Discrete Structures: Counting

Amotz Bar-Noy

Department of Computer and Information Science
Brooklyn College
How to count to 1000 on two hands

https://www.youtube.com/watch?v=1SMmc9gQmHQ

“Fascinating” properties of the numbers 1 to 9

https://www.youtube.com/watch?v=ByZLqOF-Jjk
Magic Squares

- **Solve The 3x3 Magic Square Completely - There Can Only Be One!**
  - [https://www.youtube.com/watch?v=zPnN046OM34](https://www.youtube.com/watch?v=zPnN046OM34)

- **The magic, myth and math of magic squares:**
  - [https://www.youtube.com/watch?v=-Tbd3dz1RnY](https://www.youtube.com/watch?v=-Tbd3dz1RnY)

- **Magic Square Party Trick:**
  - [https://www.youtube.com/watch?v=aQxCnmhqZko](https://www.youtube.com/watch?v=aQxCnmhqZko)
Euler’s Identity

\[ e^{\pi i} + 1 = 0 \quad \text{e}^{\pi i} = -1 \]

\[ e^{\pi i} \]

\[ \pi \text{ tells } i: \text{ get real!} \]
\[ i \text{ answers to } \pi: \text{ be rational!} \]
\[ e \text{ tells both of them: } \text{ join me and we will be } -1 \]

https://www.youtube.com/watch?v=IUTGFQpKaPU

Euler’s Identity

- Euler’s proof and some history:
  - https://www.youtube.com/watch?v=sKtloBAuP74&t=233s

- Euler’s proof a shorter version:
  - https://www.youtube.com/watch?v=NXrBoWOBvIY

- For dummies (by Mathologer):
  - https://www.youtube.com/watch?v=-dhHrg-KbJ0
Representing Numbers: Decimal, Binary, ...
Combinations and Permutations

Introduction with Cartoon Slides

http://tinytram.com/math/combinatorics/

Videos


https://www.youtube.com/watch?v=hVqq3nm0IHs

https://www.youtube.com/watch?v=LM5iOHKo_Fc&index=7&list=PLMyAzUai9V3ox_LDw154GRkNxovx6NqQX
Binomial Coefficients

Definition
There are \( \binom{n}{k} \) different subsets of \( k \) elements from a set of \( n \) elements.

Theorem:
\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]
Proof

There are \( n \) ways to select the first element, there are \( n - 1 \) ways to select the second element, and so on. In particular, there are \((n - k + 1)\) ways to select the \( k \)th element.

In total, there are

\[
g(n, k) = n(n - 1)(n - 2) \cdots (n - k + 1) = \frac{n!}{(n - k)!}
\]

ways to select an **ordered** list of \( k \) elements from a set of \( n \) elements.

Each set of \( k \) elements can be represented with \( k! \) different ordered lists. Therefore, there are

\[
\frac{g(n, k)}{k!} = \frac{n!}{k!(n - k)!}
\]

ways to select a subset of \( k \) elements from a set of \( n \) elements.
Binomial Coefficients Representations

\[ \binom{n}{k} = \frac{n^k}{k!} = \frac{n(n-1)(n-2) \cdots (n-k+1)}{k(k-1)(k-2) \cdots 2 \cdot 1} = \prod_{i=1}^{k} \frac{n+1-i}{i} = \frac{n!}{k!(n-k)!} \]
Example

Let \( S = \{ R, B, G, Y \} \) be a set of the 4 colors Red, Blue, Green, Yellow.

- There are \( \binom{4}{0} = 1 \) ways to choose zero colors from \( S \):
  - \( \{\emptyset\} \).

- There are \( \binom{4}{1} = 4 \) ways to choose one color from \( S \):
  - \( \{\{R\}, \{B\}, \{G\}, \{Y\}\} \).

- There are \( \binom{4}{2} = 6 \) ways to choose two colors from \( S \):
  - \( \{\{R, B\}, \{R, G\}, \{R, Y\}, \{B, G\}, \{B, Y\}, \{G, Y\}\} \).

- There are \( \binom{4}{3} = 4 \) ways to choose three colors from \( S \):
  - \( \{\{R, B, G\}, \{R, B, Y\}, \{R, G, Y\}, \{B, G, Y\}\} \).

- There are \( \binom{4}{4} = 1 \) ways to choose four colors from \( S \):
  - \( \{\{R, B, G, Y\}\} \).
Small $k$

\[
\binom{n}{0} = \frac{n!}{0!n!} = 1
\]
\[
\binom{n}{1} = \frac{n!}{1!(n-1)!} = n
\]
\[
\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}
\]
\[
\binom{n}{3} = \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{6}
\]
Symmetry

\[
\binom{n}{k} = \binom{n}{n-k}
\]

Combinatorial Proof I

- Selecting a subset of \( k \) elements from a set of \( n \) elements is equivalent to selecting a subset of \( n-k \) elements from a set of \( n \) elements.
- Therefore, the number of subsets of size \( k \) is equal to the number of subsets of size \( n-k \).

