Discrete Structures: Counting and Combinatorics

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Numbers

How to count to 1000 on two hands

https://www.youtube.com/watch?v=1SMmc9gQmHQ

“Fascinating” properties of the numbers 1 to 9

https://www.youtube.com/watch?v=ByZLqOF-Jjk

There are 365 days and 12 months in a year

365 = 10^2 + 11^2 + 12^2 = 13^2 + 14^2

It is the only integer for which there exists an integer n such that

\[(n - 2)^2 + (n - 1)^2 + n^2 = (n + 1)^2 + (n + 2)^2\]
Can Numbers be Interesting?

The taxi-cub number

- 1729 is the smallest number that is the sum of two cubes in two different ways:

\[ 1729 = 12^3 + 1^3 = 10^3 + 9^3 \]

Story

- This number is called a taxicab number, because in a discussion between the mathematicians G. H. Hardy and Srinivasa Ramanujan about interesting and dull numbers, the former remarked that the number 1729 of the taxicab he had ridden seemed uninteresting, and the latter immediately answered that it is interesting, being the smallest number that is the sum of two cubes in two different ways.
All the Positive Integers Are Interesting!

Proof

- The "proof" is by contradiction.
- If there exists a non-empty set of uninteresting natural numbers, then there must be a smallest uninteresting number.
- But the smallest uninteresting number is itself interesting because it is the smallest uninteresting number: a contradiction.

Online resources

- https://www.youtube.com/watch?v=Ysd1XhqMbe8
- https://en.m.wikipedia.org/wiki/Interesting_number_paradox
Euler’s Identity

The identity

\[ e^{\pi i} + 1 = 0 \quad e^{\pi i} = -1 \]

A joke???

\( \pi \) tells \( i \): get real!
\( i \) answers to \( \pi \): be rational!
\( e \) tells both of them: join me and we will be - one!

https://www.youtube.com/watch?v=IUTGFQpKaPU

Online resources

- https://www.youtube.com/watch?v=sKtloBAuP74&t=233s
- https://www.youtube.com/watch?v=NXrBoWOBvIY
- https://www.youtube.com/watch?v=-dhHrg-KbJ0

Which is larger \( e^\pi \) or \( \pi^e \)?

- https://www.youtube.com/watch?v=JE_YeVlSqLY
- https://www.youtube.com/watch?v=I7wiS9rH2h0
Representing Numbers: Decimal, Binary, ... 

Different Bases For Numbers

- Online videos:
  - https://www.youtube.com/watch?v=1srwWeMe3BE
  - https://www.youtube.com/watch?v=aW3qCcH6Dao
  - https://www.youtube.com/watch?v=Fpm-E5v6ddc

- A short tutorial:
  - https://www.tutorialspoint.com/computer_logical_organization/number_system_conversion.htm

The Josephus Problem

- https://www.youtube.com/watch?v=uCsD3ZGzMgE
Combinations and Permutations

Introduction with Cartoon Slides

- http://tinytram.com/math/combinatorics/

Online resources

- https://www.youtube.com/watch?v=uNSlQvDzCVw&feature=youtu.be
- https://www.youtube.com/watch?v=hVqq3nm0IHs
- https://youtu.be/LM5iOHKo_Fc?list=PLMyAzUai9V3ox_LDw154GRkNxovx6NgQX
Four Counting Types

Setting

- **Input:** A universal set $S = \{s_1, s_2, \ldots, s_n\}$ with $n$ distinct objects.
- **Goal:** Count the number of some structures with some parameters containing objects from $S$.

Structures

- **Permutations:** In how many ways can all the objects of $S$ be ordered?
- **Lists with repetitions:** How many lists with $1 \leq k$ objects from $S$ exist where repetitions are allowed?
- **Lists without repetitions:** How many lists with $1 \leq k \leq n$ objects from $S$ exist where repetitions are not allowed?
- **Subsets:** How many subsets of $S$ exist?
Example: $S = \{ R, B, G, M \}$

There are $3! = 6$ permutations of $\{ R, B, G \}$

$RGB, RGB, BRG, BGR, GRB, GBR$

There are $4! = 24$ permutations of $S = \{ R, B, G, M \}$

$RBGM, RBMG, RGBM, RGMB, RMBG, RMGB, BRGM, BRMG, BGRM, BGMR, BMRG, BMGR, GRBM, GRMB, GBRM, GBMR, GMRB, GMBR, MRBG, MRGB, MBRG, MBGR, MGRB, MGBR$
Example: \( S = \{ R, B, G, M \} \)

There are \( 4^2 = 16 \) lists with repetitions of length 2

\[
RR \quad RB \quad RG \quad RM \\
BR \quad BB \quad BG \quad BM \\
GR \quad GB \quad GG \quad GM \\
MR \quad MB \quad MG \quad MM
\]

There are \( 4^4 = 256 \) lists with repetitions of length 4

\[
RRRR \quad RRRB \quad RRRG \quad RRRM \quad \ldots \\
\ldots \quad BBBR \quad BBBB \quad BBBG \quad BBBM \quad \ldots \\
\ldots \quad GGGR \quad GGBB \quad GGGG \quad GGM \quad \ldots \\
\ldots \quad MMMR \quad MMMB \quad MMMG \quad MMMM
\]
Example: $S = \{R, B, G, M\}$

There are $4 \cdot 3 = 12$ lists without repetitions of length 2

- $RB$, $RG$, $RM$
- $BR$, $BG$, $BM$
- $GR$, $GB$, $GM$
- $MR$, $MB$, $MG$

There are $4 \cdot 3 \cdot 2 = 24$ lists without repetitions of length 3

- $RBG$, $RBM$, $RGB$, $RGM$, $RMB$, $RMG$
- $BRG$, $BRM$, $BGR$, $BGM$, $BMR$, $BMG$
- $GRB$, $GRM$, $GBR$, $GBM$, $GMR$, $GMB$
- $MRB$, $MRG$, $MBR$, $MBG$, $MGR$, $MGB$
Example: $S = \{ R, B, G, M \}$

$S$ has 4 subsets of size 1

\{ R \} \quad \{ B \} \quad \{ G \} \quad \{ M \}

$S$ has 6 subsets of size 2

\{ RB \} \quad \{ RG \} \quad \{ RM \}
\{ BG \} \quad \{ BM \} \quad \{ GM \}

$S$ has 4 subsets of size 3

\{ RBG \} \quad \{ RBM \} \quad \{ RGM \} \quad \{ BGM \}
Permutations

Definition
- An $n$-permutation $\pi$ is a 1-1 function from the set of numbers \{1, 2, \ldots n\} to itself
- $\pi(i) \neq \pi(j)$ for a permutation $\pi = (\pi(1), \pi(2), \ldots, \pi(n))$ for all $1 \leq i \neq j \leq n$

Counting permutations
- There are $n! = n(n−1)(n−2) \cdots 2 \cdot 1$ permutations for $n \geq 1$

Proof
- There are $n$ options for $\pi(1)$
- There are $n−1$ options for $\pi(2)$
- $\vdots$
- There are 2 options for $\pi(n−1)$
- There is only 1 option for $\pi(n)$
Permutations: Examples

Small values

- $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, $5! = 120$, $6! = 720$, ...

