1. For \( n \geq 1 \), let \( L = (\ell_1, \ell_2 \ldots \ell_n) \) be a list with repetitions of length \( n \). For \( 1 \leq i \leq j \leq n \), the sub-list \( L(i, j) \) of \( L \) is the list \((\ell_i, \ell_{i+1}, \ldots, \ell_j)\). The sub-list \( L(i, j) \) is a prefix-list of \( L \) if \( i = 1 \) and it is a suffix-list of \( L \) if \( j = n \).

Example: Consider the list \( L = (A, B, A, C, D, D, A) \) of length \( n = 7 \). One of its sub-lists is \( L(2, 4) = (B, A, C) \) and another one is \( L(3, 6) = (A, C, D, D) \). The sub-lists \( L(1, 2) = (A, B) \) and \( L(1, 1) = (A) \) are two of the prefix-lists of \( L \) and the sub-lists \( L(5, 7) = (D, D, A) \) and \( L(7, 7) = (A) \) are two of the suffix-lists of \( L \). The sub-list \( L(1, 7) \) (which is \( L \) itself) is both a prefix-list and a suffix-list of \( L \).

(a) For \( n \geq 1 \), how many prefix-lists of \( L \) are there as a function of \( n \)?

Answer: \( n \)

Proof: The \( n \) prefix-lists of \( L \) are \( L(1, 1), L(1, 2), \ldots, L(1, n) \).

(b) For \( n \geq 1 \), how many suffix-lists of \( L \) are there as a function of \( n \)?

Answer: \( n \)

Proof: The \( n \) suffix-lists of \( L \) are \( L(1, n), L(2, n), \ldots, L(n, n) \).

(c) For \( n \geq 1 \), how many sub-lists of \( L \) are there as a function of \( n \)?

Answer: \( \frac{n(n+1)}{2} \)

Proof I:
- \( L(1, 1) \) is the only sub-list that ends with the index 1.
- \( L(1, 2) \) and \( L(2, 2) \) are the only two sub-lists that end with the index 2.
- In general, for \( 1 \leq i \leq n \), there are only \( i \) sub-lists that end with the index \( i \):
  \[ L(1, i), L(2, i), \ldots, L(i, i) \]
- In particular, there are \( n \) sub-lists (the \( n \) suffix-lists) that ends with the index \( n \):
  \[ L(1, n), L(2, n), \ldots, L(n, n) \]
- Since a sub-list must end with an index \( i \) for some \( 1 \leq u \leq n \), it follows that in total the number of sub-lists is
  \[ 1 + 2 + \cdots + n = \frac{n(n + 1)}{2} \]

Proof II: Each choice of two indices \( 1 \leq i < j \leq n \) corresponds to a sub-list of \( L \) of length at least two. There are \( \binom{n}{2} \) such choices. In addition, \( L(1, 1), L(2, 2), \ldots, L(n, n) \) are the \( n \) sub-lists of size 1. In total the number of sub-lists is
  \[ \binom{n}{2} + n = \frac{n(n - 1)}{2} + n = \frac{n^2 - n + 2n}{2} = \frac{n^2 + n}{2} = \frac{(n + 1)n}{2} = \binom{n + 1}{2} \]
2. For a positive integer \( n \geq 3 \), simplify the following expression:

\[
n \binom{n}{2} - 3 \binom{n}{3}
\]

The goal is to “get rid” of the two binomial coefficients and replace the above expression with one simple term which is a function of \( n \).

**Answer:** \( n(n - 1) \).

**Proof:**

\[
n \cdot \binom{n}{2} - 3 \cdot \binom{n}{3} = n \cdot \frac{n(n - 1)}{2} - 3 \cdot \frac{n(n - 1)(n - 2)}{6}
\]

\[
= \frac{n(n - 1)n}{2} - \frac{n(n - 1)(n - 2)}{2}
\]

\[
= \frac{n(n - 1)(n - (n - 2))}{2}
\]

\[
= \frac{n(n - 1) \cdot 2}{2}
\]

\[
= n(n - 1)
\]
3. Prove by induction that $7^n - 1$ is divisible by 6 for all integers $n \geq 1$.

Proof by induction:

- **Induction base.** $7^1 - 1 = 6 = 6 \cdot 1$ for $n = 1$.
- **Induction hypothesis.** Assume that $7^k - 1 = 6 \cdot q$ for $k \geq 1$ and an integer $q$.
- **Inductive step.** Prove that $7^{k+1} - 1 = 6 \cdot p$ for $n > 1$ and an integer $p$.

\[
7^{k+1} - 1 = 7 \cdot 7^k - 1 \quad \text{(*) algebra *)}
\]
\[
= 6 \cdot 7^k + (7^k - 1) \quad \text{(*) algebra *)}
\]
\[
= 6 \cdot 7^k + 6 \cdot q \quad \text{(*) induction hypothesis *)}
\]
\[
= 6(7^k + q) \quad \text{(*) algebra *)}
\]

Hence, $7^{k+1} = 6 \cdot p$ for integer $p = 7^n - 1 + q$ and is therefore divisible by 6.

