Discrete Structures: Counting and Combinatorics

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Numbers

How to count to 1000 on two hands

https://www.youtube.com/watch?v=lSMmc9gQmHQ

“Fascinating” properties of the numbers 1 to 9

https://www.youtube.com/watch?v=ByZLqOF-Jjk

There are 365 days in a year

- \( 365 = 10^2 + 11^2 + 12^2 = 13^2 + 14^2 \)
- It is the only integer for which there exists an integer \( n \) such that

\[
(n - 2)^2 + (n - 1)^2 + n^2 = (n + 1)^2 + (n + 2)^2
\]
Can Numbers be Interesting?

The taxi-cub number

- 1729 is the smallest number that is the sum of two cubes in two different ways:

\[ 1729 = 12^3 + 1^3 = 10^3 + 9^3 \]

Story

- This number is called a taxicab number, because in a discussion between the mathematicians G. H. Hardy and Srinivasa Ramanujan about interesting and dull numbers, the former remarked that the number 1729 of the taxicab he had ridden seemed uninteresting, and the latter immediately answered that it is interesting, being the smallest number that is the sum of two cubes in two different ways.
All the Positive Integers Are Interesting!

Proof

- The ”proof” is by contradiction.
- If there exists a non-empty set of uninteresting natural numbers, then there must be a smallest uninteresting number.
- But the smallest uninteresting number is itself interesting because it is the smallest uninteresting number: a contradiction.

Online resources

- https://www.youtube.com/watch?v=Ysd1XhqMbe8
- https://en.m.wikipedia.org/wiki/Interesting_number_paradox
Euler’s Identity

The identity

\[ e^{\pi i} + 1 = 0 \quad e^{\pi i} = -1 \]

A joke???

\( \pi \) tells \( i \): get real!
\( i \) answers to \( \pi \): be rational!
\( e \) tells both of them: join me and we will be - one!

https://www.youtube.com/watch?v=IUTGFQpKaPU

Online resources

- https://www.youtube.com/watch?v=sKtloBAuP74&t=233s
- https://www.youtube.com/watch?v=NXrBoWOBvIY
- https://www.youtube.com/watch?v=-dhHrg-KbJ0

Which is larger \( e^\pi \) or \( \pi^e \)?

- https://www.youtube.com/watch?v=JE_YeVlSqLY
Representing Numbers: Decimal, Binary, …

Different Bases For Numbers

- Online videos:
  - https://www.youtube.com/watch?v=1srwWeMe3BE
  - https://www.youtube.com/watch?v=aW3qCcH6Dao
  - https://www.youtube.com/watch?v=Fpm-E5v6ddc

- A short tutorial:
  - https://www.tutorialspoint.com/computer_logical_organization/number_system_conversion.htm

The Josephus Problem

- https://www.youtube.com/watch?v=uCsD3ZGzMgE
Combinations and Permutations

Introduction with Cartoon Slides

http://tinytram.com/math/combinatorics/

Online resources

https://www.youtube.com/watch?v=uNS1QvDzCVw&feature=youtu.be

https://www.youtube.com/watch?v=hVqq3nm0IHs

https://youtu.be/LM5iOHKo_Fc?list=PLMyAzUai9V3ox_LDw154GRkNxovx6NqQX
Four Counting Types

Setting

- **Input:** A universal set $S = \{s_1, s_2, \ldots, s_n\}$ with $n$ distinct objects.
- **Goal:** Count the number of some structures with some parameters containing objects from $S$.

Structures

- **Permutations:** In how many ways can all the objects of $S$ be ordered?
- **Lists with repetitions:** How many lists with $1 \leq k \leq n$ objects from $S$ exist where repetitions are allowed?
- **Lists without repetitions:** How many lists with $1 \leq k \leq n$ objects from $S$ exist where repetitions are not allowed?
- **Subsets:** How many subsets of $S$ exist?
Example: $\mathcal{S} = \{R, B, G, M\}$

There are $3! = 6$ permutations of $\{R, B, G\}$

- RGB, RGB, BRG, BGR, GRB, GBR

There are $4! = 24$ permutations of $\mathcal{S} = \{R, B, G, M\}$

- RBGM, RBMG, RGBM, RGMB, RMBG, RMGB, BRGM, BRMG, BGRM, BGMR, BMRG, BMGR, GRBM, GRMB, GBRM, GBMR, GMRB, GMRB, MRBG, MRGB, MBRG, MBGR, MGRB, MGBR
Example: \( S = \{ R, B, G, M \} \)

There are \( 4^2 = 16 \) lists with repetitions of length 2

\[
RR \quad RB \quad RG \quad RM \\
BR \quad BB \quad BG \quad BM \\
GR \quad GB \quad GG \quad GM \\
MR \quad MB \quad MG \quad MM
\]

There are \( 4^4 = 256 \) lists with repetitions of length 4

\[
RRRR \quad RRRB \quad RRRG \quad RRRM \quad \ldots \\
\ldots \quad BBBR \quad BBBB \quad BBBG \quad BBBM \quad \ldots \\
\ldots \quad GGGR \quad GGGB \quad GGGG \quad GGGM \quad \ldots \\
\ldots \quad MMMR \quad MMMB \quad MMMG \quad MMMM
\]
Example: \( S = \{R, B, G, M\} \)

There are \(4 \cdot 3 = 12\) lists without repetitions of length 2

- \(RB\)  \(RG\)  \(RM\)
- \(BR\)  \(BG\)  \(BM\)
- \(GR\)  \(GB\)  \(GM\)
- \(MR\)  \(MB\)  \(MG\)

There are \(4 \cdot 3 \cdot 2 = 24\) lists without repetitions of length 3

- \(RBG\)  \(RBM\)  \(RGB\)  \(RGM\)  \(RMB\)  \(RMG\)
- \(BRG\)  \(BRM\)  \(BGR\)  \(BGM\)  \(BMR\)  \(BMG\)
- \(GRB\)  \(GRM\)  \(GBR\)  \(GBM\)  \(GMR\)  \(GMB\)
- \(MRB\)  \(MRG\)  \(MBR\)  \(MBG\)  \(MGR\)  \(MGB\)
Example: \( S = \{ R, B, G, M \} \)

\( S \) has 4 subsets of size 1

\[
\{ R \} \quad \{ B \} \quad \{ G \} \quad \{ M \}
\]

\( S \) has 6 subsets of size 2

\[
\{ RB \} \quad \{ RG \} \quad \{ RM \} \\
\{ BG \} \quad \{ BM \} \quad \{ GM \}
\]

\( S \) has 4 subsets of size 3

\[
\{ RGB \} \quad \{ RBM \} \quad \{ RGM \} \quad \{ BGM \}
\]
Permutations

Definition

- An $n$-permutation $\pi$ is a 1-1 function from the set of numbers $\{1, 2, \ldots, n\}$ to itself
- $\pi(i) \neq \pi(j)$ for a permutation $\Pi = (\pi(1), \pi(2), \ldots, \pi(n))$ for all $1 \leq i \neq j \leq n$