The 10 digits

- There are $10! = 3628800$ different ways to arrange the 10 digits 0, 1, ..., 9
- If numbers with less than 10 digits have leading zeros, then there are 10 billions ($10000000000 = 10^{10}$) numbers with 10 digits
- Only 0.036288% of these numbers contain all the 10 digits

The 26 letters

- There are $26! = 403291461126605635584000000$ arrangements of the 26 letters of the English alphabet
- $26! \approx 4 \times 10^{27}$ which is about 400 millions billions of billions
There Are Exactly $10!$ Seconds in 6 Weeks

### Preprocessing

- There are $60 = 3 \times 4 \times 5$ seconds in one minute
- There are $60 = 2 \times \sqrt{9} \times 10$ minutes in one hour
- There are $24 = \sqrt{9} \times 8$ hours in a day
- There are 7 days in a week
- There are 6 weeks

### $10!$ seconds

The number $S$ of seconds in 6 weeks is therefore

$$S = (3 \times 4 \times 5) \times (2 \times \sqrt{9} \times 10) \times (\sqrt{9} \times 8) \times 7 \times 6$$

$$= 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10$$

$$= 10!$$
Derangements

Definition

- A derangement is a permutation $\pi$ on the numbers 1, 2, $\ldots$ $n$ such that $\pi(i) \neq i$ for all $1 \leq i \leq n$
- A derangement is a permutation $\pi$ on the numbers 1, 2, $\ldots$ $n$ without fixed points ($\neg \exists_{1 \leq i \leq n}(\pi(i) = i)$)

Examples

- (21) is the only derangement for $n = 2$
- (231) and (312) are the only two derangements for $n = 3$ while the other four permutations (123), (132), (213), and (321) are not derangements because each contains at least one fixed point

The 9 derangements for $n = 4$ out of the 24 permutations:

- (1234) (1243) (1324) (1342) (1423) (1432)
- (2134) (2143) (2314) (2341) (2413) (2431)
- (3124) (3142) (3214) (3241) (3412) (3421)
- (4123) (4132) (4213) (4231) (4312) (4321)
Counting Derangements

**Notation**
- The number of derangements of $n$ is $!n$ (the subfactorial of $n$)

**Small $n$**
- $!1 = 0$, $!2 = 1$, $!3 = 2$, $!4 = 9$, $!5 = 44$, $!6 = 265$ ...

**Recursive formula**
- $!1 = 0$, $!2 = 1$, and for $n \geq 3$:
  \[
  !n = (n - 1)(!(n - 1) + !(n - 2))
  \]
  - $!3 = 2(1 + 0) = 2$
  - $!4 = 3(2 + 1) = 9$
  - $!5 = 4(9 + 2) = 44$
  - $!6 = 5(44 + 9) = 265$
Derangements

Non-recursive formulas

- \( n! = \left\lfloor \frac{n!}{e} \right\rfloor \approx \frac{n!}{2.718} \) where \([x]\) is the nearest integer to \(x\)
- \( n! = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} = n!(1/2 - 1/6 + 1/24 - 1/120 + \cdots) \)

Corollary

- About \(1/e \approx 0.367879\) of the permutations are derangements
- About 63% of the permutations have at least one fixed point

An online resource

- https://www.youtube.com/watch?v=pbXg5EI5t4c
Lists With Repetitions

**Definition**
- \( \mathcal{L} = (\ell_1, \ell_2, \ldots, \ell_k) \) is an ordered list of \( k \) objects from the set \( S = \{s_1, s_2, \ldots, s_n\} \) if \( \ell_i \in S \) for all \( 1 \leq i \leq k \)
- \( \ell_i \) could be equal to \( \ell_j \) for \( 1 \leq i < j \leq k \)

**Counting the number of lists with repetitions**
- There are \( n^k \) lists of length \( k \) from a set of size \( n \)

**Proof**
- For each \( 1 \leq i \leq k \), there are \( n \) options for \( \ell_i \)
- For the \( k \) possible indices \( 1 \leq i \leq k \), there are \( n^k \) options for \( (\ell_1, \ell_2, \ldots, \ell_k) \)
Lists With Repetitions: Examples

Numbers
- Assuming numbers have leading zeros, then there are 10 billions \(10000000000 = 10^{10}\) numbers \((\text{lists})\) with 10 digits

Letters
- There are \(26^3 = 17576\) possible three-letter words in English and \(26^4 = 456976\) possible four-letter words in English
- There are less than 200000 words in the Oxford English Dictionary!

Codes
- There are \(10^4 = 10000\) possible codes for a 4-digit lock
- The codes: 0000, 0001 \(\ldots\) 4567 \(\ldots\) 7766 \(\ldots\) 9998, 9999
Nesting Loops

Pseudocode

function $f(n)$ (* integer $n \geq 1$ *)

$c = 0$

for $i = 1$ to $n$
do

for $j = 1$ to $n$
do

for $k = 1$ to $n$
do

print $(i, j, k)$

$c := c + 1$

Observations

- The function $f(n)$ prints all possible lists with repetitions $(i, j, k)$ for which $i, j, k \in \{1, 2, \ldots, n\}$
- The value of $c$ at the end is $n^3$
Strings

**Definition**
- An $n$-ary string of length $k$ is an ordered list $D = (d_1, d_2, \ldots, d_k)$ such that $d_i \in \{0, 1, \ldots, n - 1\}$ for all $1 \leq i \leq k$
- In a binary string $d_i = 0$ or $d_i = 1$ for all $1 \leq i \leq k$

**Counting strings**
- There are $n^k$ strings of length $k$
- There are $2^k$ binary strings of length $k$

**Example: the 16 binary strings of length 4**

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**Example: the 27 ternary strings of length 3**

