1. Let $A, B, C,$ and $D$ be four non-empty sets such that

- The intersection between $A$ and $B$ is not empty: $A \cap B \neq \emptyset$.
- The intersection between $B$ and $C$ is not empty: $B \cap C \neq \emptyset$.
- The intersection between $C$ and $D$ is not empty: $C \cap D \neq \emptyset$.
- The intersection between $A$ and $C$ is empty: $A \cap C = \emptyset$.
- The intersection between $B$ and $D$ is empty: $B \cap D = \emptyset$.
- The intersection between $A$ and $D$ is empty: $A \cap D = \emptyset$.

(a) Draw the Venn-diagram for the sets $A, B, C,$ and $D$.

(b) How many objects appear in at least two sets given the following data.
- $|A| = 5$.
- $|B| = 10$.
- $|C| = 15$.
- $|D| = 20$.
- $|A \cup B \cup C \cup D| = 40$.

**Answer:** The following is the inclusion-exclusion principle for four sets:

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D|$$

$$- |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D|$$

$$+ |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D|$$

$$- |A \cap B \cap C \cap D|$$

This implies that

$$|A| + |B| + |C| + |D| - |A \cup B \cup C \cup D| = 5 + 10 + 15 + 20 - 40 = 10$$

is the number of objects that appear in at least two sets because all the other combinations of sets represent intersections of at least two sets.

**Remark 1:** The answer in part (b) is correct for any four sets with or without the six listed restrictions about the pairwise intersections.

**Remark 2:** There are several possibilities for the number of objects in each zone in the Venn-diagram from part (a). Below is one of them.
2. (a) **Proposition:** \((x \land y) \lor z \not\equiv x \land (y \lor z)\)

**Proof:** Consider an assignment in which \(x = F\) and \(z = T\). Regardless of the assignment to \(y\), it follows that \((x \land y) \lor z = T\) because \(z = T\) and \(x \land (y \lor z) = F\) because \(x = F\) which implies the proposition.

(b) **Proposition:** \((x \lor y) \land z \not\equiv x \lor (y \land z)\)

**Proof:** Consider an assignment in which \(x = T\) and \(z = F\). Regardless of the assignment to \(y\), it follows that \((x \lor y) \land z = F\) because \(z = F\) and \(x \lor (y \land z) = T\) because \(x = T\) which implies the proposition.

**Remark:** Note that in the proofs of both part (a) and part (b) there is no need to generate the whole truth table. In order to prove that the two expressions are not equivalent, it is enough to show that the expressions are not the same for at least one assignment.
3. Prove that the following identity is correct by induction on \( n \geq 1 \) or by any other method.

\[
\sum_{i=1}^{2n} (2i - 1) = 4n^2
\]

Note that the sum starts with \( i = 1 \) but ends with \( i = 2n \) and not with \( i = n \). As a result, the inductive steps adds two new terms to the sum compared with the number of terms in the sum assumed by the induction hypothesis.

**Proof by induction:**

- **Notations.**
  
  \[
  L(n) = (2 \cdot 1 - 1) + (2 \cdot 2 - 1) + \cdots + (2 \cdot 2n - 1) = 1 + 3 + \cdots + (4n - 1)
  
  R(n) = 4n^2
  \]

- **Induction base.** Prove that \( L(1) = R(1) \):
  
  \[
  L(1) = 1 + 3 = 4 \\
  R(1) = 4 \cdot 1^2 = 4
  \]

- **Induction hypothesis.** Assume that \( L(k) = R(k) \) for \( k \geq 1 \):
  
  \[
  L(k) = 1 + 3 + \cdots + 4k - 1 \\
  R(k) = 4k^2
  \]

- **Inductive step.** Prove that \( L(k+1) = R(k+1) \) for \( k \geq 1 \):
  
  \[
  L(k + 1) = 1 + 3 + \cdots + (4k - 1) + (4k + 1) + (4k + 3) \\
  = L(k) + (4k + 1) + (4k + 3) \\
  = L(k) + (8k + 4) \\
  = R(k) + (8k + 4) \\
  = 4k^2 + 8k + 4 \\
  = 4(k^2 + 2k + 1) \\
  = 4(k + 1)^2 \\
  = R(k + 1)
  \]