Counting permutations

- There are $n! = n(n-1)(n-2)\cdots2\cdot1$ permutations for $n \geq 1$

Proof

- There are $n$ options for $\pi(1)$
- There are $n-1$ options for $\pi(2)$
- $\vdots$
- There are 2 options for $\pi(n-1)$
- There is only 1 option for $\pi(n)$
Permutations: Examples

**Small values**
- $1! = 1, \quad 2! = 2, \quad 3! = 6, \quad 4! = 24, \quad 5! = 120, \quad 6! = 720, \ldots$

**The 10 digits**
- There are $10! = 3628800$ different ways to arrange the 10 digits $0, 1, \ldots, 9$
- If numbers with less than 10 digits have leading zeros, then there are 10 billions ($10000000000 = 10^{10}$) numbers with 10 digits
- Only 0.036288% of these numbers contain all the 10 digits

**The 26 letters**
- There are $26! = 403291461126605635584000000$ arrangements of the 26 letters of the English alphabet
- $26! \approx 4 \times 10^{27}$ which is about 400 millions billions of billions
Derangements

Definition

- A derangement is a permutation $\pi$ on the numbers $1, 2, \ldots, n$ such that $\pi(i) \neq i$ for all $1 \leq i \leq n$.
- A derangement is a permutation $\pi$ on the numbers $1, 2, \ldots, n$ without fixed points ($\neg \exists_{1 \leq i \leq n}(\pi(i) = i)$).

Examples

- $(21)$ is the only derangement for $n = 2$.
- $(231)$ and $(312)$ are the only two derangements for $n = 3$ while the other four permutations $(123), (132), (213),$ and $(321)$ are not derangements because each contains at least one fixed point.
- The 9 derangements for $n = 4$ out of the 24 permutations:
  - $(1234)$, $(1243)$, $(1324)$, $(1342)$, $(1423)$, $(1432)$,
  - $(2134)$, $(2143)$, $(2314)$, $(2341)$, $(2413)$, $(2431)$,
  - $(3124)$, $(3142)$, $(3214)$, $(3241)$, $(3412)$, $(3421)$,
  - $(4123)$, $(4132)$, $(4213)$, $(4231)$, $(4312)$, $(4321)$.
Counting Derangements

Notation
- The number of derangements of \( n \) is \(!n\) (the subfactorial of \( n \))

Small \( n \)
- \(!1 = 0, !2 = 1, !3 = 2, !4 = 9, !5 = 44, !6 = 265 \ldots\)

Recursive formula
- \(!1 = 0, !2 = 1, \) and for \( n \geq 3\):
  \[ !n = (n - 1)(!(n - 1) + !(n - 2)) \]
  - \(!3 = 2(!2 + !1) = 2(1 + 0) = 2\)
  - \(!4 = 3(!3 + !2) = 3(2 + 1) = 9\)
  - \(!5 = 4(!4 + !3) = 4(9 + 2) = 44\)
  - \(!6 = 5(!5 + !4) = 5(44 + 9) = 265\)
Derangements

Non-recursive formulas

- \( !n = \left\lfloor \frac{n!}{e} \right\rfloor \approx \frac{n!}{2.718} \) where \([x]\) is the nearest integer to \(x\)
- \( !n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} = n!(1/2 - 1/6 + 1/24 - 1/120 + \cdots) \)

Corollary

- About \(1/e \approx 0.367879\) of the permutations are derangements
- About 63% of the permutations have at least one fixed point

An online resource

https://www.youtube.com/watch?v=pbXg5EI5t4c
Lists With Repetitions

**Definition**
- \( \mathcal{L} = (\ell_1, \ell_2, \ldots, \ell_k) \) is an ordered list of \( k \) objects from the set \( S = \{s_1, s_2, \ldots, s_n\} \) if \( \ell_i \in S \) for all \( 1 \leq i \leq k \).
- \( \ell_i \) could be equal to \( \ell_j \) for \( 1 \leq i < j \leq k \).

**Counting the number of lists with repetitions**
- There are \( n^k \) lists of length \( k \) from a set of size \( n \).

**Proof**
- For each \( 1 \leq i \leq k \), there are \( n \) options for \( \ell_i \).
- For the \( k \) possible indices \( 1 \leq i \leq k \), there are \( n^k \) options for \( (\ell_1, \ell_2, \ldots, \ell_k) \).
Lists With Repetitions: Examples

Numbers
- Assuming numbers have leading zeros, then there are 10 billions \((10000000000 = 10^{10})\) numbers (lists) with 10 digits

Letters
- There are \(26^3 = 17576\) possible three-letter words in English and \(26^4 = 456976\) possible four-letter words in English
- There are less than 200000 words in the Oxford English Dictionary!

Codes
- There are \(10^4 = 10000\) possible codes for a 4-digit lock
- The codes: 0000, 0001 \ldots 4567 \ldots 7766 \ldots 9998, 9999
Nesting Loops

Pseudocode

function \( f(n) \) (* integer \( n \geq 1 \) *)
\[
    c = 0 \\
    \text{for } i = 1 \text{ to } n \text{ do} \\
    \quad \text{for } j = 1 \text{ to } n \text{ do} \\
    \quad \quad \text{for } k = 1 \text{ to } n \text{ do} \\
    \quad \quad \quad \text{print } (i, j, k) \\
    \quad \quad c := c + 1
\]

Observations

- The function \( f(n) \) prints all possible lists with repetitions \((i, j, k)\) for which \(i, j, k \in \{1, 2, \ldots, n\}\)
- The value of \(c\) at the end is \(n^3\)
Strings

Definition
- An $n$-ary string of length $k$ is an ordered list $D = (d_1, d_2, \ldots, d_k)$ such that $d_i \in \{0, 1, \ldots, n-1\}$ for all $1 \leq i \leq k$
- In a binary string $d_i = 0$ or $d_i = 1$ for all $1 \leq i \leq k$

Counting strings
- There are $n^k$ strings of length $k$
- There are $2^k$ binary strings of length $k$

Example: the 16 binary strings of length 4

0000 0001 0010 0011 0100 0101 0110 0111
1000 1001 1010 1011 1100 1101 1110 1111

Example: the 27 ternary strings of length 3

000 001 002 010 011 012 020 021 022
100 101 102 110 111 112 120 121 122
200 201 202 210 211 212 220 221 222
Non Homogeneous Lists

Definition
- In a non-homogeneous list, for $1 \leq i \leq k$, the entry $\ell_i$ gets its value from a different domain of objects denoted by $S_i$
- $\mathcal{L} = (\ell_1, \ell_2, \ldots, \ell_k)$ is an ordered non-homogeneous list of $k$ objects if $\ell_i \in S_i$ for all $1 \leq i \leq k$

Counting the number of non-homogeneous lists
- Assume $n_i$ is the size $S_i$
- Then there are $n_1 n_2 \cdots n_k$ non-homogeneous lists of length $k$