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Non Homogeneous Lists

Definition

- In a non-homogeneous list, for $1 \leq i \leq k$, the entry $\ell_i$ gets its value from a different domain of objects denoted by $S_i$
- $\mathcal{L} = (\ell_1, \ell_2, \ldots, \ell_k)$ is an ordered non-homogeneous list of $k$ objects if $\ell_i \in S_i$ for all $1 \leq i \leq k$

Counting the number of non-homogeneous lists

- Assume $n_i$ is the size $S_i$
- Then there are $n_1 n_2 \cdots n_k$ non-homogeneous lists of length $k$

Proof

- For each $1 \leq i \leq k$, there are $n_i$ options for $\ell_i$
- For the $k$ possible $1 \leq i \leq k$, there are $n_1 n_2 \cdots n_k$ options for $(\ell_1, \ell_2, \ldots, \ell_k)$
Non Homogeneous Lists: Examples

Passwords

- There are $26^2 \cdot 10^4 = 6760000$ possible passwords of length 6 that must start with 2 letters and end with 4 digits
- $S_1 = S_2 = \{A, B, \ldots, Z\}$ and $S_3 = S_4 = S_5 = S_6 = \{0, 1, \ldots, 9\}$
- AA0000, AA0001 $\ldots$ CZ9999, DA0000 $\ldots$ ZZ9998, ZZ9999

Taxi licenses

- There were only $10 \cdot 26 \cdot 10^2 = 26000$ possible taxi license numbers in New York city that must start with a digit followed by a letter and end with two digits
- $S_2 = \{A, B, \ldots, Z\}$ and $S_1 = S_3 = S_4 = \{0, 1, \ldots, 9\}$
- 0A00, 0A01 $\ldots$ 5L99, 5M00 $\ldots$ 9Z98, 9Z99
- To add licenses, there are now licenses like $5L99_B$
Pseudocode

function \( f(r, s, t) \) (* integers \( r, s, t \geq 1 \) *)

\[
c = 0
\]

for \( i = 1 \) to \( r \) do

  for \( j = 1 \) to \( s \) do

    for \( k = 1 \) to \( t \) do

      print \((i, j, k)\)

      \( c := c + 1 \)

Observations

- The function \( f(r, s, t) \) prints all possible non-homogeneous lists \((i, j, k)\) for which \( i \in \{1, 2, \ldots, r\}, j \in \{1, 2, \ldots, s\}, \) and \( k \in \{1, 2, \ldots, t\} \)

- The value of \( c \) at the end is \( r \cdot s \cdot t \)
Lists Without Repetitions

**Definition**

\[ \mathcal{L} = (\ell_1, \ell_2, \ldots, \ell_k) \] is an ordered list without repetitions of \( k \leq n \) objects from the set \( S = \{s_1, s_2, \ldots, s_n\} \)

- \( \ell_i \in S \) for all \( 1 \leq i \leq k \) and
- \( \ell_i \neq \ell_j \) for \( 1 \leq i < j \leq k \)

**Counting the number of lists without repetitions**

There are \( n^k = n(n-1)(n-2) \cdots (n-k+1) = \frac{n!}{(n-k)!} \) lists without repetitions of length \( k \) on the numbers \( 1, 2, \ldots, n \)

**Proof**

- There are \( n \) options for \( \ell_1 \)
- There are \( n-1 \) options for \( \ell_2 \)
  
  
  
- There are \( n-k+1 \) options for \( \ell_k \)
- In total there are \( n(n-1)(n-2) \cdots (n-k+1) \) options
Remarks

- $k \leq n$ because there are only $n$ options for each $\ell_i$
- Permutations are lists without repetitions for which $k = n$

Three-digit numbers

- There are $720 = 10 \cdot 9 \cdot 8$ three-digit numbers for which all the digits are different
- 012, 013 … 309, 310 … 598, 601 … 986, 987

Three-letter words

- There are $26 \cdot 25 \cdot 24 = 15600$ possible three-letter words in English with three different letters
- The Scrabble Dictionary (OWL2) recognizes only 1015 three-letter words (words with repetitions)
Nesting Loops

Pseudocode

function $f(n)$ (* integer $n \geq 1$ *)

\[
\begin{align*}
&c = 0 \\
&\text{for } i = 1 \text{ to } n \text{ do} \\
&\quad \text{for } j = 1 \text{ to } n \text{ if } j \neq i \text{ do} \\
&\quad\quad \text{for } k = 1 \text{ to } n \text{ if } (k \neq i \text{ and } k \neq j) \text{ do} \\
&\quad\quad\quad \text{print } (i, j, k) \\
&\quad\quad \text{c := c + 1}
\end{align*}
\]

Observations

- The function $f(n)$ prints all possible lists without repetitions $(i, j, k)$ for which $i, j, k \in \{1, 2, \ldots, n\}$ are three distinct numbers
- The value of $c$ at the end is $n(n - 1)(n - 2)$
Subsets

**Definition**
- For \( n \geq 1 \) and \( 0 \leq k \leq n \), a set of \( n \) objects has \( \binom{n}{k} \) different subsets of \( k \) objects.
- Equivalently, there are \( \binom{n}{k} \) different ways to select \( k \) objects from a set of \( n \) objects.

**Notations**
- \( \binom{n}{k} \) is called “\( n \) choose \( k \)”
- Additional notations to \( \binom{n}{k} \) are \( C(n, k) \), \( C_{n,k} \), \( C^k_n \), and \( nC_k \)

**Special cases**
- \( \binom{n}{0} = 1 \) for \( k = 0 \): the empty set is the only subset with 0 objects and there is only one way to select 0 objects.
- \( \binom{n}{n} = 1 \) for \( k = n \): the entire set is the only subset with \( n \) objects and there is only one way to select all the \( n \) objects.
Example of Subsets

All the subsets of the set \( S = \{ R, B, G, M \} \)

- There is only \( \binom{4}{0} = 1 \) way to choose zero colors from \( S \):
  - \( \emptyset \)
- There are \( \binom{4}{1} = 4 \) ways to choose one color from \( S \):
  - \( \{ R \}, \{ B \}, \{ G \}, \{ M \} \)
- There are \( \binom{4}{2} = 6 \) ways to choose two colors from \( S \):
  - \( \{ R, B \}, \{ R, G \}, \{ R, M \}, \{ B, G \}, \{ B, M \}, \{ G, M \} \)
- There are \( \binom{4}{3} = 4 \) ways to choose three colors from \( S \):
  - \( \{ R, B, G \}, \{ R, B, M \}, \{ R, G, M \}, \{ B, G, M \} \)
- There is only \( \binom{4}{4} = 1 \) way to choose four colors from \( S \):
  - \( \{ R, B, G, M \} \)
A Formula for $\binom{n}{k}$

