1. A length-$n$ ternary string is a list with repetitions $(d_1, d_2, \ldots, d_n)$ such that $d_i \in \{0, 1, 2\}$ for all $1 \leq i \leq n$. Assume $n \geq 2$. Justify your answers for each of the following five questions.

(a) How many length-$n$ ternary strings are there?

Answer: $3^n$

Explanation: There are $k^n$ lists with repetitions of length $n$ when the objects are taken from a set of size $k$. In ternary sets $k = 3$ and therefore the answer is $k^n = 3^n$.

(b) In how many length-$n$ ternary strings $d_1 = d_n$?

Answer: $3^{n-1}$

Explanation: After $d_1, d_2, \ldots, d_{n-1}$ are determined $d_n$ must be equal to $d_1$. As a result, the answer is the number of ternary strings of length $n - 1$ which is $3^{n-1}$.

(c) In how many length-$n$ ternary strings $d_1 \neq d_n$?

Answer: $2 \cdot 3^{n-1}$

Explanation: In a ternary string either $d_1 = d_n$ or $d_1 \neq d_n$. Therefore, the answer to this part plus the answer to part (b) is the answer to part (a). Hence the answer is $3^n - 3^{n-1} = 2 \cdot 3^{n-1}$.

(d) How many length-$n$ ternary strings do not contain 0?

Answer: $2^n$

Explanation: In this part, the objects in the list are taken from the set $\{1, 2\}$ and therefore $k = 2$. Hence, there are $2^n$ such strings. Observe that this is the number of binary strings in which 2 replaces 0.

(e) How many length-$n$ ternary strings contain at least one 0?

Answer: $3^n - 2^n$

Explanation: In a ternary string either 0 appears or not. Therefore, the answer to this part plus the answer to part (d) is the answer to part (a). Hence the answer is $3^n - 2^n$. 
2. Assume \( n \geq 2 \). Express the following sum of three binomial coefficients terms with one binomial coefficient term

\[
\binom{n}{1} + \binom{n}{2} + \binom{n+1}{1}
\]

**Hint:** Can be solved with algebra by replacing each \( \binom{n}{k} \) term with \( \frac{n!}{k!(n-k)!} \). Applying twice the recursive formula \( \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \) avoids the algebra.

**Answer:** \( \binom{n+2}{2} \).

An algebraic proof:

\[
\binom{n}{1} + \binom{n}{2} + \binom{n+1}{1} = \frac{n!}{1!(n-1)!} + \frac{n!}{2!(n-2)!} + \frac{(n+1)!}{1!n!}
\]

\[
= n + \frac{n(n-1)}{2} + (n+1)
\]

\[
= \frac{2n + n^2 - n + 2(n+2)}{2}
\]

\[
= \frac{n^2 + 3n + 2}{2}
\]

\[
= \frac{(n+2)(n+1)}{2}
\]

\[
= \binom{n+2}{2}
\]

A proof based on the recursive formula: Let \( x = \binom{n}{1} \), \( y = \binom{n}{2} \), \( z = \binom{n+1}{1} \), \( w = x + y \), and \( v = x + y + z \). First compute \( w \) based on the recursive formula

\[
w = x + y = \binom{n}{2} + \binom{n}{1} = \binom{n+1}{2}
\]

Then to get \( v \) as a single binomial coefficient term, compute \( w + z \) based on the recursive formula

\[
v = w + z = \binom{n+1}{2} + \binom{n+1}{1} = \binom{n+2}{2}
\]
3. Prove by induction that \( n! > 5n \) for all integers \( n \geq 4 \).

The inequality is false for \( 1 \leq n \leq 3 \) and true for \( 4 \leq n \leq 6 \)

\[
\begin{align*}
1! &= 1 \ < \ 5 \cdot 1 = 5 \\
2! &= 2 \ < \ 5 \cdot 2 = 10 \\
3! &= 6 \ < \ 5 \cdot 3 = 15 \\
4! &= 24 \ > \ 5 \cdot 4 = 20 \\
5! &= 120 \ > \ 5 \cdot 5 = 25 \\
6! &= 720 \ > \ 5 \cdot 6 = 30 \\
\end{align*}
\]

Proof by induction:

- **Induction base.** \( 4! = 24 > 20 = 5 \cdot 4 \) for \( n = 4 \).
- **Induction hypothesis.** Assume that \( k! > 5k \) for \( k \geq 4 \).
- **Inductive step.** Prove that \( (k+1)! > 5(k+1) \) for \( k \geq 4 \).

\[
\begin{align*}
(k+1)! &= (k+1)k! \\
> (k+1) \cdot 5k \\
= 5(k+1)k \\
> 5(k+1) \\
\end{align*}
\]

**Another proof:** The claim is correct for \( n = 4 \) since \( 4! = 24 > 5 \cdot 4 = 20 \) and for \( n = 5 \) since \( 5! = 120 > 5 \cdot 5 = 25 \). Assume now that \( n \geq 6 \). As a result, \( n! = n(n-1)(n-2)\cdots2 \) is greater than the partial product \( n(n-1) \). The proof follows because \( n - 1 \geq 5 \) implies that

\[n! > n(n-1) \geq 5n\]
4. Match each one of the following 5 recursive formulas with one of the possible 5 solutions:

\[-n \ ; \ 2n \ ; \ n^2 \ ; \ 2^n \ ; \ n!\]

(a) \(T(0) = 1\ ; \ T(n) = 2T(n-1)\) for \(n > 0\)

Solution: \(T(n) = 2^n\).