Proof
- For each $1 \leq i \leq k$, there are $n_i$ options for $\ell_i$
- For the $k$ possible $1 \leq i \leq k$, there are $n_1 n_2 \cdots n_k$ options for $(\ell_1, \ell_2, \ldots, \ell_k)$
Non Homogeneous Lists: Examples

Passwords
- There are $26^2 \cdot 10^4 = 6760000$ possible passwords of length 6 that must start with 2 letters and end with 4 digits
- $S_1 = S_2 = \{A, B, \ldots, Z\}$ and $S_3 = S_4 = S_5 = S_6 = \{0, 1, \ldots, 9\}$
- AA0000, AA0001 ... CZ9999, DA0000 ... ZZ9998, ZZ9999

Taxi licenses
- There were only $10 \cdot 26 \cdot 10^2 = 26000$ possible taxi license numbers in New York city that must start with a digit followed by a letter and end with two digits
- $S_2 = \{A, B, \ldots, Z\}$ and $S_1 = S_3 = S_4 = \{0, 1, \ldots, 9\}$
- 0A00, 0A01 ... 5L99, 5M00 ... 9Z98, 9Z99
- To add licenses, there are now licenses like 5L99$_B$
Nesting Loops

Pseudocode

function $f(r, s, t)$ (* integers $r, s, t \geq 1$ *)

\[ c = 0 \]

for $i = 1$ to $r$ do

\[ \text{for } j = 1 \text{ to } s \text{ do} \]

\[ \text{for } k = 1 \text{ to } t \text{ do} \]

print $(i, j, k)$

\[ c := c + 1 \]

Observations

- The function $f(r, s, t)$ prints all possible non-homogeneous lists $(i, j, k)$ for which $i \in \{1, 2, \ldots, r\}$, $j \in \{1, 2, \ldots, s\}$, and $k \in \{1, 2, \ldots, t\}$
- The value of $c$ at the end is $r \cdot s \cdot t$
Lists Without Repetitions

Definition

- \( \mathcal{L} = (\ell_1, \ell_2, \ldots, \ell_k) \) is an ordered list without repetitions of \( k \leq n \) objects from the set \( S = \{s_1, s_2, \ldots, s_n\} \)
  - \( \ell_i \in S \) for all \( 1 \leq i \leq k \) and
  - \( \ell_i \neq \ell_j \) for \( 1 \leq i < j \leq k \)

Counting the number of lists without repetitions

- There are \( n^k = n(n-1)(n-2) \cdots (n-k+1) = \frac{n!}{(n-k)!} \) lists without repetitions of length \( k \) on the numbers \( 1, 2, \ldots, n \)

Proof

- There are \( n \) options for \( \ell_1 \)
- There are \( n-1 \) options for \( \ell_2 \)
  
  
  
  
- There are \( n-k+1 \) options for \( \ell_k \)
- In total there are \( n(n-1)(n-2) \cdots (n-k+1) \) options
Remarks

- $k \leq n$ because there are only $n$ options for each $\ell_i$
- Permutations are lists without repetitions for which $k = n$

Three-digit numbers

- There are $720 = 10 \cdot 9 \cdot 8$ three-digit numbers for which all the digits are different
- 012, 013 ... 309, 310 ... 598, 601 ... 986, 987

Three-letter words

- There are $26 \cdot 25 \cdot 24 = 15600$ possible three-letter words in English with three different letters
- The Scrabble Dictionary (OWL2) recognizes only 1015 three-letter words (words with repetitions)
Nesting Loops

Pseudocode

function $f(n)$ (* integer $n \geq 1$ *)

$c = 0$

for $i = 1$ to $n$ do

    for $j = 1$ to $n$ if $j \neq i$ do

        for $k = 1$ to $n$ if ($k \neq i$ and $k \neq j$) do

            print $(i, j, k)$

        $c := c + 1$

Observations

- The function $f(n)$ prints all possible lists without repetitions $(i, j, k)$ for which $i, j, k \in \{1, 2, \ldots, n\}$ are three distinct numbers
- The value of $c$ at the end is $n(n - 1)(n - 2)$
Subsets

Notation

- For $n \geq 1$ and $0 \leq k \leq n$, there are $\binom{n}{k}$ different subsets of $k$ objects out of a set of $n$ objects.

Special cases

- For $k = 0$, the empty set is the only subset with 0 objects and therefore, $\binom{n}{0} = 1$.
- For $k = n$, the entire set is the only subset with $n$ objects and therefore, $\binom{n}{n} = 1$.

Theorem

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Online calculator

https://www.omnicalculator.com/math/binomial-coefficient
Example

All the subsets of the set $S = \{ R, B, G, M \}$

- There is only $\binom{4}{0} = 1$ way to choose zero colors from $S$:
  * $\emptyset$

- There are $\binom{4}{1} = 4$ ways to choose one color from $S$:
  * $\{ R \}, \{ B \}, \{ G \}, \{ M \}$

- There are $\binom{4}{2} = 6$ ways to choose two colors from $S$:
  * $\{ R, B \}, \{ R, G \}, \{ R, M \}, \{ B, G \}, \{ B, M \}, \{ G, M \}$

- There are $\binom{4}{3} = 4$ ways to choose three colors from $S$:
  * $\{ R, B, G \}, \{ R, B, M \}, \{ R, G, M \}, \{ B, G, M \}$

- There is only $\binom{4}{4} = 1$ way to choose four colors from $S$:
  * $\{ R, B, G, M \}$
\( \binom{n}{k} = \frac{n!}{k!(n-k)!} \) For Small \( k \)

\[
\begin{align*}
\binom{n}{0} & = \frac{n!}{0!n!} = 1 \\
\binom{n}{1} & = \frac{n!}{1!(n-1)!} = n \\
\binom{n}{2} & = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2} \\
\binom{n}{3} & = \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{6} \\
\binom{n}{4} & = \frac{n!}{4!(n-4)!} = \frac{n(n-1)(n-2)(n-3)}{24} \\
\binom{n}{5} & = \frac{n!}{5!(n-5)!} = \frac{n(n-1)(n-2)(n-3)(n-4)}{120}
\end{align*}
\]
\[
\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}
\]