**Second proof:** Recall that

\[
\sum_{i=1}^{n} i = \frac{n(n+1)}{2}
\]

This identity implies the following,

\[
L(n) = \sum_{i=1}^{2n} (2i - 1) \\
= \sum_{i=1}^{2n} (2i) + \sum_{i=1}^{2n} (-1) \\
= 2 \sum_{i=1}^{2n} i - 2n \\
= 2 \frac{2n(2n+1)}{2} - 2n \\
= 2n(2n+1) - 2n \\
= 4n^2 + 2n - 2n \\
= 4n^2 \\
= R(n)
\]

(* definition of \( R(n) \) *)
4. Solve the following recursive formula by finding the closed form expression for $T(n)$ as a function of $n$. Assume that $n$ is a positive integer.

$$T(n) = \begin{cases} 
3 & \text{for } n = 1 \\
6 & \text{for } n = 2 \\
2T(n - 1) - T(n - 2) & \text{for } n > 2
\end{cases}$$

Small values of $n$:

- $T(1) = 3$
- $T(2) = 6$
- $T(3) = 2 \cdot 6 - 3 = 12 - 3 = 9$
- $T(4) = 2 \cdot 9 - 6 = 18 - 6 = 12$
- $T(5) = 2 \cdot 12 - 9 = 24 - 9 = 15$
- $T(6) = 2 \cdot 15 - 12 = 30 - 12 = 18$

Solution: $T(n) = 3n$ for $n \geq 1$.

Proof by induction:

- **Induction base.**
  
  $T(1) = 3 = 3 \cdot 1$
  
  $T(2) = 6 = 3 \cdot 2$

- **Induction hypothesis.** For $n \geq 3$,
  
  $T(n - 1) = 3(n - 1) = 3n - 3$
  
  $T(n - 2) = 3(n - 2) = 3n - 6$

- **Inductive step.** For $n \geq 3$,
  
  $$T(n) = 2T(n - 1) - T(n - 2) \quad (\text{\ast definition of } T(n) \ast)$$
  
  $$= 2(3n - 3) - (3n - 6) \quad (\text{\ast induction hypothesis \ast})$$
  
  $$= 6n - 6 - 3n + 6 \quad (\text{\ast algebra \ast})$$
  
  $$= 3n \quad (\text{\ast algebra \ast})$$
5. Alice, Bob, and Charlie play the following game with a fair 6-face dice and a fair coin.

- In the first round they throw the dice. If they throw 6 they win and the game ends.
- Otherwise, if they throw 1, 2, 3, 4, or 5 they flip the coin.
- If they flip heads they win and if they flip tails they lose and the game ends.

(a) What is the probability that a player flips the coin?

**Answer:** A player flips the coin only if the player throws 1, 2, 3, 4, or 5. This happens with probability \( \frac{5}{6} \).

(b) What is the winning probability?

**Answer:** When a player throws 6 the player wins without flipping the coin and this happens with probability \( \frac{1}{6} \). With probability \( \frac{5}{6} \) the player flips the coin and then the player wins with probability \( \frac{1}{2} \). It follows that the winning probability is

\[
\frac{1}{6} + \frac{5}{6} \cdot \frac{1}{2} = \frac{1}{6} + \frac{5}{12} = \frac{7}{12}
\]

(c) Bob lost the game. What is the probability that Bob’s dice showed 1?

**Answer:** Bob’s dice did not show 6 because Bob lost the game. Since the other five options are equally possible, given that Bob lost, the probability that Bob’s dice showed 1 is \( \frac{1}{5} \).

(d) Alice won the game. What is the probability that Alice did not flip the coin?

**Answer:** With probability \( \frac{1}{6} \) Alice won without flipping the coin and with probability \( \frac{5}{6} \cdot \frac{1}{2} = \frac{5}{12} \) Alice won after flipping heads. Therefore, given that Alice won, the probability that Alice did not flip the coin is

\[
\frac{1}{6} + \frac{5}{12} = \frac{1}{7} + \frac{2}{7}
\]

(e) Charlie replaced the fair coin with a fake one whose both sides are heads. What is the probability that Charlie wins the game?

