Discrete Structures: Counting

Amotz Bar-Noy
Department of Computer and Information Science
Brooklyn College
Numbers

How to count to 1000 on two hands

https://www.youtube.com/watch?v=1SMmc9gQmHQ

“Fascinating” properties of the numbers 1 to 9

https://www.youtube.com/watch?v=ByZLqOF-Jjk
Magic Squares

- Solve The 3x3 Magic Square Completely - There Can Only Be One!
  - https://www.youtube.com/watch?v=zPnN046OM34

- The magic, myth and math of magic squares:
  - https://www.youtube.com/watch?v=-Tbd3dzlRnY

- Magic Square Party Trick:
  - https://www.youtube.com/watch?v=aQxCnmhqZko
Euler’s Identity

\[ e^{\pi i} + 1 = 0 \quad e^{\pi i} = -1 \]

\[ \pi \text{ tells } i: \text{ get real!} \]
\[ i \text{ answers to } \pi: \text{ be rational!} \]
\[ e \text{ tells both of them: } join \text{ me and we will be } -1 \]


https://www.youtube.com/watch?v=IUTGFQpKaPU
Euler’s Identity

- Euler’s proof and some history:
  - https://www.youtube.com/watch?v=sKtloBAuP74&t=233s

- Euler’s proof a shorter version:
  - https://www.youtube.com/watch?v=NXrBoWOBvIY

- For dummies (by Mathologer):
  - https://www.youtube.com/watch?v=-dhHrg-KbJ0
Representing Numbers: Decimal, Binary, …

Different Bases For Numbers
https://www.patreon.com/posts/different-bases-15635190

The Josephus Problem
https://www.youtube.com/watch?v=uCsD3ZGzMgE
Combinations and Permutations

Introduction with Cartoon Slides

http://tinytram.com/math/combinatorics/

Videos


https://www.youtube.com/watch?v=hVqq3nm0IHs

https://www.youtube.com/watch?v=LM5iOHKo_Fc&index=7&list=PLMyAzUai9V3ox_LDw154GRkNxovx6NqQX
Binomial Coefficients

Definition
For \( n \geq 1 \) and \( 0 \leq k \leq n \), there are \( \binom{n}{k} \) different subsets of \( k \) objects out of a set of \( n \) objects.

Remarks
- For \( k = 0 \), the empty set is the only subset with 0 objects. Therefore, \( \binom{n}{0} = 0 \)
- For \( k = n \), the entire set is the only subset with \( n \) objects. Therefore, \( \binom{n}{n} = 0 \)

Theorem:
\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]
Let $S = \{R, B, G, Y\}$ be a set of the 4 colors Red, Blue, Green, Yellow.

- There are $\binom{4}{0} = 1$ ways to choose zero colors from $S$:
  - $\{\emptyset\}$.

- There are $\binom{4}{1} = 4$ ways to choose one color from $S$:
  - $\{\{R\}, \{B\}, \{G\}, \{Y\}\}$.

- There are $\binom{4}{2} = 6$ ways to choose two colors from $S$:
  - $\{\{R, B\}, \{R, G\}, \{R, Y\}, \{B, G\}, \{B, Y\}, \{G, Y\}\}$.

- There are $\binom{4}{3} = 4$ ways to choose three colors from $S$:

- There are $\binom{4}{4} = 1$ ways to choose four colors from $S$:
  - $\{\{R, B, G, Y\}\}$. 
Small $k$

\[
\begin{align*}
\binom{n}{0} &= \frac{n!}{0!n!} = 1 \\
\binom{n}{1} &= \frac{n!}{1!(n-1)!} = n \\
\binom{n}{2} &= \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2} \\
\binom{n}{3} &= \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{6}
\end{align*}
\]
Proof of Theorem

There are $n$ ways to select the first object, there are $n - 1$ ways to select the second object, and so on. In particular, there are $(n - k + 1)$ ways to select the $k$th object.

In total, there are

$$g(n, k) = n(n - 1)(n - 2) \cdots (n - k + 1) = \frac{n!}{(n - k)!}$$

ways to select an ordered list of $k$ objects from a set of $n$ objects.

