
= 3n^2 + 6n \quad (* \text{algebra} *)
= 3n(n+2) \quad (* \text{algebra} *)
= R(n) \quad (* \text{definition of } R(n) *)
\]

Third proof: Recall that the sum of an arithmetic progression with \( n \) numbers is the sum of the first and the last numbers multiplied by \( n/2 \). The identity is about the sum of the following arithmetic progression with \( n \) numbers

\[ 9, 15, 21, \ldots, 6n + 3 \]

As a result the sum of this arithmetic progression is

\[
L(n) = 9 + 15 + 21 + \cdots + (6n + 3) \quad (* \text{definition of } L(n) *)
= \frac{n}{2} (9 + (6n + 3)) \quad (* \text{sum of arithmetic progression} *)
= \frac{n(6n + 12)}{2} \quad (* \text{algebra} *)
= \frac{6n(n + 2)}{2} \quad (* \text{algebra} *)
= 3n(n + 2) \quad (* \text{algebra} *)
= R(n) \quad (* \text{definition of } R(n) *)
\]
7. Prove the following identity by induction on $n \geq 1$ or by any other method.

$$\sum_{i=1}^{n} (i \cdot i!) = (n+1)! - 1$$

**Proof by induction:**

- **Notations.**

$$L(n) = 1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n!$$

$$R(n) = (n+1)! - 1$$

- **Induction base.** Prove that $L(1) = R(1)$:

$$L(1) = 1 \cdot 1! = 1$$

$$R(1) = (1+1)! - 1 = 2! - 1 = 1$$

- **Induction hypothesis.** Assume that $L(n-1) = R(n-1)$ for $n > 1$:

$$L(n-1) = 1 \cdot 1! + 2 \cdot 2! + \cdots + (n-1) \cdot (n-1)!$$

$$R(n-1) = ((n-1) + 1)! - 1$$

$$= n! - 1$$

- **Inductive step.** Prove that $L(n) = R(n)$ for $n > 1$:

$$L(n) = 1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n!$$

(* definition of $L(n)$ *)

$$= L(n-1) + n \cdot n!$$

(* definition of $L(n-1)$ *)

$$= R(n-1) + n \cdot n!$$

(* induction hypothesis *)

$$= (n! - 1) + n \cdot n!$$

(* evaluation of $R(n-1)$ *)

$$= (n \cdot n! + n!) - 1$$

(* algebra *)

$$= (n+1)n! - 1$$

(* algebra *)

$$= (n+1)! - 1$$

(* definition of factorial *)

$$= R(n)$$

(* definition of $R(n)$ *)
Second proof:

\[ L(n) = \sum_{i=1}^{n} (i \cdot i!) \quad \text{(} \ast \text{ definition of } L(n) \ast) \]

\[ = \sum_{i=1}^{n} (i \cdot i! + i! - i!) \quad \text{(} \ast \text{ algebra } \ast) \]

\[ = \sum_{i=1}^{n} ((i + 1)i! - i!) \quad \text{(} \ast \text{ algebra } \ast) \]

\[ = \sum_{i=1}^{n} ((i + 1)! - i!) \quad \text{(} \ast \text{ definition of factorial } \ast) \]

\[ = (2! - 1!) + (3! - 2!) + \cdots + ((n + 1)! - n!) \quad \text{(} \ast \text{ expanding the summation } \ast) \]

\[ = (n + 1)! - 1! \quad \text{(} \ast \text{ cancellations } \ast) \]

\[ = (n + 1)! - 1 \quad \text{(} \ast 1! = 1 \ast) \]

\[ = R(n) \quad \text{(} \ast \text{ definition of } R(n) \ast) \]