**Theorem**

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

**Equivalent formulas**

$\binom{n}{k} = \frac{n^k}{k!}$

$$= \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots2\cdot1}$$

$$= \frac{n}{1} \cdot \frac{(n-1)}{2} \cdot \frac{n-2}{3} \cdots \frac{(n-k+1)}{k}$$

$$= \prod_{i=1}^{k} \left( \frac{n+1-i}{i} \right)$$
Proof of Theorem

\( \binom{n}{k} = \frac{n!}{k!(n-k)!} \)

- There are \( n \) ways to select the first object, there are \( n - 1 \) ways to select the second object, and so on . . .
- There are \( (n - k + 1) \) ways to select the last \( k^{th} \) object
- In total, there are

\[
g(n, k) = n(n - 1)(n - 2) \cdots (n - k + 1) = \frac{n!}{(n - k)!}
\]

ways to select an ordered list of \( k \) objects from a set of \( n \) objects
- Each subset of \( k \) objects is selected in \( k! \) different ordered lists and therefore, there are

\[
g(n, k) \cdot \frac{1}{k!} = \frac{n!}{k!(n - k)!}
\]

ways to select a subset of \( k \) objects from a set of \( n \) objects
Nesting Loops

Pseudocode

function \( f(n) \) (* integer \( n \geq 1 \) *)

\[
c = 0
for \ i = 1 \ to \ n - 2 \ do
  for \ j = i + 1 \ to \ n - 1 \ do
    for \ k = j + 1 \ to \ n \ do
      \text{print} \ (i, j, k)
      c := c + 1
\]

Observations

- The function \( f(n) \) prints all possible subsets \( \{i, j, k\} \) from the set \( \{1, 2, \ldots, n\} \) such that \( i < j < k \)
- The value of \( c \) at the end is \( \binom{n}{3} = \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{6} \)
Calculating \( \binom{n}{k} \) For Some Small Values of \( n \) and \( k \)

\( n = 4 \) and \( k = 2 \)

\[
\binom{4}{2} = \frac{4!}{2!2!} = \frac{24}{2 \cdot 2} = \frac{24}{4} = 6
\]

\( n = 5 \) and \( k = 3 \)

\[
\binom{5}{3} = \frac{5!}{3!2!} = \frac{120}{6 \cdot 2} = \frac{120}{12} = 10
\]

\( n = 6 \) and \( k = 3 \)

\[
\binom{6}{3} = \frac{6!}{3!3!} = \frac{720}{6 \cdot 6} = \frac{720}{36} = 20
\]

Online calculator

https://www.omnicalculator.com/math/binomial-coefficient
\[ \binom{n}{k} = \frac{n!}{k!(n-k)!} \] For Small \( k \)

<table>
<thead>
<tr>
<th>( k )</th>
<th>( \binom{n}{k} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \frac{n!}{0!n!} = 1 )</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{n!}{1!(n-1)!} = n )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2} )</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{6} )</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{n!}{4!(n-4)!} = \frac{n(n-1)(n-2)(n-3)}{24} )</td>
</tr>
<tr>
<td>5</td>
<td>( \frac{n!}{5!(n-5)!} = \frac{n(n-1)(n-2)(n-3)(n-4)}{120} )</td>
</tr>
</tbody>
</table>
\[
\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2} \quad \text{For Small } n
\]

<table>
<thead>
<tr>
<th>(n)</th>
<th>(\binom{n}{2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>(\frac{2!}{2!0!} = \frac{2}{2 \cdot 1} = 1 = \frac{2 \cdot 1}{2})</td>
</tr>
<tr>
<td>3</td>
<td>(\frac{3!}{2!1!} = \frac{6}{2 \cdot 1} = 3 = \frac{3 \cdot 2}{2})</td>
</tr>
<tr>
<td>4</td>
<td>(\frac{4!}{2!2!} = \frac{24}{2 \cdot 2} = 6 = \frac{4 \cdot 3}{2})</td>
</tr>
<tr>
<td>5</td>
<td>(\frac{5!}{2!3!} = \frac{120}{2 \cdot 6} = 10 = \frac{5 \cdot 4}{2})</td>
</tr>
<tr>
<td>6</td>
<td>(\frac{6!}{2!4!} = \frac{720}{2 \cdot 24} = 15 = \frac{6 \cdot 5}{2})</td>
</tr>
<tr>
<td>7</td>
<td>(\frac{7!}{2!5!} = \frac{5040}{2 \cdot 120} = 21 = \frac{7 \cdot 6}{2})</td>
</tr>
</tbody>
</table>
(n)_3 = \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{6} \quad \text{For Small } n

\begin{align*}
(3)_3 &= \frac{3!}{3!0!} = \frac{6}{6 \cdot 1} = 1 = \frac{3 \cdot 2 \cdot 1}{6} \\
(4)_3 &= \frac{4!}{3!1!} = \frac{24}{6 \cdot 1} = 4 = \frac{4 \cdot 3 \cdot 2}{6} \\
(5)_3 &= \frac{5!}{3!2!} = \frac{120}{6 \cdot 2} = 10 = \frac{5 \cdot 4 \cdot 3}{6} \\
(6)_3 &= \frac{6!}{3!3!} = \frac{720}{6 \cdot 6} = 20 = \frac{6 \cdot 5 \cdot 4}{6} \\
(7)_3 &= \frac{7!}{3!4!} = \frac{5040}{6 \cdot 24} = 35 = \frac{7 \cdot 6 \cdot 5}{6} \\
(8)_3 &= \frac{8!}{3!5!} = \frac{40320}{6 \cdot 120} = 56 = \frac{8 \cdot 7 \cdot 6}{6}
\end{align*}
Theorem

\[
\binom{n}{k} = \binom{n}{n-k}
\]

Algebraic Proof

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}
\]
Theorem

\[ \binom{n}{k} = \binom{n}{n-k} \]

Combinatorial Proof

- Selecting a subset of \( k \) objects from a set of \( n \) objects is equivalent to selecting the complement subset of the \( n - k \) objects not in the set.
- Therefore, the number of subsets of size \( k \) is equal to the number of subsets of size \( n - k \).
Example: \( S = \{C, R, B, G, M\} \)