Each set of $k$ objects can be represented with $k!$ different ordered lists. Therefore, there are

$$\frac{g(n, k)}{k!} = \frac{n!}{k!(n - k)!}$$

ways to select a subset of $k$ objects from a set of $n$ objects.
Binomial Coefficients Representations

\[
\begin{align*}
\binom{n}{k} &= \frac{n^k}{k!} \\
&= \frac{n(n-1)(n-2) \cdots (n-k+1)}{k(k-1)(k-2) \cdots 2 \cdot 1} \\
&= \prod_{i=1}^{k} \frac{n+1-i}{i} \\
&= \frac{n!}{k!(n-k)!}
\end{align*}
\]
Symmetry

\[
\binom{n}{k} = \binom{n}{n-k}
\]

Combinatorial Proof

- Selecting a subset of \( k \) objects from a set of \( n \) objects is equivalent to selecting a subset of \( n - k \) objects from a set of \( n \) objects.
- Therefore, the number of subsets of size \( k \) is equal to the number of subsets of size \( n - k \).

Algebraic Proof

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}
\]
Recursive Formula

for all integers $n \geq 0$ \hspace{1cm} (n_0) = (n_n) = 1

for all integers $1 \leq k \leq n - 1$ \hspace{1cm} (n_k) = (n_{k-1}) + (n_{k-1})
Consider a set with the $n$ objects $R = \{x_1, x_2, \ldots, x_n\}$.

There are two options for selecting a subset $S$ of $R$ with $k$ objects.
- $x_n \in S$: There are $\binom{n-1}{k-1}$ different ways to select additional $k - 1$ objects out of $x_1, x_2, \ldots, x_{n-1}$.
- $x_n \notin S$: There are $\binom{n-1}{k}$ different ways to select $k$ objects out of $x_1, x_2, \ldots, x_{n-1}$.

In total the number of ways to select $k$ objects from a set of $n$ objects $\binom{n}{k}$ is also

$$\binom{n-1}{k-1} + \binom{n-1}{k}$$
Recursive Formula: Algebraic Proof

\[
\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!}
\]

\[
= \frac{k(n-1)!}{k(k-1)!(n-k)!} + \frac{(n-k)(n-1)!}{(n-k)k!(n-k-1)!}
\]

\[
= \frac{k(n-1)!}{k!(n-k)!} + \frac{(n-k)(n-1)!}{k!(n-k)!}
\]

\[
= \frac{k(n-1)! + (n-k)(n-1)!}{k!(n-k)!}
\]

\[
= \frac{(k + (n-k))(n-1)!}{k!(n-k)!}
\]

\[
= \frac{n(n-1)!}{k!(n-k)!}
\]

\[
= \frac{n!}{k!(n-k)!}
\]

\[
= \binom{n}{k}
\]
Recursive Formula: Combinatorial Proof II

- \( \binom{n}{k} \) is the number of bit strings of length \( n \) containing \( k \) 1’s.

- Some start with a 1 and the rest start with a 0.

- Bit strings which start with a 1:
  * After the 1, out of the remaining \( n - 1 \) bits, exactly \( k - 1 \) must be 1’s.
  * There are exactly \( \binom{n-1}{k-1} \) such bit strings.

- Bit strings which start with a 0.
  * After the 0, out of the remaining \( n - 1 \) bits, exactly \( k \) must be 1’s.
  * There are exactly \( \binom{n-1}{k} \) such bit strings.

Thus, the number of bit strings of length \( n \) containing \( k \) 1’s is also

\[
\binom{n-1}{k-1} + \binom{n-1}{k}.
\]
One Recursive Step

\[
\binom{n}{k} = \frac{k}{n-k} \binom{n-1}{k-1}
\]

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)!}{k(k-1)!(n-k)!} = \frac{n}{k} \cdot \frac{(n-1)!}{(k-1)!(n-k)!} = \frac{n}{k} \binom{n-1}{k-1}
\]
Another One Recursive Step

\[
\binom{n}{k} = \frac{n+1-k}{k} \binom{n}{k-1}
\]

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{(n+1-k)n!}{k(k-1)!(n+1-k)(n-k)!} = \frac{n+1-k}{k} \cdot \frac{n!}{(k-1)!(n+1-k)!}
\]

\[
= \frac{n+1-k}{k} \binom{n}{k-1}
\]
Pascal’s Triangle

Short visualized definition:
https://www.youtube.com/watch?v=Zo2JrPjiJHc

The mathematical secrets of Pascals triangle
https://www.youtube.com/watch?v=XMriWTvPXHI

The Math of “The 12 Days Of Christmas”:
https://www.youtube.com/watch?v=fC8W4s6N9HQ