**Another proof:** The following is an identity for any positive real numbers $x \geq y$ and an integer $n \geq 1$,

\[
x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + x^2y^{n-3} + xy^{n-2} + y^{n-1})
\]

For $x = 7$ and $y = 1$ the above identity is equivalent to the following identity,

\[
7^n - 1 = 7^n - 1^n
\]
\[
= (7 - 1)(7^{n-1} + 7^{n-2} \cdot 1 + 7^{n-3} \cdot 1^2 + \cdots + 7^2 \cdot 1^{n-3} + 7 \cdot 1^{n-2} + 1^{n-1})
\]
\[
= (7 - 1)(7^{n-1} + 7^{n-2} + 7^{n-3} + \cdots + 49 + 7 + 1)
\]
\[
= 6(7^{n-1} + 7^{n-2} + \cdots + 7 + 1)
\]

This implies that $7^n - 1$ is divisible by 6.
4. For each one of the following three closed-form expressions, define a recursive formula with an initial value such that the solution to the recursive formula is the closed-form expression. That is, for each expression, define \( T(n) \) as a function of \( T(n-1) \) and define \( T(1) \).

(a) \( T(n) = 2n \) for an integer \( n \geq 1 \).

Answer:

\[
T(n) = \begin{cases} 
  2 & \text{for } n = 1 \\
  T(n-1) + 2 & \text{for } n \geq 1 
\end{cases}
\]

**Proof by induction** \( T(n) = 2n \) for \( n \geq 1 \):

- **Induction base.** \( T(1) = 2 \cdot 1 = 2 \) for \( n = 1 \).
- **Induction hypothesis.** Assume that \( T(n-1) = 2(n-1) \) for \( n > 1 \).
- **Inductive step.** Prove that \( T(n) = 2n \) for \( n > 1 \):

\[
\begin{align*}
T(n) &= T(n-1) + 2 \\
     &= 2(n-1) + 2 \\
     &= 2n - 2 + 2 \\
     &= 2n
\end{align*}
\]

(b) \( T(n) = 2^n \) for an integer \( n > 1 \).

Answer:

\[
T(n) = \begin{cases} 
  2 & \text{for } n = 1 \\
  2T(n-1) & \text{for } n > 1 
\end{cases}
\]

**Proof by induction that** \( T(n) = 2^n \) for \( n \geq 1 \):

- **Induction base.** \( T(1) = 2^1 = 2 \) for \( n = 1 \).
- **Induction hypothesis.** Assume that \( T(n-1) = 2^{n-1} \) for \( n > 1 \).
- **Inductive step.** Prove that \( T(n) = 2^n \) for \( n > 1 \):

\[
\begin{align*}
T(n) &= 2T(n-1) \\
     &= 2 \cdot 2^{n-1} \\
     &= 2^n
\end{align*}
\]

(c) \( T(n) = n! \) for an integer \( n \geq 1 \).

Answer:

\[
T(n) = \begin{cases} 
  1 & \text{for } n = 1 \\
  nT(n-1) & \text{for } n > 1 
\end{cases}
\]

**Proof by induction that** \( T(n) = n! \) for \( n \geq 1 \):

- **Induction base.** \( T(1) = 1! = 1 \) for \( n = 1 \).
- **Induction hypothesis.** Assume that \( T(n-1) = (n-1)! \) for \( n > 1 \).
- **Inductive step.** Prove that \( T(n) = n! \) for \( n > 1 \):

\[
\begin{align*}
T(n) &= nT(n-1) \\
     &= n \cdot (n-1)! \\
     &= n!
\end{align*}
\]
5. Solve the following recursive formula and prove that your solution is correct.

$$T(n) = \begin{cases} 
1 & \text{for } n = 0 \\
2T(n - 1) + 1 & \text{for } n \geq 1
\end{cases}$$

Small values of $n$:

- $n = 0$: $T(0) = 1 = 2^1 - 1$.
- $n = 1$: $T(1) = 2T(0) + 1 = 2 \cdot 1 + 1 = 3 = 2^2 - 1$.
- $n = 2$: $T(2) = 2T(1) + 1 = 2 \cdot 3 + 1 = 7 = 2^3 - 1$.
- $n = 3$: $T(3) = 2T(2) + 1 = 2 \cdot 7 + 1 = 15 = 2^4 - 1$.
- $n = 4$: $T(4) = 2T(3) + 1 = 2 \cdot 15 + 1 = 31 = 2^5 - 1$.

Answer: $T(n) = 2^{n+1} - 1$.

Proof by induction:

- **Induction base.** $T(0) = 2^{0+1} - 1 = 1$ for $n = 0$.
- **Induction hypothesis.** Assume that $T(n - 1) = 2^n - 1$ for $n > 0$.
- **Inductive step.** Prove that $T(n) = 2^{n+1} - 1$ for $n > 0$:

$$
\begin{align*}
T(n) &= 2T(n - 1) + 1 \\
     &= 2(2^n - 1) + 1 \\
     &= 2^{n+1} - 2 + 1 \\
     &= 2^{n+1} - 1
\end{align*}
$$