Intuition: In every recursive step the value is doubled by two.

Proof:
- *Induction base.* \(T(0) = 1 = 2^0\) for \(n = 0\).
- *Induction hypothesis.* Assume that \(T(n-1) = 2^{n-1}\) for \(n > 0\).
- *Inductive step.* Prove that \(T(n) = 2^n\) for \(n > 0\):

\[
T(n) = 2T(n-1) \\
= 2 \cdot 2^{n-1} \\
= 2^n
\]

(b) \(T(0) = 0\ ; \ T(n) = T(n - 1) - 1\) for \(n > 0\)

Solution: \(T(n) = -n\).

Intuition: In every recursive step the value is decremented by one.

Proof:
- *Induction base.* \(T(0) = 0 = -0\) for \(n = 0\).
- *Induction hypothesis.* Assume that \(T(n-1) = -(n - 1)\) for \(n > 0\).
- *Inductive step.* Prove that \(T(n) = -n\) for \(n > 0\):

\[
T(n) = T(n - 1) - 1 \\
= -(n - 1) - 1 \\
= -n + 1 - 1 \\
= -n
\]

(c) \(T(0) = 0\ ; \ T(n) = T(n - 1) + (2n - 1)\) for \(n > 0\)

Solution: \(T(n) = n^2\).

Intuition: In the \(n^{th}\) recursive step the value is incremented by \(2n - 1\) which is the difference between \(n^2\) and \((n - 1)^2\).

Proof:
- *Induction base.* \(T(0) = 0 = 0^2\) for \(n = 0\).
- *Induction hypothesis.* Assume that \(T(n-1) = (n - 1)^2\) for \(n > 0\).
- *Inductive step.* Prove that \(T(n) = n^2\) for \(n > 0\):

\[
T(n) = T(n - 1) + (2n - 1) \\
= (n - 1)^2 + (2n - 1) \\
= n^2 - 2n + 1 + 2n - 1 \\
= n^2
\]
(d) $T(0) = 1$ ; $T(n) = nT(n-1)$ for $n > 0$

Solution: $T(n) = n!$.

Intuition: In the $n^{th}$ recursive step the value is multiplied by $n$.

Proof:
- Induction base. $T(0) = 1 = 0!$ for $n = 0$.
- Induction hypothesis. Assume that $T(n - 1) = (n - 1)!$ for $n > 0$.
- Inductive step. Prove that $T(n) = n!$ for $n > 0$:

\[
T(n) = n \cdot T(n - 1) \\
= n \cdot (n - 1)! \\
= n!
\]

(e) $T(0) = 0$ ; $T(n) = T(n - 1) + 2$ for $n > 0$

Solution: $T(n) = 2n$.

Intuition: In every recursive step the value is incremented by 2.

Proof:
- Induction base. $T(0) = 0 = 2 \cdot 0$ for $n = 0$.
- Induction hypothesis. Assume that $T(n - 1) = 2(n - 1)$ for $n > 0$.
- Inductive step. Prove that $T(n) = 2n$ for $n > 0$:

\[
T(n) = T(n - 1) + 2 \\
= 2(n - 1) + 2 \\
= 2n - 2 + 2 \\
= 2n
\]
5. Prove that every integer greater than 7 is a sum of a nonnegative multiple of 3 and a nonnegative multiple of 5.

**Intuition:** If the claim is true for \( k \) because \( k = 3p + 5q \) for nonnegative integers \( p \) and \( q \), then it is true for \( k + 3 \) because \( k + 3 = 3(p + 1) = 5q \). To apply induction, the inductive step for \( n \) will be based on the induction hypothesis for \( n - 3 \) while the base case should be established for the three consecutive integers 8, 9, and 10.

**Proof:**
- **Induction base.** Verify correctness for \( n = 8, \ n = 9, \) and \( n = 10. \)
  - \( 8 = 1 \cdot 3 + 1 \cdot 5. \)
  - \( 9 = 3 \cdot 5 + 0 \cdot 5. \)
  - \( 10 = 0 \cdot 3 + 2 \cdot 5. \)
- **Induction hypothesis.** Assume that \( n - 3 = 3q + 5p \) for \( n > 10 \) and nonnegative integers \( p \) and \( q. \)
- **Inductive step.** Prove that \( n = 3(q + 1) + 5p\) for \( n > 10: \)

\[
\begin{align*}
n &= (n - 3) + 3 \\
&= (3p + 5q) + 3 \\
&= 3(q + 1) + 5q
\end{align*}
\]