For Small \( n \)

\[
\begin{align*}
\binom{2}{2} &= \frac{2!}{2!0!} = \frac{2}{2 \cdot 1} = 1 = \frac{2 \cdot 1}{2} \\
\binom{3}{2} &= \frac{3!}{2!1!} = \frac{6}{2 \cdot 1} = 3 = \frac{3 \cdot 2}{2} \\
\binom{4}{2} &= \frac{4!}{2!2!} = \frac{24}{2 \cdot 2} = 6 = \frac{4 \cdot 3}{2} \\
\binom{5}{2} &= \frac{5!}{2!3!} = \frac{120}{2 \cdot 6} = 10 = \frac{5 \cdot 4}{2} \\
\binom{6}{2} &= \frac{6!}{2!4!} = \frac{720}{2 \cdot 24} = 15 = \frac{6 \cdot 5}{2} \\
\binom{7}{2} &= \frac{7!}{2!5!} = \frac{5040}{2 \cdot 120} = 21 = \frac{7 \cdot 6}{2}
\end{align*}
\]
\[
\binom{n}{3} = \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{6} \quad \text{For Small } n
\]

\[
\begin{align*}
\binom{3}{3} &= \frac{3!}{3!0!} = \frac{6}{6 \cdot 1} = 1 = \frac{3 \cdot 2 \cdot 1}{6} \\
\binom{4}{3} &= \frac{4!}{3!1!} = \frac{24}{6 \cdot 1} = 4 = \frac{4 \cdot 3 \cdot 2}{6} \\
\binom{5}{3} &= \frac{5!}{3!2!} = \frac{120}{6 \cdot 2} = 10 = \frac{5 \cdot 4 \cdot 3}{6} \\
\binom{6}{3} &= \frac{6!}{3!3!} = \frac{720}{6 \cdot 6} = 20 = \frac{6 \cdot 5 \cdot 4}{6} \\
\binom{7}{3} &= \frac{7!}{3!4!} = \frac{5040}{6 \cdot 24} = 35 = \frac{7 \cdot 6 \cdot 5}{6} \\
\binom{8}{3} &= \frac{8!}{3!5!} = \frac{40320}{6 \cdot 120} = 56 = \frac{8 \cdot 7 \cdot 6}{6}
\end{align*}
\]
Proof of Theorem

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]

- There are \( n \) ways to select the first object, there are \( n - 1 \) ways to select the second object, and so on ...  
- There are \( (n - k + 1) \) ways to select the last \( k \text{th} \) object  
- In total, there are 
  \[
g(n, k) = n(n - 1)(n - 2) \cdots (n - k + 1) = \frac{n!}{(n - k)!}
\]
  ways to select an ordered list of \( k \) objects from a set of \( n \) objects  
- Each subset of \( k \) objects is selected in \( k! \) different ordered lists and therefore, there are 
  \[
  \frac{g(n, k)}{k!} = \frac{n!}{k!(n - k)!}
  \]
  ways to select a subset of \( k \) objects from a set of \( n \) objects
Different Representations and Notations for \( \binom{n}{k} \)

**Formulas**

\[
\binom{n}{k} = \frac{n^k}{k!} \\
= \frac{n(n-1)(n-2) \cdots (n-k+1)}{k(k-1)(k-2) \cdots 2 \cdot 1} \\
= \frac{n \cdot (n-1) \cdot n-2 \cdots (n-k+1)}{1 \cdot 2 \cdot 3 \cdots k} \\
= \prod_{i=1}^{k} \frac{n+1-i}{i} \\
= \frac{n!}{k!(n-k)!}
\]

**Additional notations:** \( C \) stands for combinations

\[ C(n, k) \quad C_{n,k} \quad C_n^k \quad nC_k \]
Nesting Loops

Pseudocode

function $f(n)$ (* integer $n \geq 1$ *)
$$c = 0$$
for $i = 1$ to $n - 2$ do
  for $j = i + 1$ to $n - 1$ do
    for $k = j + 1$ to $n$ do
      print $(i, j, k)$
      $c := c + 1$

Observations

- The function $f(n)$ prints all possible subsets $\{i, j, k\}$ from the set $\{1, 2, \ldots, n\}$ such that $i < j < k$
- The value of $c$ at the end is $\binom{n}{3} = \frac{n(n-1)(n-2)}{6}$
Symmetry

Theorem

\[
\binom{n}{k} = \binom{n}{n-k}
\]

Algebraic Proof

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}
\]
Symmetry

Theorem

\[ \binom{n}{k} = \binom{n}{n-k} \]

Combinatorial Proof

- Selecting a subset of \( k \) objects from a set of \( n \) objects is equivalent to selecting the complement subset of the \( n - k \) objects not in the set.
- Therefore, the number of subsets of size \( k \) is equal to the number of subsets of size \( n - k \).
Example: \( S = \{C, R, B, G, M\} \)

Matching the \( \binom{5}{2} = 10 \) two-subsets with the \( \binom{5}{3} = 10 \) three-subsets:

\[
\begin{align*}
\{G, M\} & \leftrightarrow \{C, R, B\} \\
\{B, M\} & \leftrightarrow \{C, R, G\} \\
\{B, G\} & \leftrightarrow \{C, R, M\} \\
\{R, M\} & \leftrightarrow \{C, B, G\} \\
\{R, G\} & \leftrightarrow \{C, B, M\} \\
\{R, B\} & \leftrightarrow \{C, G, M\} \\
\{C, M\} & \leftrightarrow \{R, B, G\} \\
\{C, G\} & \leftrightarrow \{R, B, M\} \\
\{C, B\} & \leftrightarrow \{R, G, M\} \\
\{C, R\} & \leftrightarrow \{B, G, M\}
\end{align*}
\]
Subsets as Binary Strings

**Theorem**

The number of subsets of a set $S$ with $n$ objects is the same as the number of binary strings of length $n$ which is $2^n$

**Proof**

- A subset $R \subseteq S$ can be represented by the binary string $(b_1, b_2, \ldots, b_n)$ in which $b_i = 1$ if $s_i \in R$ and $b_i = 0$ if $s_i \notin R$

- A binary string $(b_1, b_2, \ldots, b_n)$ can be represented by a subset $R \subseteq S$ such that $s_i \in R$ if $b_i = 1$ and $s_i \notin R$ if $b_i = 0$

- Thus, there is a **one-to-one mapping** from the set $2^S$ of all the subsets of $S$ to the set of all binary strings of length $n$

- Therefore, $|2^S| = 2^n$
Example: The $2^4 = 16$ Subsets of $\{R, B, G, M\}$

- $\emptyset \equiv (0, 0, 0, 0)$
- $\{R\} \equiv (1, 0, 0, 0)$
- $\{B\} \equiv (0, 1, 0, 0)$
- $\{G\} \equiv (0, 0, 1, 0)$
- $\{M\} \equiv (0, 0, 0, 1)$
- $\{R, B\} \equiv (1, 1, 0, 0)$
- $\{R, G\} \equiv (1, 0, 1, 0)$
- $\{R, M\} \equiv (1, 0, 0, 1)$
- $\{B, G\} \equiv (0, 1, 1, 0)$
- $\{B, M\} \equiv (0, 1, 0, 1)$
- $\{G, M\} \equiv (0, 0, 1, 1)$
- $\{R, B, G\} \equiv (1, 1, 1, 0)$
- $\{R, B, M\} \equiv (1, 1, 0, 1)$
- $\{R, G, M\} \equiv (1, 0, 1, 1)$
- $\{B, G, M\} \equiv (0, 1, 1, 1)$
- $\{R, B, G, M\} \equiv (1, 1, 1, 1)$
Subsets as Binary Strings

Corollary
- For $0 \leq k \leq n$, there are $\binom{n}{k}$ binary strings of length $n$ with exactly $k$ ones

