CISC 2210 – Introduction to Discrete Structures

Final Exam

May 19, 2022

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Structure, problem selection, and credit:

• You have 2 hours to complete the exam.

• In the exam you will find nine problems partitioned into four parts. You need to do six problems to get the maximum credit of 100.

**Part 1 Sets and Logic:** Do either problem 1 or problem 2 for 25 credits.

**Part 2 Induction and Recursion:** Do either problem 3 or problem 4 for 25 credits.

**Part 3 Probability and Counting:** No choices. Do problem 5 for 25 credits.

**Part 4 Algorithms and Graphs:** Do both problems 6 and 7 for 10 credits each and then do either problem 8 or problem 9 for 20 credits. The total credit for this part is 40 = 10 + 10 + 20.

• Note that you must do Problems 5, 6, 7 which are marked with an *.

• The maximum credit is 115 = 25 + 25 + 25 + 40. However the maximum final grade is 100. All credits above 100 are ignored.

• In Parts 1, 2, 4, you may answer more problems if you choose. If you do so, the best answer in each part will be considered for the total credit.

• You will get only partial credit if you fail to justify or prove your answers. You will get 20% of the credit for any problem if you leave the allocated space for the answer empty. You will get zero credit for wrong answers.

Honor code: Students are expected to do this exam by themselves without any external help from other people, the Internet, books, or notes. Cheaters will be punished severely. At minimum, they will fail the exam, but they may fail the whole class. In addition, students who cheat risk disciplinary measures by Brooklyn College and CUNY.
Part 1: Sets and Logic

Do either Problem 1 or Problem 2 for maximum 25 credits.

Problem 1

Let $A$, $B$, $C$, and $D$ be four non empty sets such that

- The intersection between $A$ and $B$ is not empty: $A \cap B \neq \emptyset$.
- The intersection between $B$ and $C$ is not empty: $B \cap C \neq \emptyset$.
- The intersection between $C$ and $D$ is not empty: $C \cap D \neq \emptyset$.
- The intersection between $A$ and $C$ is empty: $A \cap C = \emptyset$.
- The intersection between $B$ and $D$ is empty: $B \cap D = \emptyset$.
- The intersection between $A$ and $D$ is empty: $A \cap D = \emptyset$.

1. Draw the Venn-diagram for the sets $A$, $B$, $C$, and $D$. 
2. How many objects appear in at least two sets given the following data. Justify your answer.

- $|A| = 5$.
- $|B| = 10$.
- $|C| = 15$.
- $|D| = 20$.
- $|A \cup B \cup C \cup D| = 40$. 
Problem 2

Prove the following two statements.
For each statement, assign TRUE or FALSE values to the three variables $x$, $y$, and $z$ such that the TRUE/FALSE value of the left hand side of the expression is different from the TRUE/FALSE value of the right hand side of the expression. Explain your answer.

1. $(x \land y) \lor z \not\equiv x \land (y \lor z)$

2. $(x \lor y) \land z \not\equiv x \lor (y \land z)$
Part 2: Induction and Recursion

Do either Problem 3 or Problem 4 for maximum 25 credits.

Problem 3

Prove that the following identity is correct by induction on \( n \geq 1 \) or by any other method.

\[
\sum_{i=1}^{2n} (2i - 1) = 4n^2
\]

Note that the sum starts with \( i = 1 \) but ends with \( i = 2n \) and not with \( i = n \). As a result, the inductive steps adds two new terms to the sum compared with the number of terms in the sum assumed by the induction hypothesis.
Problem 4

Solve the following recursive formula by finding the closed form expression for $T(n)$ as a function of $n$. Assume that $n$ is a positive integer.

$$T(n) = \begin{cases} 
3 & \text{for } n = 1 \\
6 & \text{for } n = 2 \\
2T(n-1) - T(n-2) & \text{for } n > 2 
\end{cases}$$

Prove the correctness of your answer. A correct guess without a proof will get only partial credit.
Part 3: Probability and Counting

Do Problem 5 for maximum 25 credits.