Matching the \( \binom{5}{2} = 10 \) two-subsets with the \( \binom{5}{3} = 10 \) three-subsets:

- \( \{G, M\} \leftrightarrow \{C, R, B\} \)
- \( \{B, M\} \leftrightarrow \{C, R, G\} \)
- \( \{B, G\} \leftrightarrow \{C, R, M\} \)
- \( \{R, M\} \leftrightarrow \{C, B, G\} \)
- \( \{R, G\} \leftrightarrow \{C, B, M\} \)
- \( \{R, B\} \leftrightarrow \{C, G, M\} \)
- \( \{C, M\} \leftrightarrow \{R, B, G\} \)
- \( \{C, G\} \leftrightarrow \{R, B, M\} \)
- \( \{C, B\} \leftrightarrow \{R, G, M\} \)
- \( \{C, R\} \leftrightarrow \{B, G, M\} \)
Theorem

The number of subsets of a set \( S \) with \( n \) objects is the same as the number of binary strings of length \( n \) which is \( 2^n \)

Proof

A subset \( R \subseteq S \) can be represented by the binary string \((b_1, b_2, \ldots, b_n)\) in which \( b_i = 1 \) if \( s_i \in R \) and \( b_i = 0 \) if \( s_i \notin R \)

A binary string \((b_1, b_2, \ldots, b_n)\) can be represented by a subset \( R \subseteq S \) such that \( s_i \in R \) if \( b_i = 1 \) and \( s_i \notin R \) if \( b_i = 0 \)

Thus, there is a one-to-one mapping from the set \( 2^S \) of all the subsets of \( S \) to the set of all binary strings of length \( n \)

Therefore, \(|2^S| = 2^n\)
Example: The $2^4 = 16$ Subsets of $\{ R, B, G, M \}$

\[
\begin{align*}
\emptyset &\equiv (0, 0, 0, 0) & \{ B, G \} &\equiv (0, 1, 1, 0) \\
\{ R \} &\equiv (1, 0, 0, 0) & \{ B, M \} &\equiv (0, 1, 0, 1) \\
\{ B \} &\equiv (0, 1, 0, 0) & \{ G, M \} &\equiv (0, 0, 1, 1) \\
\{ G \} &\equiv (0, 0, 1, 0) & \{ R, B, G \} &\equiv (1, 1, 1, 0) \\
\{ M \} &\equiv (0, 0, 0, 1) & \{ R, B, M \} &\equiv (1, 1, 0, 1) \\
\{ R, B \} &\equiv (1, 1, 0, 0) & \{ R, G, M \} &\equiv (1, 0, 1, 1) \\
\{ R, G \} &\equiv (1, 0, 1, 0) & \{ B, G, M \} &\equiv (0, 1, 1, 1) \\
\{ R, M \} &\equiv (1, 0, 0, 1) & \{ R, B, G, M \} &\equiv (1, 1, 1, 1)
\end{align*}
\]
Corollary

For $0 \leq k \leq n$, there are $\binom{n}{k}$ binary strings of length $n$ with exactly $k$ ones.

Proof

- By definition, a set of size $n$ has $\binom{n}{k}$ subsets of size $k$.
- The one-to-one mapping in the proof of the theorem maps all the sets of size $k$ to all the binary strings with exactly $k$ ones.

Special cases

- The null set $\emptyset$ is equivalent to the all-0 binary string $(0, 0, \ldots, 0)$.
- The set itself is equivalent to the all-1 binary string $(1, 1, \ldots, 1)$.
- A binary string with a singleton 1 is equivalent to a singleton subset $\{x\}$ that contains one of the objects $x$ from the set.
Recursive Formula

Recursion

for all integers $n \geq 0$  
\[
\binom{n}{0} = \binom{n}{n} = 1
\]

for all integers $1 \leq k \leq n - 1$  
\[
\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}
\]

Examples

- $6 = \binom{4}{2} = \binom{3}{1} + \binom{3}{2} = 3 + 3 = 6$
- $10 = \binom{5}{3} = \binom{4}{2} + \binom{4}{3} = 6 + 4 = 10$
- $20 = \binom{6}{3} = \binom{5}{2} + \binom{5}{3} = 10 + 10 = 20$
- $15 = \binom{6}{4} = \binom{5}{3} + \binom{5}{4} = 10 + 5 = 15$
- $35 = \binom{7}{4} = \binom{6}{3} + \binom{6}{4} = 20 + 15 = 35$
Recursive Formula: Combinatorial Proof I

\[ \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} : \text{Informal proof by induction} \]

- Consider a set with \( n \) objects \( S = \{s_1, s_2, \ldots, s_n\} \)
- There are two options for selecting a subset \( R \) of \( S \) with \( k \) objects:
  - \( s_n \in R \): There are \( \binom{n-1}{k-1} \) different ways to select additional \( k - 1 \) objects out of \( s_1, s_2, \ldots, s_{n-1} \)
  - \( s_n \notin R \): There are \( \binom{n-1}{k} \) different ways to select \( k \) objects out of \( s_1, s_2, \ldots, s_{n-1} \)
- In total the number of ways to select \( k \) objects from a set of \( n \) objects \( \binom{n}{k} \) is also

\[ \binom{n-1}{k-1} + \binom{n-1}{k} \]
Subsets of $S = \{ R, B, G, M \}$

The six subsets of size 2 from the set $S = \{ R, B, G, M \}$

- There are $\binom{4}{2} = 6$ ways to choose two colors from $S$:
  * $\{ R, B \}, \{ R, G \}, \{ R, M \}, \{ B, G \}, \{ B, M \}, \{ G, M \}$

- There are $\binom{3}{1} = 3$ ways to choose two colors from $S$ where one of them is Magenta:
  * $\{ R, M \}, \{ B, M \}, \{ G, M \}$

- There are $\binom{3}{2} = 3$ ways to choose two colors from $S$ none of them is Magenta:
  * $\{ R, B \}, \{ R, G \}, \{ B, G \}$
Subsets of $S = \{C, R, B, G, M\}$

The ten subsets of size 3 from the set $S = \{C, R, B, G, M\}$

- There are $\binom{5}{3} = 10$ ways to choose three colors from $S$:
  * $\{R, B, G\}$, $\{R, B, M\}$, $\{R, G, M\}$, $\{B, G, M\}$

- There are $\binom{4}{2} = 6$ ways to choose three colors from $S$ where one of them is Magenta:

- There are $\binom{4}{3} = 4$ ways to choose three colors from $S$ none of them is Magenta:
  * $\{C, R, B\}$, $\{C, R, G\}$, $\{C, B, G\}$, $\{R, B, G\}$
Recursive Formula: Algebraic Proof

\[
\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}
\]

\[
\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!}
\]

\[
= \frac{k(n-1)!}{k(k-1)!(n-k)!} + \frac{(n-k)(n-1)!}{(n-k)k!(n-k-1)!}
\]

\[
= \frac{k(n-1)!}{k!(n-k)!} + \frac{(n-k)(n-1)!}{k!(n-k)!}
\]

\[
= \frac{k(n-1)! + (n-k)(n-1)!}{k!(n-k)!}
\]

\[
= \frac{(k + (n-k))(n-1)!}{k!(n-k)!}
\]

\[
= \frac{n(n-1)!}{k!(n-k)!}
\]

\[
= \frac{n!}{k!(n-k)!}
\]

\[
= \binom{n}{k}
\]
Recursive Formula: Combinatorial Proof II

\[ \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} : \text{Informal proof by induction} \]

- \( \binom{n}{k} \) is the number of binary strings of length \( n \) containing \( k \) 1’s
- Some strings start with a 1 while the rest start with a 0
- Binary strings which start with a 1:
  - After the 1, out of the remaining \( n - 1 \) bits \( k - 1 \) bits must be 1
  - There are \( \binom{n-1}{k-1} \) such binary strings
- Binary strings which start with a 0:
  - After the 0, out of the remaining \( n - 1 \) bits \( k \) must be 1
  - There are \( \binom{n-1}{k} \) such binary strings
- Thus, the number of binary strings of length \( n \) containing \( k \) 1’s is
  \[ \binom{n-1}{k-1} + \binom{n-1}{k} \]
Example

The 10 binary strings of length 5 with exactly 2 ones

- $(11000)$, $(10100)$, $(10010)$, $(10001)$, $(01100)$
- $(01010)$, $(01001)$, $(00110)$, $(00101)$, $(00011)$

- There are $\binom{4}{1} = 4$ strings that start with 1:
  - $(11000)$, $(10100)$, $(10010)$, $(10001)$

- There are $\binom{4}{2} = 6$ strings that start with 0:
  - $(01100)$, $(01010)$, $(01001)$, $(00110)$, $(00101)$, $(00011)$

- The total number of strings is $\binom{5}{2} = \binom{4}{1} + \binom{4}{2} = 4 + 6 = 10$
Example

The 20 binary strings of length 6 with exactly 3 zeros

(000111) (001011) (001101) (001110) (010011)
(010101) (010110) (011001) (011010) (011100)
(100011) (100101) (100110) (101001) (101010)
(101100) (110001) (110010) (110100) (111000)

- There are $\binom{5}{2} = 10$ strings that start with 0:
  (000111) (001011) (001101) (001110) (010011)
  (010101) (010110) (011001) (011010) (011100)

- There are $\binom{5}{3} = 10$ strings that start with 1:
  (100011) (100101) (100110) (101001) (101010)
  (101100) (110001) (110010) (110100) (111000)

- The total number of strings is $\binom{6}{3} = \binom{5}{2} + \binom{5}{3} = 10 + 10 = 20$
**Recursive Formula**

**Theorem**

For $1 \leq k \leq n$,

$$\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}$$

**Proof**

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$= \frac{n(n-1)!}{k(k-1)!(n-k)!}$$

$$= \frac{n}{k} \cdot \frac{(n-1)!}{(k-1)!(n-k)!}$$

$$= \frac{n}{k} \binom{n-1}{k-1}$$
3rd Recursive Formula

**Theorem**
- For $1 \leq k \leq n$,

\[
\binom{n}{k} = \frac{n}{n-k} \binom{n-1}{k}
\]

**Proof**

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} \\
= \frac{n(n-1)!}{k!(n-k)(n-k-1)!} \\
= \frac{n}{n-k} \cdot \frac{(n-1)!}{k!(n-k-1)!} \\
= \frac{n}{n-k} \binom{n-1}{k}
\]
**4th Recursive Formula**

**Theorem**

For $1 \leq k \leq n$,

\[
\binom{n}{k} = \frac{n + 1 - k}{k} \binom{n}{k - 1}
\]

**Proof**

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{(n + 1 - k)n!}{k(k-1)!(n+1-k)(n-k)!} = \frac{n+1-k}{k} \cdot \frac{n!}{(k-1)!(n+1-k)!} = \frac{n+1-k}{k} \binom{n}{k-1}
\]
Examples

\[ \binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35 \]

- \[ \binom{7}{3} = \binom{6}{2} + \binom{6}{3} = 15 + 20 = 35 \]
- \[ \binom{7}{3} = \frac{7}{3} \cdot \binom{6}{2} = \frac{7}{3} \cdot \frac{6 \cdot 5}{2 \cdot 1} = \frac{7}{3} \cdot 15 = 35 \]
- \[ \binom{7}{3} = \frac{7}{4} \cdot \binom{6}{3} = \frac{7}{4} \cdot \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = \frac{7}{4} \cdot 20 = 35 \]
- \[ \binom{7}{3} = \frac{5}{3} \cdot \binom{7}{2} = \frac{5}{3} \cdot \frac{7 \cdot 6}{2 \cdot 1} = \frac{5}{3} \cdot 21 = 35 \]

\[ \binom{8}{3} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56 \]

- \[ \binom{8}{3} = \binom{7}{2} + \binom{7}{3} = 21 + 35 = 56 \]
- \[ \binom{8}{3} = \frac{8}{3} \cdot \binom{7}{2} = \frac{8}{3} \cdot \frac{7 \cdot 6}{2 \cdot 1} = \frac{8}{3} \cdot 21 = 56 \]
- \[ \binom{8}{3} = \frac{8}{5} \cdot \binom{7}{3} = \frac{8}{5} \cdot \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = \frac{8}{5} \cdot 35 = 56 \]
- \[ \binom{8}{3} = \frac{6}{3} \cdot \binom{8}{2} = 2 \cdot \frac{8 \cdot 7}{2 \cdot 1} = 2 \cdot 28 = 56 \]
Another Proof for the Main Recursive Formula

\[
\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}
\]

- The 2\textsuperscript{nd} recursive formula implies that
  \[
  \binom{n-1}{k-1} = k \binom{n}{k}
  \]