**Answer:** Charlie wins with probability 1. This is because Charlie wins either by throwing 6 or by flipping the fake coin after throwing a number smaller than 6.

**Some answers using counting arguments:** Assume for simplicity that players always flip the coin even after winning by throwing 6. The following table shows \( W \) (win) or \( L \) (lose) for each of the twelve possible options of throwing the dice and then flipping the coin.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>heads</td>
<td>W</td>
<td>W</td>
<td>W</td>
<td>W</td>
<td>W</td>
<td>W</td>
</tr>
<tr>
<td>tails</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>W</td>
</tr>
</tbody>
</table>

(b) There are 12 entries in the table out of which 7 entries are marked with \( W \). Therefore, the winning probability is \( \frac{7}{12} \).

(c) There are 5 entries that are marked with \( L \) out of which only one belongs to the column associated with throwing 1. Therefore, the probability that Bob lost the game because the dice showed 1 is \( \frac{1}{5} \).

(d) There are 7 entries that are marked with \( W \) out of which 2 entries belong to a win without flipping the coin. Therefore, the probability that Alice won the game without flipping the coin is \( \frac{2}{7} \).
6. Let $P$ be a problem whose input is an array of size $n$ for $n \geq 1$.

- Algorithm $A$ solves $P$ with complexity $\Theta(n)$.
- Algorithm $B$ solves $P$ with complexity $\Theta(\log(n))$.
- Algorithm $C$ solves $P$ with complexity $\Theta(n \log(n))$.
- Algorithm $D$ solves $P$ with complexity $\Theta(n^2)$.
- Algorithm $E$ solves $P$ with complexity $\Theta(1)$.

Order the five algorithms from the most efficient to the least efficient.

**Answer:** The following is the hierarchy among the five functions:

$$1 = o(\log(n)) = o(n) = o(n \log(n)) = o(n^2)$$

Therefore, using $X < Y$ to indicate that algorithm $X$ is more efficient than algorithm $Y$, it follows that the order among the five algorithms from the most efficient one to the least efficient one is as follows:

$$E < B < A < C < D$$

In other words, the most efficient algorithm is Algorithm $E$, the second most efficient algorithm is Algorithm $B$, the third most efficient algorithm is Algorithm $A$, the fourth most efficient algorithm is Algorithm $C$, while the least efficient algorithm is Algorithm $D$. 
7. Below are all the 11 non-isomorphic graphs with 4 vertices.

(a) Mark all the graphs that are trees.

**Answer:** $G_5$ and $G_7$.

These two graphs are the only connected graphs with 3 edges and without cycles.

(b) Mark all the graphs that are not connected.

**Answer:** $G_1$, $G_2$, $G_3$, $G_4$, and $G_6$.

$G_3$ has two isolated edges while the other four graphs each has at least one isolated vertex. The rest of the graphs do not have isolated edges or isolated vertices and therefore they are connected.
8. Alice and Bob play the following search game.

- Alice selects an integer \( x \) between 1 and \( n \): \( 1 \leq x \leq n \).
- Bob must find \( x \) by asking questions of the type “is \( x \leq i \)” for some integer \( 1 \leq i \leq n \).
- Alice may (but does not have to) lie in one of her answers. She must answer correctly the rest of the questions even if a question is repeated. However, Bob does not know in advance which answer is a lie.

In your own words (use a high-level description and try to avoid codes), describe an algorithm for Bob to find the integer \( x \) selected by Alice.

What is the worst-case complexity (number of questions asked) of your algorithm?

**Answer:** Bob can find \( x \) with at most \( \lceil \log_2(n) \rceil + 1 \) questions.

**Observation I:** If Bob repeats a question twice and gets the same answer then the answer is correct. This is true because Alice may lie at most once.

**Observation II:** If Bob repeats a question three times and gets one answer twice and the opposite answer once then the majority answer is the correct answer. This is true because Alice may lie at most once.

**Algorithm:** Bob searches for \( x \) by following the binary-search procedure. However, Bob repeats each procedure’s question twice and continues applying the procedure if both answers are identical. If the two answers are different, then Bob repeats this procedure’s question for the third time and adopts the majority answer. Once Alice lies, Bob proceeds with the binary-search procedure as is without repeating questions.