Problem 5

Alice, Bob, and Charlie play the following game with a fair 6-face dice and a fair coin.
- In the first round they throw the dice. If they throw 6 they win and the game ends.
- Otherwise, if they throw 1, 2, 3, 4, or 5 they flip the coin.
- If they flip heads they win and if they flip tails they lose and the game ends.

Justify your answers to the following five questions.

Hint: In questions 3 and 4 the probability is a conditional probability.

1. What is the probability that a player flips the coin?

2. What is the winning probability?
3. Bob lost the game. What is the probability that Bob’s dice showed 1?

4. Alice won the game. What is the probability that Alice did not flip the coin?

5. Charlie replaced the fair coin with a fake one whose both sides are heads. What is the probability that Charlie wins the game?
Part 4: Algorithms and Graphs

Do both Problems 6 and Problem 7 for maximum 10 credits each. Then do either Problem 8 or Problem 9 for maximum 20 credits.

Problem 6

Let $P$ be a problem whose input is an array of size $n$ for $n \geq 1$.

- Algorithm $A$ solves $P$ with complexity $\Theta(n)$.
- Algorithm $B$ solves $P$ with complexity $\Theta(\log(n))$.
- Algorithm $C$ solves $P$ with complexity $\Theta(n \log(n))$.
- Algorithm $D$ solves $P$ with complexity $\Theta(n^2)$.
- Algorithm $E$ solves $P$ with complexity $\Theta(1)$.

Order the five algorithms from the most efficient to the least efficient. Justify your answer.
Problem 7

Below are all the 11 non-isomorphic graphs with 4 vertices.

1. Mark all the graphs that are trees.
2. Mark all the graphs that are not connected.
Problem 8

Alice and Bob play the following search game.
- Alice selects an integer $x$ between 1 and $n$: $1 \leq x \leq n$.
- Bob must find $x$ by asking questions of the type “is $x \leq i$?” for some integer $1 \leq i \leq n$.
- Alice may (but does not have to) lie in one of her answers. She must answer correctly the rest of the questions even if a question is repeated. However, Bob does not know in advance which answer is a lie.

In your own words (use a high-level description and try to avoid codes), describe an algorithm for Bob to find the integer $x$ selected by Alice.

What is the worst-case complexity (number of questions asked) of your algorithm?

Justify the correctness of your algorithm and the correctness of your complexity claim.
Problem 9

A **balloon-graph** is a graph that is composed of a cycle that is connected to a path.

- All the edges of the graph are either part of the cycle or part of the path.
- Exactly one vertex belongs to the cycle and to the path.
- Some vertices (at least 2) belong only to the cycle.
- The rest of the vertices (at least 1) belong only to the path.

A balloon-graph must contain at least 4 vertices. See above the only possible balloon-graph with 4 vertices and two of the four possible non-isomorphic balloon-graphs with 7 vertices. One of them has a path of length 3 that is connected to a cycle of size 5 and one of them has a path of length 4 that is connected to a cycle of size 4.

1. Illustrate the two possible non-isomorphic balloon-graphs with 5 vertices.

2. Illustrate the three possible non-isomorphic balloon-graphs with 6 vertices.
3. For $n \geq 4$, how many non-isomorphic balloon-graphs with $n$ vertices exist? Justify your answer.

4. For $n \geq 4$, all possible balloon-graphs have the same number of edges. What is this number? Prove your answer.
5. For $n \geq 4$, the degree sequence of all possible balloon-graphs is the same. Justify your answers to the following five questions.

(a) How many vertices in a balloon-graph with $n$ vertices have degree 0?
(b) How many vertices in a balloon-graph with $n$ vertices have degree 1?
(c) How many vertices in a balloon-graph with $n$ vertices have degree 2?
(d) How many vertices in a balloon-graph with $n$ vertices have degree 3?
(e) How many vertices in a balloon-graph with $n$ vertices have degree greater than 3?