Proof
- By definition, a set of size $n$ has $\binom{n}{k}$ subsets of size $k$
- The one-to-one mapping in the proof of the theorem maps all the sets of size $k$ to all the binary strings with exactly $k$ ones

Special cases
- The null set $\emptyset$ is equivalent to the all-0 binary string $(0, 0, \ldots, 0)$
- The set itself is equivalent to the all-1 binary string $(1, 1, \ldots, 1)$
- A binary string with a singleton 1 is equivalent to a singleton subset $\{x\}$ that contains one of the objects $x$ from the set
An Identity

**Theorem**

\[
\binom{n}{2} + \binom{n+1}{2} = n^2
\]

**Proof**

\[
\begin{align*}
\binom{n}{2} + \binom{n+1}{2} &= \frac{n(n-1)}{2} + \frac{(n+1)n}{2} \\
&= \frac{n(n-1) + (n+1)n}{2} \\
&= \frac{n((n-1) + (n+1))}{2} \\
&= \frac{n \cdot 2n}{2} \\
&= n^2
\end{align*}
\]

**Examples**

\[
\begin{align*}
\binom{3}{2} + \binom{4}{2} &= 3 + 6 = 9 = 3^2 \\
\binom{5}{2} + \binom{6}{2} &= 10 + 15 = 25 = 5^2
\end{align*}
\]
Recursive Formula

Recursion

for all integers $n \geq 0$ \( \binom{n}{0} = \binom{n}{n} = 1 \)

for all integers $1 \leq k \leq n - 1$ \( \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \)

Examples

- $6 = \binom{4}{2} = \binom{3}{1} + \binom{3}{2} = 3 + 3 = 6$
- $10 = \binom{5}{3} = \binom{4}{2} + \binom{4}{3} = 6 + 4 = 10$
- $20 = \binom{6}{3} = \binom{5}{2} + \binom{5}{3} = 10 + 10 = 20$
- $15 = \binom{6}{4} = \binom{5}{3} + \binom{5}{4} = 10 + 5 = 15$
- $35 = \binom{7}{4} = \binom{6}{3} + \binom{6}{4} = 20 + 15 = 35$
Recursive Formula: Combinatorial Proof I

\[
\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} : \text{Informal proof by induction}
\]

- Consider a set with \( n \) objects \( S = \{ s_1, s_2, \ldots, s_n \} \)
- There are two options for selecting a subset \( R \) of \( S \) with \( k \) objects:
  * \( s_n \in R \): There are \( \binom{n-1}{k-1} \) different ways to select additional \( k - 1 \) objects out of \( s_1, s_2, \ldots, s_{n-1} \)
  * \( s_n \notin R \): There are \( \binom{n-1}{k} \) different ways to select \( k \) objects out of \( s_1, s_2, \ldots, s_{n-1} \)
- In total the number of ways to select \( k \) objects from a set of \( n \) objects \( \binom{n}{k} \) is also

\[
\binom{n-1}{k-1} + \binom{n-1}{k}
\]
Example

The six subsets of size 2 from the set $S = \{R, B, G, M\}$

- There are $\binom{4}{2} = 6$ ways to choose two colors from $S$:
  - $\{R, B\}, \{R, G\}, \{R, M\}, \{B, G\}, \{B, M\}, \{G, M\}$

- There are $\binom{3}{1} = 3$ ways to choose two colors from $S$ where one of them is Magenta:
  - $\{R, M\}, \{B, M\}, \{G, M\}$

- There are $\binom{3}{2} = 3$ ways to choose two colors from $S$ none of them is Magenta:
  - $\{R, B\}, \{R, G\}, \{B, G\}$
The ten subsets of size 3 from the set $S = \{ C, R, B, G, M \}$

- There are $\binom{5}{3} = 10$ ways to choose three colors from $S$:
  - $\{ R, B, G \}$, $\{ R, B, M \}$, $\{ R, G, M \}$, $\{ B, G, M \}$

- There are $\binom{4}{2} = 6$ ways to choose three colors from $S$ where one of them is Magenta:

- There are $\binom{4}{3} = 4$ ways to choose three colors from $S$ none of them is Magenta:
Recursive Formula: Algebraic Proof

\[
{n \choose k} = {n-1 \choose k-1} + {n-1 \choose k}
\]

\[
{n-1 \choose k-1} + {n-1 \choose k} = \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!}
\]

\[
= \frac{k(n-1)!}{k(k-1)!(n-k)!} + \frac{(n-k)(n-1)!}{(n-k)k!(n-k-1)!}
\]

\[
= \frac{k(n-1)!}{k!(n-k)!} + \frac{(n-k)(n-1)!}{k!(n-k)!}
\]

\[
= \frac{k(n-1)! + (n-k)(n-1)!}{k!(n-k)!}
\]

\[
= \frac{(k + (n-k))(n-1)!}{k!(n-k)!}
\]

\[
= \frac{n(n-1)!}{k!(n-k)!}
\]

\[
= \frac{n!}{k!(n-k)!}
\]

\[
= {n \choose k}
\]
Recursive Formula: Combinatorial Proof II

\( \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \): Informal proof by induction

- \( \binom{n}{k} \) is the number of binary strings of length \( n \) containing \( k \) 1’s
- Some strings start with a 1 while the rest start with a 0
- Binary strings which start with a 1:
  - After the 1, out of the remaining \( n-1 \) bits \( k-1 \) bits must be 1.
  - There are \( \binom{n-1}{k-1} \) such binary strings
- Binary strings which start with a 0:
  - After the 0, out of the remaining \( n-1 \) bits \( k \) must be 1
  - There are \( \binom{n-1}{k} \) such binary strings

Thus, the number of binary strings of length \( n \) containing \( k \) 1’s is

\[ \binom{n-1}{k-1} + \binom{n-1}{k} \]
Example

The 10 binary strings of length 5 with exactly 2 ones

(11000) (10100) (10010) (10001) (01100)
(01010) (01001) (00110) (00101) (00011)

- There are \( \binom{4}{1} = 4 \) strings that start with 1:
  (11000) (10100) (10010) (10001)
- There are \( \binom{4}{2} = 6 \) strings that start with 0:
  (01100) (01010) (01001) (00110) (00101) (00011)
- The total number of strings is \( \binom{5}{2} = \binom{4}{1} + \binom{4}{2} = 4 + 6 = 10 \)
Example

The 20 binary strings of length 6 with exactly 3 zeros

(000111) (001011) (001101) (001110) (010011)
(010101) (010110) (011001) (011010) (011100)
(100011) (100101) (100110) (101001) (101010)
(101100) (110001) (110010) (110100) (111000)

- There are \( \binom{5}{2} = 10 \) strings that start with 0:
  (000111) (001011) (001101) (001110) (010011)
  (010101) (010110) (011001) (011010) (011100)

- There are \( \binom{5}{3} = 10 \) strings that start with 1:
  (100011) (100101) (100110) (101001) (101010)
  (101100) (110001) (110010) (110100) (111000)

- The total number of strings is \( \binom{6}{3} = \binom{5}{2} + \binom{5}{3} = 10 + 10 = 20 \)
Theorem