What You Don’t Know About Pascal’s Triangle
https://www.youtube.com/watch?v=J0I1NuxUcpQ

Pascal’s Triangle - Numberphile
https://www.youtube.com/watch?v=0iMtlus-afo

Summary of facts:
https://www.mathsisfun.com/pascals-triangle.html
\[(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = x^n + nx^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{2} x^2 y^{n-2} + nxy^{n-1} + y^n\]
Proof

By definition,

\[(x + y)^n = (x + y) \cdot (x + y) \cdot (x + y) \cdots (x + y) .\]

Using the distributive property to get a product \(x^k y^{n-k}\):
- select \(k\) of the \(n\) terms to contribute an \(x\) to the product,
- select the other \(n - k\) terms to contribute a \(y\) to the product.

The coefficient of \(x^k y^{n-k}\) is therefore \(\binom{n}{k}\):
- the number of ways to select \(k\) objects from a set of size \(n\).

Summing over all possible values of \(k\) from 0 to \(n\) implies that

\[(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} .\]
Examples

\[(x + y)^1 = \binom{1}{0}x^1y^0 + \binom{1}{1}x^0y^1 = x + y\]

\[(x + y)^2 = \binom{2}{0}x^2y^0 + \binom{2}{1}x^1y^1 + \binom{2}{2}x^0y^2 = x^2 + 2xy + y^2\]

\[(x + y)^3 = \binom{3}{0}x^3y^0 + \binom{3}{1}x^2y^1 + \binom{3}{2}x^1y^2 + \binom{3}{3}x^0y^3 = x^3 + 3x^2y + 3xy^2 + y^3\]

\[(x + y)^4 = \binom{4}{0}x^4y^0 + \binom{4}{1}x^3y^1 + \binom{4}{2}x^2y^2 + \binom{4}{3}x^1y^3 + \binom{4}{4}x^0y^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\]
Corollary

\[(1 + x)^n = \binom{n}{0}x^0 + \binom{n}{1}x^1 + \binom{n}{2}x^2 + \cdots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n\]

\[= \sum_{k=0}^{n} \binom{n}{k}x^k\]

Example

\[(1 + x)^4 = \binom{4}{0}x^0 + \binom{4}{1}x^1 + \binom{4}{2}x^2 + \binom{4}{3}x^3 + \binom{4}{4}x^4\]

\[= 1 + 4x + 6x^2 + 4x^3 + x^4\]
Sum of All Binomial Coefficients for a Given $n$

**Examples**

$$2^0 = \binom{0}{0} = 1$$

$$2^1 = \binom{1}{0} + \binom{1}{1} = 1 + 1 = 2$$

$$2^2 = \binom{2}{0} + \binom{2}{1} + \binom{2}{2} = 1 + 2 + 1 = 4$$

$$2^3 = \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 1 + 3 + 3 + 1 = 8$$

$$2^4 = \binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 1 + 4 + 6 + 4 + 1 = 16$$
Sum of All Binomial Coefficients for a Given \( n \)

**General Case**

\[
2^n = (1 + 1)^n = \binom{n}{0}1^n0 + \binom{n}{1}1^{n-1}1^1 + \binom{n}{2}1^{n-2}1^2 + \cdots + \binom{n}{n}1^01^n
\[
= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}
\[
= \sum_{k=0}^{n} \binom{n}{k}
\]
Computing $3^n$ with Powers of 2

$$3^n = (1 + 2)^n$$

$$= \binom{n}{0}1^n2^0 + \binom{n}{1}1^{n-1}2^1 + \binom{n}{2}1^{n-2}2^2 + \cdots + \binom{n}{n}1^02^n$$

$$= \binom{n}{0}2^0 + \binom{n}{1}2^1 + \binom{n}{2}2^2 + \cdots + \binom{n}{n}2^n$$

$$= \sum_{k=0}^{n} \binom{n}{k}2^k$$
### Computing $3^n$ with Powers of 2

#### $n = 3$

\[
27 = 3^3 = \binom{3}{0} 2^0 + \binom{3}{1} 2^1 + \binom{3}{2} 2^2 + \binom{3}{3} 2^3 \\
= 1 \cdot 1 + 3 \cdot 2 + 3 \cdot 4 + 1 \cdot 8 \\
= 1 + 6 + 12 + 8
\]