- The 3\textsuperscript{rd} recursive formula implies that
  \[
  \binom{n-1}{k} = \frac{n-k}{n} \binom{n}{k}
  \]

- Therefore,
  \[
  \binom{n-1}{k-1} + \binom{n-1}{k} = \frac{k}{n} \binom{n}{k} + \frac{n-k}{n} \binom{n}{k} = \frac{(k + n - k)}{n} \binom{n}{k} = \binom{n}{k}
  \]
$$(x + y)^n$$

**Theorem**

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$= \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \cdots + \binom{n}{k} x^{n-k} y^k + \cdots + \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} x^0 y^n$$

$$= x^n + nx^{n-1} y + \frac{n(n-1)}{2} x^{n-2} y^2 + \cdots + \frac{n(n-1)}{2} x^2 y^{n-2} + nxy^{n-1} + y^n$$

**The binomial coefficients**

- Based on the above theorem, $$\binom{n}{k}$$ is called a binomial coefficient
Proof

By definition,

\[(x + y)^n = (x + y) \cdot (x + y) \cdot \cdots (x + y) \cdot (x + y)\]

Using the distributive laws to get the product \(x^{n-k}y^k\):

* select \(k\) of the \(n\) terms to contribute a \(y\) to the product
* select the other \(n - k\) terms to contribute an \(x\) to the product

The coefficient of \(x^{n-k}y^k\) is therefore \(\binom{n}{k}\):

* the number of ways to select \(k\) objects from a set of size \(n\)

Summing over all possible values of \(k\) from 0 to \(n\) implies that

\[(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k\]
\[(x + y)^n\]

**Small \( n \)**

\[(x + y)^1 = \binom{1}{0} x^1 y^0 + \binom{1}{1} x^0 y^1 = x + y\]

\[(x + y)^2 = \binom{2}{0} x^2 y^0 + \binom{2}{1} x^1 y^1 + \binom{2}{2} x^0 y^2 = x^2 + 2xy + y^2\]

\[(x + y)^3 = \binom{3}{0} x^3 y^0 + \binom{3}{1} x^2 y^1 + \binom{3}{2} x^1 y^2 + \binom{3}{3} x^0 y^3 = x^3 + 3x^2 y + 3xy^2 + y^3\]

\[(x + y)^4 = \binom{4}{0} x^4 y^0 + \binom{4}{1} x^3 y^1 + \binom{4}{2} x^2 y^2 + \binom{4}{3} x^1 y^3 + \binom{4}{4} x^0 y^4 = x^4 + 4x^3 y + 6x^2 y^2 + 4xy^3 + y^4\]

**Example**

\[(3 + 2)^4 = 3^4 + 4 \cdot 3^3 \cdot 2 + 6 \cdot 3^2 \cdot 2^2 + 4 \cdot 3 \cdot 2^3 + 2^4 = 81 + 216 + 216 + 96 + 16 = 625 = 5^4\]
\[(1 + y)^n\]

**Corollary**

\[
(1 + y)^n = \binom{n}{0} y^0 + \binom{n}{1} y^1 + \binom{n}{2} y^2 + \cdots + \binom{n}{n-1} y^{n-1} + \binom{n}{n} y^n
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} y^k
\]

**Example**

\[
(1 + y)^4 = \binom{4}{0} y^0 + \binom{4}{1} y^1 + \binom{4}{2} y^2 + \binom{4}{3} y^3 + \binom{4}{4} y^4
\]

\[
= 1 + 4y + 6y^2 + 4y^3 + y^4
\]

\[
(1 + 4)^4 = 1 + 4 \cdot 4 + 6 \cdot 4^2 + 4 \cdot 4^3 + 4^4
\]

\[
= 1 + 16 + 96 + 256 + 256 = 625 = 5^4
\]
\[(x + 1)^n\]

**Corollary**

\[
(x + 1)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2} + \cdots + \binom{n}{n-1}x^1 + \binom{n}{n}x^0
\]

\[
= \sum_{k=0}^{n} \binom{n}{k}x^{n-k}
\]

\[
= \sum_{k=0}^{n} \binom{n}{n-k}x^{n-k}
\]

\[
= \sum_{\ell=0}^{n} \binom{n}{\ell}x^{\ell}
\]

**Example**

\[
(x + 1)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3 + \binom{4}{2}x^2 + \binom{4}{1}x^1 + \binom{4}{0}x^0
\]

\[
= x^4 + 4x^3 + 6x^2 + 4x + 1
\]
Sum of All Binomial Coefficients for a Given \( n \)

Examples

\[
\begin{align*}
\binom{0}{0} &= 1 = 2^0 \\
\binom{1}{0} + \binom{1}{1} &= 1 + 1 = 2 = 2^1 \\
\binom{2}{0} + \binom{2}{1} + \binom{2}{2} &= 1 + 2 + 1 = 4 = 2^2 \\
\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} &= 1 + 3 + 3 + 1 = 8 = 2^3 \\
\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} &= 1 + 4 + 6 + 4 + 1 = 16 = 2^4
\end{align*}
\]
Theorem

\[ \sum_{k=0}^{n} \binom{n}{k} = 2^n \]

Proof

\[ 2^n = (1 + 1)^n \]

\[ = \binom{n}{0}1^n1^0 + \binom{n}{1}1^{n-1}1^1 + \binom{n}{2}1^{n-2}1^2 + \cdots + \binom{n}{n}1^01^n \]

\[ = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} \]

\[ = \sum_{k=0}^{n} \binom{n}{k} \]
Number of Subsets

**Theorem**
- A set of size $n$ has $2^n$ subsets

**Proof**
- By definition, a set of size $n$ has $\binom{n}{k}$ subsets of size $k$ for $0 \leq k \leq n$
- Therefore, the number of subsets of a set of size $n$ is

$$
\sum_{k=0}^{k=n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}
$$
- By the previous theorem this sum equals $2^n$
Computing $3^n$ with Powers of 2

**Corollary**

\[ 3^n = (1 + 2)^n \]

\[ = \binom{n}{0}1^n2^0 + \binom{n}{1}1^{n-1}2^1 + \binom{n}{2}1^{n-2}2^2 + \cdots + \binom{n}{n}1^02^n \]

\[ = \binom{n}{0}2^0 + \binom{n}{1}2^1 + \binom{n}{2}2^2 + \cdots + \binom{n}{n}2^n \]

\[ = \sum_{k=0}^{n} \binom{n}{k}2^k \]
Computing $3^n$ with Powers of 2

$n = 3$

$27 = 3^3 = \binom{3}{0}2^0 + \binom{3}{1}2^1 + \binom{3}{2}2^2 + \binom{3}{3}2^3$