Algebraic Proof

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}.
\]
for all integers $n \geq 0$ \hspace{1cm} \binom{n}{0} = \binom{n}{n} = 1

for all integers $1 \leq k \leq n - 1$ \hspace{1cm} \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$
Consider a set with the \( n \) elements \( R = \{ x_1, x_2, \ldots, x_n \} \).

There are two options for selecting a subset \( S \) of \( R \) with \( k \) elements.

- \( x_n \in S \): There are \( \binom{n-1}{k-1} \) different ways to select additional \( k - 1 \) elements out of \( x_1, x_2, \ldots, x_{n-1} \).
- \( x_n \notin S \): There are \( \binom{n-1}{k} \) different ways to select \( k \) elements out of \( x_1, x_2, \ldots, x_{n-1} \).

In total the number of ways to select \( k \) elements from a set of \( n \) elements \( \binom{n}{k} \) is also

\[
\binom{n-1}{k-1} + \binom{n-1}{k}
\]
Recursive Formula: Algebraic Proof

\[
\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!}
\]

\[
= \frac{k(n-1)!}{k(k-1)!(n-k)!} + \frac{(n-k)(n-1)!}{(n-k)k!(n-k-1)!}
\]

\[
= \frac{k(n-1)!}{k!(n-k)!} + \frac{(n-k)(n-1)!}{k!(n-k)!}
\]

\[
= \frac{k(n-1)! + (n-k)(n-1)!}{k!(n-k)!}
\]

\[
= \frac{(k + (n-k))(n-1)!}{k!(n-k)!}
\]

\[
= \frac{n(n-1)!}{k!(n-k)!}
\]

\[
= \frac{n!}{k!(n-k)!}
\]

\[
= \binom{n}{k}
\]
Recursive Formula: Combinatorial Proof II

- \( \binom{n}{k} \) is the number of bit strings of length \( n \) containing \( k \) 1’s.

Some start with a 1 and the rest start with a 0.

Bit strings which start with a 1:
- After the 1, out of the remaining \( n - 1 \) bits, exactly \( k - 1 \) must be 1’s.
- There are exactly \( \binom{n-1}{k-1} \) such bit strings.

Bit strings which start with a 0.
- After the 0, out of the remaining \( n - 1 \) bits, exactly \( k \) must be 1’s.
- There are exactly \( \binom{n-1}{k} \) such bit strings.

Thus, the number of bit strings of length \( n \) containing \( k \) 1’s is also

\[ \binom{n-1}{k-1} + \binom{n-1}{k} . \]
One Recursive Step

\[
\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}
\]

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]

\[
= \frac{n(n-1)!}{k(k-1)!(n-k)!}
\]

\[
= \frac{n}{k} \cdot \frac{(n-1)!}{(k-1)!(n-k)!}
\]

\[
= \frac{n}{k} \binom{n-1}{k-1}
\]
Another One Recursive Step

\[
\binom{n}{k} = \frac{n + 1 - k}{k} \binom{n}{k - 1}
\]

\[
\binom{n}{k} = \frac{n!}{k!(n - k)!} = \frac{(n + 1 - k)n!}{k(k - 1)!(n + 1 - k)(n - k)!}
\]

\[
= \frac{n + 1 - k}{k} \cdot \frac{n!}{(k - 1)!(n + 1 - k)!}
\]

\[
= \frac{n + 1 - k}{k} \binom{n}{k - 1}
\]
Pascal’s Triangle

- Short visualized definition:
  - https://www.youtube.com/watch?v=Zo2JrPjijHc

- The mathematical secrets of Pascals triangle
  - https://www.youtube.com/watch?v=XMriWTvPXHI

- The Math of “The 12 Days Of Christmas”:
  - https://www.youtube.com/watch?v=fC8W4s6N9HQ

- What You Don’t Know About Pascal’s Triangle
  - https://www.youtube.com/watch?v=J0I1NuxUcpQ

- Pascal’s Triangle - Numberphile
  - https://www.youtube.com/watch?v=OiMtlus-afo

- Summary of facts:
  - https://www.mathsisfun.com/pascals-triangle.html
Theorem

\[(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k\]

\[= x^n + nx^{n-1}y + \binom{n}{2} x^{n-2}y^2 + \cdots + \binom{n}{2} x^2y^{n-2} + nxy^{n-1} + y^n\]
Proof

By definition,

\[(x + y)^n = (x + y) \cdot (x + y) \cdot (x + y) \cdots (x + y) .\]

Using the distributive property to get a product \(x^k y^{n-k}\):
- select \(k\) of the \(n\) terms to contribute an \(x\) to the product,
- select the other \(n - k\) terms to contribute a \(y\) to the product.

The coefficient of \(x^k y^{n-k}\) is therefore \(\binom{n}{k}\):
- the number of ways to select \(k\) elements from a set of size \(n\).