For $1 \leq k \leq n$,

$$\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}$$

Proof

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$= \frac{n(n-1)!}{k(k-1)!(n-k)!}$$

$$= \frac{n \cdot (n-1)!}{k \cdot (k-1)!(n-k)!}$$

$$= \frac{n(n-1)}{k(k-1)} \cdot \frac{(n-1)!}{(n-k)!}$$

$$= \frac{n}{k} \binom{n-1}{k-1}$$
3rd Recursive Formula

**Theorem**

For \(1 \leq k \leq n\),

\[
\binom{n}{k} = \frac{n}{n-k} \binom{n-1}{k}
\]

**Proof**

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)!}{k!(n-k)(n-k-1)!} = \frac{n}{n-k} \cdot \frac{(n-1)!}{k!(n-k-1)!} = \frac{n}{n-k} \binom{n-1}{k}
\]
Theorem

For $1 \leq k \leq n$,

$$\binom{n}{k} = \frac{n + 1 - k}{k} \binom{n}{k - 1}$$

Proof

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$= \frac{(n+1-k)n!}{k(k-1)!(n+1-k)(n-k)!}$$

$$= \frac{n+1-k}{k} \cdot \frac{n!}{(k-1)!(n+1-k)!}$$

$$= \frac{n+1-k}{k} \binom{n}{k-1}$$
Examples

\( \binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35 \)

\begin{itemize}
  \item \( \binom{7}{3} = \binom{6}{2} + \binom{6}{3} = 15 + 20 = 35 \)
  \item \( \binom{7}{3} = \frac{7}{3} \cdot \binom{6}{2} = \frac{7}{3} \cdot \frac{6 \cdot 5}{2 \cdot 1} = \frac{7}{3} \cdot 15 = 35 \)
  \item \( \binom{7}{3} = \frac{7}{4} \cdot \binom{6}{3} = \frac{7}{4} \cdot \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = \frac{7}{4} \cdot 20 = 35 \)
  \item \( \binom{7}{3} = \frac{5}{3} \cdot \binom{7}{2} = \frac{5}{3} \cdot \frac{7 \cdot 6}{2 \cdot 1} = \frac{5}{3} \cdot 21 = 35 \)
\end{itemize}

\( \binom{8}{3} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56 \)

\begin{itemize}
  \item \( \binom{8}{3} = \binom{7}{2} + \binom{7}{3} = 21 + 35 = 56 \)
  \item \( \binom{8}{3} = \frac{8}{3} \cdot \binom{7}{2} = \frac{8}{3} \cdot \frac{7 \cdot 6}{2 \cdot 1} = \frac{8}{3} \cdot 21 = 56 \)
  \item \( \binom{8}{3} = \frac{8}{5} \cdot \binom{7}{3} = \frac{8}{5} \cdot \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = \frac{8}{5} \cdot 35 = 56 \)
  \item \( \binom{8}{3} = \frac{6}{3} \cdot \binom{8}{2} = 2 \cdot \frac{8 \cdot 7}{2 \cdot 1} = 2 \cdot 28 = 56 \)
\end{itemize}
Another Proof for the Main Recursive Formula

\[
\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}
\]

- The 2\textsuperscript{nd} recursive formula implies that

\[
\binom{n-1}{k-1} = \frac{k}{n} \binom{n}{k}
\]

- The 3\textsuperscript{rd} recursive formula implies that

\[
\binom{n-1}{k} = \frac{n-k}{n} \binom{n}{k}
\]

Therefore,

\[
\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{k}{n} \binom{n}{k} + \frac{n-k}{n} \binom{n}{k} = \frac{(k+n-k)}{n} \binom{n}{k} = \binom{n}{k}
\]
\[ (x + y)^n \]

**Theorem**

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k
\]

\[ = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{k} x^{n-k} y^k + \cdots + \binom{n}{n-1} xy^{n-1} + \binom{n}{n} y^n \]

\[ = x^n + nx^{n-1} y + \frac{n(n-1)}{2} x^{n-2} y^2 + \cdots + \frac{n(n-1)}{2} x^2 y^{n-2} + nxy^{n-1} + y^n \]

**The binomial coefficients**

- Based on the above theorem, \( \binom{n}{k} \) is called a binomial coefficient.
\((x + y)^n\)

**Proof**

- By definition,
  \[
  (x + y)^n = (x + y) \cdot (x + y) \cdots (x + y) \cdot (x + y)
  \]
  
- Using the distributive laws to get the product \(x^k y^{n-k}\):
  
  - select \(k\) of the \(n\) terms to contribute an \(x\) to the product
  - select the other \(n - k\) terms to contribute a \(y\) to the product

- The coefficient of \(x^k y^{n-k}\) is therefore \(\binom{n}{k}\):
  
  - the number of ways to select \(k\) objects from a set of size \(n\)

- Summing over all possible values of \(k\) from 0 to \(n\) implies that
  \[
  (x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}
  \]
Examples

\[(x + y)^1 = \binom{1}{0}x^1y^0 + \binom{1}{1}x^0y^1 = x + y\]

\[(x + y)^2 = \binom{2}{0}x^2y^0 + \binom{2}{1}x^1y^1 + \binom{2}{2}x^0y^2 = x^2 + 2xy + y^2\]

\[(x + y)^3 = \binom{3}{0}x^3y^0 + \binom{3}{1}x^2y^1 + \binom{3}{2}x^1y^2 + \binom{3}{3}x^0y^3 = x^3 + 3x^2y + 3xy^2 + y^3\]

\[(x + y)^4 = \binom{4}{0}x^4y^0 + \binom{4}{1}x^3y^1 + \binom{4}{2}x^2y^2 + \binom{4}{3}x^1y^3 + \binom{4}{4}x^0y^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\]
Corollary

\[(1 + y)^n = \binom{n}{0} y^0 + \binom{n}{1} y^1 + \binom{n}{2} y^2 + \cdots + \binom{n}{n-1} y^{n-1} + \binom{n}{n} y^n = \sum_{k=0}^{n} \binom{n}{k} y^k\]

Example

\[(1 + y)^4 = \binom{4}{0} y^0 + \binom{4}{1} y^1 + \binom{4}{2} y^2 + \binom{4}{3} y^3 + \binom{4}{4} y^4 = 1 + 4y + 6y^2 + 4y^3 + y^4\]
Sum of All Binomial Coefficients for a Given $n$

**Examples**

$2^0 = \binom{0}{0} = 1$

$2^1 = \binom{1}{0} + \binom{1}{1} = 1 + 1 = 2$

$2^2 = \binom{2}{0} + \binom{2}{1} + \binom{2}{2} = 1 + 2 + 1 = 4$

$2^3 = \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 1 + 3 + 3 + 1 = 8$

$2^4 = \binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 1 + 4 + 6 + 4 + 1 = 16$
Theorem

\[ \sum_{k=0}^{n} \binom{n}{k} = 2^n \]

Proof

\[ 2^n = (1 + 1)^n \]

\[ = \binom{n}{0}1^n1^0 + \binom{n}{1}1^{n-1}1^1 + \binom{n}{2}1^{n-2}1^2 + \cdots + \binom{n}{n}1^01^n \]

\[ = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} \]

\[ = \sum_{k=0}^{n} \binom{n}{k} \]
Theorem

A set of size \( n \) has \( 2^n \) subsets

Proof

By definition, a set of size \( n \) has \( \binom{n}{i} \) subsets of size \( i \) for \( 0 \leq i \leq n \).