**Correctness:** Bob finds \( x \) because the binary-search procedure finds \( x \). Observation I and Observation II guarantee that Bob never makes mistakes.

**Complexity:** Assume first that \( n \) is a power of 2.

In the worst case, Alice lies to the first or the second occurrence of the last question of the binary-search procedure when Bob is reduced to a range of size 2 (either \([x, x + 1]\) or \([x - 1, x]\)). This forces Bob to repeat the last question of the procedure three times. In this case, Bob repeats each one of the first \( \log_2(n) - 1 \) procedure’s questions twice while repeating the last procedure’s question three times for the following total number of questions

\[
2(\log_2(n) - 1) + 3 = 2 \log_2(n) + 1
\]

Note that if Alice never lies the complexity is exactly \( 2 \log_2(n) \) because each question of the binary-search procedure is repeated twice. Moreover, if Alice lies to one of the first \( \log_2(n) - 1 \) procedure’s questions, then Bob asks fewer questions because after the lie Bob does not repeat questions.

When \( n \) can be any arbitrary integer the complexity is

\[
2 \lceil \log_2(n) \rceil + 1
\]

because the complexity of the binary-search procedure is \( \lceil \log_2(n) \rceil \).
9. A **balloon-graph** is a graph that is composed of a cycle that is connected to a path.
   - All the edges of the graph are either part of the cycle or part of the path.
   - Exactly one vertex belongs to the cycle and to the path.
   - Some vertices (at least 2) belong only to the cycle.
   - The rest of the vertices (at least 1) belong only to the path.

A balloon-graph must contain at least 4 vertices. See above the only possible balloon-graph with 4 vertices and two of the four possible non-isomorphic balloon-graphs with 7 vertices. One of them has a path of length 3 that is connected to a cycle of size 5 and one of them has a path of length 4 that is connected to a cycle of size 4.

(a) Illustrate the two possible non-isomorphic balloon-graphs with 5 vertices.

(b) Illustrate the three possible non-isomorphic balloon-graphs with 6 vertices.

(c) For \( n \geq 4 \), how many non-isomorphic balloon-graphs with \( n \) vertices exist?  
**Answer:** The size of the cycle, denoted by \( c \), determines the length of the path, denoted by \( p \). Since the cycle and the path share exactly one vertex, it follows that \( n = c + p - 1 \). A cycle must contain at least three vertices and therefore \( c \geq 3 \) while the path must contain at least two vertices and therefore \( p \geq 2 \) which implies that \( c \leq n - 1 \). As a result, there are exactly \( n - 3 \) non-isomorphic balloon-graphs each determined by \( c = 3, 4, \ldots, n - 1 \).

(d) For \( n \geq 4 \), all possible balloon-graphs have the same number of edges. What is this number?  
**Answer:** Denote by \( c \) the size of the cycle and by \( p \) the length of the path. In the previous part it was shown that \( n = c + p - 1 \). A cycle with \( c \) vertices has \( c \) edges and a path of length \( p \) has \( p - 1 \) edges. The edges of the cycle are disjoint to the edges of the path. Hence, the number of edges \( m \) of a balloon-graph with \( n \) vertices is \( c + p - 1 \). That is, \( m = n \) in a balloon-graph.

(e) For \( n \geq 4 \), the degree sequence of all possible balloon-graphs is the same. Let \( G \) be a balloon-graph with \( n \) vertices. How many vertices in \( G \) have degree 0? How many vertices in \( G \) have degree 1? How many vertices in \( G \) have degree 2? How many vertices in \( G \) have degree 3? How many vertices in \( G \) have degree greater than 3?  
**Answer:** Denote by \( c \) the size of the cycle and by \( p \) the length of the path. Let \( v \) be the vertex shared by the cycle and the path. It follows that the degree of \( v \) is 3, the degrees of any of the other \( c - 1 \) vertices in the cycle are 2, the degree of the vertex at the other end of the path is 1, and the degrees of the \( p - 2 \) inner vertices in the path are 2. Therefore, a balloon graph with \( n \) vertices has zero vertices with degree 0, one vertex with degree 1, \( n - 2 = (c - 1) + (p - 2) \) vertices with degree 2, and one vertex with degree 3. None of the vertices have degree greater than 3.