Summing over all possible values of \(k\) from 0 to \(n\) implies that

\[(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} .\]
Examples

\[(x + y)^1 = \binom{1}{0}x^1y^0 + \binom{1}{1}x^0y^1 = x + y\]

\[(x + y)^2 = \binom{2}{0}x^2y^0 + \binom{2}{1}x^1y^1 + \binom{2}{2}x^0y^2 = x^2 + 2xy + y^2\]

\[(x + y)^3 = \binom{3}{0}x^3y^0 + \binom{3}{1}x^2y^1 + \binom{3}{2}x^1y^2 + \binom{3}{3}x^0y^3 = x^3 + 3x^2y + 3xy^2 + y^3\]

\[(x + y)^4 = \binom{4}{0}x^4y^0 + \binom{4}{1}x^3y^1 + \binom{4}{2}x^2y^2 + \binom{4}{3}x^1y^3 + \binom{4}{4}x^0y^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\]
Corollary

\[(1 + x)^n = \binom{n}{0}x^0 + \binom{n}{1}x^1 + \binom{n}{2}x^2 + \cdots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n\]

\[= \sum_{k=0}^{n} \binom{n}{k}x^k\]

Example

\[(1 + x)^4 = \binom{4}{0}x^0 + \binom{4}{1}x^1 + \binom{4}{2}x^2 + \binom{4}{3}x^3 + \binom{4}{4}x^4\]

\[= 1 + 4x + 6x^2 + 4x^3 + x^4\]
Sum of All Binomial Coefficients for a Given $n$

\[
2^n = (1 + 1)^n \\
= \binom{n}{0}1^n1^0 + \binom{n}{1}1^{n-1}1^1 + \binom{n}{2}1^{n-2}1^2 + \cdots + \binom{n}{n}1^01^n \\
= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} \\
= \sum_{k=0}^{n} \binom{n}{k}
\]
Another InterestingSum

\[
3^n = (1 + 2)^n
= \binom{n}{0}1^n2^0 + \binom{n}{1}1^{n-1}2^1 + \binom{n}{2}1^{n-2}2^2 + \cdots + \binom{n}{n}1^02^n
= \binom{n}{0}2^0 + \binom{n}{1}2^1 + \binom{n}{2}2^2 + \cdots + \binom{n}{n}2^n
= \sum_{k=0}^{n} \binom{n}{k}2^k
\]
Counting Triplets

\[
\binom{n+2}{3} = \sum_{j=1}^{n} j(n+1-j)
\]

- There are \(\binom{n+2}{3}\) ordered triplets \((i < j < k)\) in the set \(\{1, \ldots, n+2\}\).
- Fix \(j' \in \{2, \ldots, n-1\}\). Observe that \(j'\) cannot be either 1 or \(n+2\).
- There are \(j' - 1\) ways to select \(i \in \{1, 2, \ldots, j' - 1\}\).
- There are \(n+2 - j'\) ways to select \(k \in \{j' + 1, j' + 2, \ldots, n+2\}\).
- Therefore, the number of triplets \((i, j, k)\) with \(j = j'\) is \((j' - 1)(n+2 - j')\).
- Consequently, the total number of triplets is \(\sum_{j'=2}^{n+1} (j' - 1)(n+2 - j')\).
- Replacing \(j'\) with \(j + 1\) implies that this number is \(\sum_{j=1}^{n+1} j(n+1-j)\).
Sum of Squares

\[
\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2
\]

There are \(\binom{2n}{n}\) ways to select \(n\) numbers from the set \(S = \{1, 2, \ldots, 2n\}\).

Partition the set \(S\) into two disjoint sets \(X = \{1, 2, \ldots, n\}\) and \(Y = \{n+1, n+2, \ldots, 2n\}\).

Every selection of \(n\) numbers from \(S\) is a selection of \(k\) numbers from \(X\) and \(n-k\) numbers from \(Y\) for some \(0 \leq k \leq n\).

For a given \(k\), there are \(f(n, k) = \binom{n}{k} \binom{n}{n-k}\) such selections.

By the symmetry of the binomial coefficient, \(f(n, k) = \binom{n}{k}^2\).

Sum \(f(n, k)\) for all \(0 \leq k \leq n\) to get all the selections.

\[
\binom{2n}{n} = \sum_{k=0}^{n} f(n, k) = \sum_{k=0}^{n} \binom{n}{k}^2.
\]
Sum of Products

\[
\binom{n}{k} = \sum_{j=0}^{k} \binom{m}{j} \binom{n-m}{k-j} \quad \text{for a given } 0 \leq m \leq n
\]

\[
\binom{2m}{m} = \sum_{j=0}^{m} \binom{m}{j}^2 \quad \text{for } n = 2m \text{ and } k = m
\]

\[
\binom{n+1}{k+1} = \sum_{m=0}^{n} \binom{m}{j} \binom{n-m}{k-j} \quad \text{for a given } 0 \leq j \leq k \leq n
\]

\[
\binom{n+1}{k+1} = \sum_{m=k}^{n} \binom{m}{k} \quad \text{for } j = k
\]
“Weighted” Sums

\[\sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1}\]

\[\sum_{k=0}^{n} k^2 \binom{n}{k} = (n + n^2)2^{n-2}\]