Therefore, the number of subsets of a set of size \( n \) is

\[
\sum_{i=0}^{i=n} \binom{n}{i} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}
\]

By the previous theorem this sum equals \( 2^n \).
Computing $3^n$ with Powers of 2

**Corollary**

$$3^n = (1 + 2)^n$$

$$= \binom{n}{0}1^n 2^0 + \binom{n}{1}1^{n-1}2^1 + \binom{n}{2}1^{n-2}2^2 + \ldots + \binom{n}{n}1^0 2^n$$

$$= \binom{n}{0}2^0 + \binom{n}{1}2^1 + \binom{n}{2}2^2 + \ldots + \binom{n}{n}2^n$$

$$= \sum_{k=0}^{n} \binom{n}{k}2^k$$
Computing \(3^n\) with Powers of 2

\[n = 3\]

\[27 = 3^3 = \binom{3}{0}2^0 + \binom{3}{1}2^1 + \binom{3}{2}2^2 + \binom{3}{3}2^3\]

\[= 1 \cdot 1 + 3 \cdot 2 + 3 \cdot 4 + 1 \cdot 8\]

\[= 1 + 6 + 12 + 8\]

\[n = 4\]

\[81 = 3^4 = \binom{4}{0}2^0 + \binom{4}{1}2^1 + \binom{4}{2}2^2 + \binom{4}{3}2^3 + \binom{4}{4}2^4\]

\[= 1 \cdot 1 + 4 \cdot 2 + 6 \cdot 4 + 4 \cdot 8 + 1 \cdot 16\]

\[= 1 + 8 + 24 + 32 + 16\]
Pascal’s Triangle

Construction rules
- The left edge and the right edge are all 1
- Construct the rows from top to bottom
- Each number is the sum of the two numbers above it diagonally

The first 7 rows
Pascal’s Triangle

The number triangle vs. the binomial coefficient triangle

```
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
```

```
(0)
(0) (1)
(2) (2) (2)
(3) (3) (3) (3)
(4) (4) (4) (4) (4)
(5) (5) (5) (5) (5) (5)
(6) (6) (6) (6) (6) (6) (6)
(7) (7) (7) (7) (7) (7) (7) (7)
```
Pascal’s Triangle

The first 13 rows

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1

1 11 55 165 330 462 462 330 165 55 11 1

1 12 66 220 495 792 924 792 495 220 66 12 1
### Pascal’s Triangle

#### The sum of the first 9 rows

<table>
<thead>
<tr>
<th>Row</th>
<th>Sum</th>
<th>Equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>$1 = 2^0$</td>
</tr>
<tr>
<td>2</td>
<td>1 + 1 = 2</td>
<td>$2 = 2^1$</td>
</tr>
<tr>
<td>3</td>
<td>1 + 2 + 1 = 4</td>
<td>$4 = 2^2$</td>
</tr>
<tr>
<td>4</td>
<td>1 + 3 + 3 + 1 = 8</td>
<td>$8 = 2^3$</td>
</tr>
<tr>
<td>5</td>
<td>1 + 4 + 6 + 4 + 1 = 16</td>
<td>$16 = 2^4$</td>
</tr>
<tr>
<td>6</td>
<td>1 + 5 + 10 + 10 + 5 + 1 = 32</td>
<td>$32 = 2^5$</td>
</tr>
<tr>
<td>7</td>
<td>1 + 6 + 15 + 20 + 15 + 6 + 1 = 64</td>
<td>$64 = 2^6$</td>
</tr>
<tr>
<td>8</td>
<td>1 + 7 + 21 + 35 + 35 + 21 + 7 + 1 = 128</td>
<td>$128 = 2^7$</td>
</tr>
<tr>
<td>9</td>
<td>1 + 8 + 28 + 56 + 70 + 56 + 28 + 8 + 1 = 256</td>
<td>$256 = 2^8$</td>
</tr>
</tbody>
</table>
Pascal’s Triangle

Online resources

- The mathematical secrets of Pascal’s triangle
  https://www.youtube.com/watch?v=XMriWTvPXHI

- The Math of “The 12 Days Of Christmas”
  https://www.youtube.com/watch?v=fC8W4s6N9HQ

- What You Don’t Know About Pascal’s Triangle
  https://www.youtube.com/watch?v=J0I1NuxUcpQ

- Pascal’s Triangle - Numberphile
  https://www.youtube.com/watch?v=0iMt1us-afo

- Summary of facts
  https://www.mathsisfun.com/pascals-triangle.html
Solving Problems

Counting rectangles in a square grid
- **Animation:**
  https://www.youtube.com/watch?v=GfODdLHwWZw

Counting paths in a rectangular grid
- **Lecture:**
  https://www.youtube.com/watch?v=fpnNaAU0iPk&list=PLmdFyQYShrjfPLdHQxuNWvh2ct666Na3z
- **Animation:**
  https://www.youtube.com/watch?v=9YUl0k2FYzc
Counting Triplets

**Theorem for** $n \geq 1$

\[
\binom{n+2}{3} = \sum_{h=1}^{n} h(n+1-h)
\]

**The binomial side**

\[
\binom{n+2}{3} = \frac{(n+2) \cdot (n+1) \cdot n}{3 \cdot 2 \cdot 1} = \frac{n^3 + 3n^2 + 2n}{6}
\]

**The sum of products side**

\[
1 \cdot n + 2 \cdot (n-1) + 3 \cdot (n-2) + \cdots + h(n+1-h) + \cdots + (n-2) \cdot 3 + (n-1) \cdot 2 + n \cdot 1
\]
Counting Triplets

\binom{n+2}{3} in Pascal’s Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1
Counting Triplets

Examples

\[
\binom{4}{3} = \frac{4 \cdot 3 \cdot 2}{3 \cdot 2 \cdot 1} = 4 \\
= 1 \cdot 2 + 2 \cdot 1 = 2 + 2 = 4
\]

\[
\binom{5}{3} = \frac{5 \cdot 4 \cdot 3}{3 \cdot 2 \cdot 1} = 10 \\
= 1 \cdot 3 + 2 \cdot 2 + 3 \cdot 1 = 3 + 4 + 3 = 10
\]

\[
\binom{6}{3} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20 \\
= 1 \cdot 4 + 2 \cdot 3 + 3 \cdot 2 + 4 \cdot 1 = 4 + 6 + 6 + 4 = 20
\]

\[
\binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35 \\
= 1 \cdot 5 + 2 \cdot 4 + 3 \cdot 3 + 4 \cdot 2 + 5 \cdot 1 = 5 + 8 + 9 + 8 + 5 = 35
\]
Counting Triplets