#### $n = 4$

\[
81 = 3^4 = \binom{4}{0} 2^0 + \binom{4}{1} 2^1 + \binom{4}{2} 2^2 + \binom{4}{3} 2^3 + \binom{4}{4} 2^4 \\
= 1 \cdot 1 + 4 \cdot 2 + 6 \cdot 4 + 4 \cdot 8 + 1 \cdot 16 \\
= 1 + 8 + 24 + 32 + 16
\]
Counting Triplets

Theorem

\[ \binom{n+2}{3} = \sum_{h=1}^{n} h(n+1-h) \]

Example

\[ \binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2} = 35 \]

\[ \binom{7}{3} = \sum_{h=1}^{5} h(n+1-h) \]

\[ = 1 \cdot 5 + 2 \cdot 4 + 3 \cdot 3 + 4 \cdot 2 + 5 \cdot 1 \]

\[ = 5 + 8 + 9 + 8 + 5 = 35 \]
Counting Triplets

Proof

- There are \( \binom{n+2}{3} \) ordered triplets \((i < j < k)\) in the set \(\{1, \ldots, n+2\}\).
- Fix the middle index \(j\).
- \(j\) cannot be either 1 or \(n+2\) and therefore \(2 \leq j \leq n+1\).
- There are \(j - 1\) ways to select \(i \in \{1, 2, \ldots, j - 1\}\) and there are \(n+2 - j\) ways to select \(k \in \{j+1, j+2, \ldots, n+2\}\).
- The number of triplets \((i, j, k)\) with \(j\) as the middle index is \((j - 1)(n+2 - j)\).
- The total number of triplets is \(\sum_{j=2}^{n+1} (j - 1)(n+2 - j)\).
- Replacing \(j\) with \(h+1\) implies that this number is \(\sum_{h=1}^{n} h(n+1 - h)\).
Sum of Squares

**Theorem**

\[
\binom{2n}{n} = \sum_{k=0}^{n} \left( \binom{n}{k} \right)^2
\]

**Example**

\[
\binom{8}{4} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2} = 70
\]

\[
\binom{8}{4} = \sum_{k=0}^{4} \left( \binom{4}{k} \right)^2
\]

\[
= \left( \binom{4}{0} \right)^2 + \left( \binom{4}{1} \right)^2 + \left( \binom{4}{2} \right)^2 + \left( \binom{4}{3} \right)^2 + \left( \binom{4}{4} \right)^2
\]

\[
= 1^2 + 4^2 + 6^2 + 4^2 + 1^2 = 1 + 16 + 36 + 16 + 1 = 70
\]
### Proof

- There are \(^{2n}_{n}\) ways to select \(n\) numbers from the set \(S = \{1, 2, \ldots, 2n\}\).
- Partition the set \(S\) into two disjoint sets \(X = \{1, 2, \ldots, n\}\) and \(Y = \{n+1, n+2, \ldots, 2n\}\).
- Every selection of \(n\) numbers from \(S\) is a selection of \(k\) numbers from \(X\) and \(n-k\) numbers from \(Y\) for some \(0 \leq k \leq n\).
- For a given \(k\), there are \(f(n, k) = \binom{n}{k} \binom{n}{n-k}\) such selections.
- By the symmetry of the binomial coefficient, \(f(n, k) = \binom{n}{k}^2\).
- Sum \(f(n, k)\) for all \(0 \leq k \leq n\) to get all the selections.

\[
\binom{2n}{n} = \sum_{k=0}^{n} f(n, k) = \sum_{k=0}^{n} \binom{n}{k}^2.
\]
Sum of Products

\[
\binom{n}{k} = \sum_{j=0}^{k} \binom{m}{j} \binom{n-m}{k-j} \quad \text{for a given } 0 \leq m \leq n
\]

\[
\binom{2m}{m} = \sum_{j=0}^{m} \left( \binom{m}{j} \right)^2 \quad \text{for } n = 2m \text{ and } k = m
\]

\[
\binom{n+1}{k+1} = \sum_{m=0}^{n} \binom{m}{j} \binom{n-m}{k-j} \quad \text{for a given } 0 \leq j \leq k \leq n
\]

\[
\binom{n+1}{k+1} = \sum_{m=k}^{n} \binom{m}{k} \quad \text{for } j = k
\]
“Weighted” Sums

\[
\sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1}
\]

\[
\sum_{k=0}^{n} k^2 \binom{n}{k} = (n + n^2)2^{n-2}
\]