**Combinatorial proof:** \( \binom{n+2}{3} = \sum_{h=1}^{n} h(n + 1 - h) \)

- There are \( \binom{n+2}{3} \) ordered triplets \((i < j < k)\) in the set \(\{1, \ldots, n+2\}\).
- Fix the middle index \(j\).
- \(j\) is neither 1 nor \(n + 2\) and therefore \(2 \leq j \leq n + 1\).
- There are \(j - 1\) ways to select \(i \in \{1, 2, \ldots, j - 1\}\) and there are \(n + 2 - j\) ways to select \(k \in \{j + 1, j + 2, \ldots, n + 2\}\).
- Therefore, the number of triplets \((i, j, k)\) with \(j\) as the middle index is \((j - 1)(n + 2 - j)\).
- The total number of triplets is \(\sum_{j=2}^{n+1} (j - 1)(n + 2 - j)\).
- Replacing \(j\) with \(h + 1\) implies that this number is \(\sum_{h=1}^{n} h(n + 1 - h)\).
Counting Triplets

The combinatorial proof using pseudocodes

- Pseudocode I: The value of $c$ at the end is $\binom{n+2}{3} = \frac{(n+2)(n+1)n}{6}$
- Pseudocode II: The value of $c$ is $\sum_{j=2}^{n+1} (j-1)(n+2-j)$

Pseudocode I

```c
function f(n) (* integer n ≥ 1 *)
    c = 0
    for i = 1 to n do
        for j = i + 1 to n + 1 do
            for k = j + 1 to n + 2 do
                c := c + 1
```

Pseudocode II

```c
function f(n) (* integer n ≥ 1 *)
    c = 0
    for j = 2 to n + 1 do
        for i = 1 to j - 1 do
            for k = j + 1 to n + 2 do
                c := c + 1
```
### Counting Triplets

**Example: $n = 5$: lexicographic order vs. proof order**

<table>
<thead>
<tr>
<th>Proof Order</th>
<th>Lexicographic Order</th>
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## Counting Triplets

**Example: \( n = 6 \): lexicographic order vs. proof order**

<table>
<thead>
<tr>
<th>Lexicographic Order</th>
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Sum of Squares of Binomial Coefficients

Theorem

\[ \binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2 \]

Example \( n = 1 \)

\[ \binom{2}{1} = 2 \]

\[ \binom{2}{1} = \sum_{k=0}^{1} \binom{1}{k}^2 \]

\[ = \binom{1}{0}^2 + \binom{1}{1}^2 \]

\[ = 1^2 + 1^2 = 1 + 1 = 2 \]
Sum of Squares of Binomial Coefficients

Theorem

\[
\binom{2n}{n} = \sum_{k=0}^{n} \left( \binom{n}{k} \right)^2
\]

Example \( n = 2 \)

\[
\binom{4}{2} = \frac{4 \cdot 3}{2 \cdot 1} = 6
\]

\[
\binom{4}{2} = \sum_{k=0}^{2} \left( \binom{2}{k} \right)^2
\]

\[
= \left( \binom{2}{0} \right)^2 + \left( \binom{2}{1} \right)^2 + \left( \binom{2}{2} \right)^2
\]

\[
= 1^2 + 2^2 + 1^2 = 1 + 4 + 1 = 6
\]
Sum of Squares of Binomial Coefficients

Theorem

\[
\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2
\]

Example \( n = 3 \)

\[
\binom{6}{3} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20
\]

\[
\binom{6}{3} = \sum_{k=0}^{3} \binom{3}{k}^2
\]

\[
= \binom{3}{0}^2 + \binom{3}{1}^2 + \binom{3}{2}^2 + \binom{3}{3}^2
\]

\[
= 1^2 + 3^2 + 3^2 + 1^2 = 1 + 9 + 9 + 1 = 20
\]
Theorem

\[
\left( \begin{array}{c} 2n \\ n \end{array} \right) = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right)^2
\]

Example \( n = 4 \)

\[
\left( \begin{array}{c} 8 \\ 4 \end{array} \right) = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = 70
\]

\[
\left( \begin{array}{c} 8 \\ 4 \end{array} \right) = \sum_{k=0}^{4} \left( \begin{array}{c} 4 \\ k \end{array} \right)^2
\]

\[
= \left( \begin{array}{c} 4 \\ 0 \end{array} \right)^2 + \left( \begin{array}{c} 4 \\ 1 \end{array} \right)^2 + \left( \begin{array}{c} 4 \\ 2 \end{array} \right)^2 + \left( \begin{array}{c} 4 \\ 3 \end{array} \right)^2 + \left( \begin{array}{c} 4 \\ 4 \end{array} \right)^2
\]

\[
= 1^2 + 4^2 + 6^2 + 4^2 + 1^2 = 1 + 16 + 36 + 16 + 1 = 70
\]
Sum of Squares of Binomial Coefficients

Combinatorial proof: \( \binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2 \)

- There are \( \binom{2n}{n} \) ways to select \( n \) numbers from the set \( S = \{1, 2, \ldots, 2n\} \)
- Partition the set \( S \) into two disjoint sets \( \mathcal{L} = \{1, 2, \ldots, n\} \) and \( \mathcal{R} = \{n+1, n+2, \ldots, 2n\} \).
- Every selection of \( n \) numbers from \( S \) is a selection of \( k \) numbers from \( \mathcal{L} \) and \( n - k \) numbers from \( \mathcal{R} \) for some \( 0 \leq k \leq n \).
- For a given \( k \), there are \( f(n, k) = \binom{n}{k} \binom{n}{n-k} \) such selections.
- By the symmetry of the binomial coefficient, \( f(n, k) = \binom{n}{k}^2 \).
- Sum \( f(n, k) \) for all \( 0 \leq k \leq n \) to get all the selections:
  \[
  \binom{2n}{n} = \sum_{k=0}^{n} f(n, k) = \sum_{k=0}^{n} \binom{n}{k}^2
  \]
**Example:** $2n = 6$: lexicographic order vs. proof order

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<tr>
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Sum of Products

\[
\binom{n}{k} = \sum_{j=0}^{k} \binom{m}{j} \binom{n-m}{k-j} \quad \text{for a given } 0 \leq m \leq n
\]

\[
\binom{2m}{m} = \sum_{j=0}^{m} \binom{m}{j}^2 \quad \text{for } n = 2m \text{ and } k = m
\]

\[
\binom{n+1}{k+1} = \sum_{m=0}^{n} \binom{m}{j} \binom{n-m}{k-j} \quad \text{for given } 0 \leq j \leq k \leq n
\]

\[
\binom{n+1}{k+1} = \sum_{m=k}^{n} \binom{m}{k} \quad \text{for } j = k
\]
"Weighted" Sums

\[
\sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1}
\]

\[
\sum_{k=0}^{n} k^2 \binom{n}{k} = (n + n^2)2^{n-2}
\]