$= 1 \cdot 1 + 3 \cdot 2 + 3 \cdot 4 + 1 \cdot 8$

$= 1 + 6 + 12 + 8$

$n = 4$

$81 = 3^4 = \binom{4}{0}2^0 + \binom{4}{1}2^1 + \binom{4}{2}2^2 + \binom{4}{3}2^3 + \binom{4}{4}2^4$

$= 1 \cdot 1 + 4 \cdot 2 + 6 \cdot 4 + 4 \cdot 8 + 1 \cdot 16$

$= 1 + 8 + 24 + 32 + 16$
Solving Problems

Counting rectangles in a square grid

- Animation: https://www.youtube.com/watch?v=GfODdLHwWzw

Counting paths in a rectangular grid

- Lecture: https://www.youtube.com/watch?v=fpNnAU0iPk&list=PLmdFyQYShrjfPLdhHQxuNWvh2ct666Na3z
- Animation: https://www.youtube.com/watch?v=9YU10k2FYzc
Pascal’s Triangle

Definition

- A **triangular** array of positive integers
- The left edge and the right edge are all 1
- Construct the rows from top to bottom
- Each number is the sum of the two numbers above it diagonally

The first 7 rows

```
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
```
Pascal’s Triangle

The number triangle vs. the binomial coefficient triangle

\[
\begin{array}{cccccccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
\binom{0}{0} \\
\binom{1}{0} & \binom{1}{1} \\
\binom{2}{0} & \binom{2}{1} & \binom{2}{2} \\
\binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} \\
\binom{4}{0} & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} & \binom{4}{4} \\
\binom{5}{0} & \binom{5}{1} & \binom{5}{2} & \binom{5}{3} & \binom{5}{4} & \binom{5}{5} \\
\binom{6}{0} & \binom{6}{1} & \binom{6}{2} & \binom{6}{3} & \binom{6}{4} & \binom{6}{5} & \binom{6}{6} \\
\binom{7}{0} & \binom{7}{1} & \binom{7}{2} & \binom{7}{3} & \binom{7}{4} & \binom{7}{5} & \binom{7}{6} & \binom{7}{7} \\
\end{array}
\]
Pascal’s Triangle

The first 13 rows

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1

1 11 55 165 330 462 462 330 165 55 11 1

1 12 66 220 495 792 924 792 495 220 66 12 1
Pascal’s Triangle

The sum of the first 9 rows

1
1 + 1
1 + 2 + 1
1 + 3 + 3 + 1
1 + 4 + 6 + 4 + 1
1 + 5 + 10 + 10 + 5 + 1
1 + 6 + 15 + 20 + 15 + 6 + 1
1 + 7 + 21 + 35 + 35 + 21 + 7 + 1
1 + 8 + 28 + 56 + 70 + 56 + 28 + 8 + 1

= 1 = 2^0
= 2 = 2^1
= 4 = 2^2
= 8 = 2^3
= 16 = 2^4
= 32 = 2^5
= 64 = 2^6
= 128 = 2^7
= 256 = 2^8
Pascal’s Triangle

Online video resources

- The mathematical secrets of Pascal’s triangle
  https://www.youtube.com/watch?v=XMriWTvPXHI

- What You Don’t Know About Pascal’s Triangle?
  https://www.youtube.com/watch?v=J0I1NuxUcpQ

- Pascal’s Triangle - Numberphile
  https://www.youtube.com/watch?v=0iMtlus-afo

Online text resources

- Summary of facts
  https://www.mathsisfun.com/pascals-triangle.html

- Mysterious Patterns in Pascal’s Triangle
Triangular Numbers

Definition

- The triangular number $T_n$ counts objects arranged in an equilateral triangle whose sides each has $n$ objects.

Illustrations

- [Link to Illustrations](https://cdn1.byjus.com/wp-content/uploads/2016/06/triangular-numbers.jpg)

Animation

- [Link to Animation](https://www.youtube.com/watch?v=TgQn8snKGtw)

Recursive definition

$$T_n = \begin{cases} 
1 & \text{for } n = 1 \\
T_{n-1} + n & \text{for } n > 1 
\end{cases}$$
Triangular Numbers

Closed-form expression

\[ T_n = n + (n - 1) + (n - 2) + \cdots + 2 + 1 = \frac{n(n + 1)}{2} = \binom{n+1}{2} \]

Proof

By induction

Visual proof for: \( n + (n - 1) + \cdots + 1 = \binom{n+1}{2} \)

https://www.youtube.com/watch?v=r5W1oGGwVUg
Triangular Numbers in Pascal’s Triangle

\[
\begin{array}{ccccccccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \\
1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 \\
1 & 9 & 36 & 84 & 126 & 126 & 84 & 36 & 9 & 1 \\
1 & 10 & 45 & 120 & 210 & 210 & 120 & 45 & 10 & 1 \\
1 & 11 & 55 & 165 & 330 & 462 & 462 & 330 & 165 & 55 & 11 & 1
\end{array}
\]
**Sum of Two Consecutive Triangular Numbers**

Identity for integer $n \geq 2$

$$T_{n-1} + T_n = \binom{n}{2} + \binom{n+1}{2} = n^2$$

**Small values of n**

- $\binom{2}{2} + \binom{3}{2} = 1 + 3 = 4 = 2^2$
- $\binom{3}{2} + \binom{4}{2} = 3 + 6 = 9 = 3^2$
- $\binom{4}{2} + \binom{5}{2} = 6 + 10 = 16 = 4^2$
- $\binom{5}{2} + \binom{6}{2} = 10 + 15 = 25 = 5^2$
- $\binom{6}{2} + \binom{7}{2} = 15 + 21 = 36 = 6^2$

**Demo**

- [https://www.youtube.com/watch?v=ONCTEp-v1hU](https://www.youtube.com/watch?v=ONCTEp-v1hU)
Sum of Two Consecutive Triangular Numbers

Identity for integer $n \geq 2$

\[ T_{n-1} + T_n = \binom{n}{2} + \binom{n+1}{2} = n^2 \]

Proof

\[
\binom{n}{2} + \binom{n+1}{2} = \frac{n(n-1)}{2} + \frac{(n+1)n}{2} = \frac{n(n-1) + (n+1)n}{2} = \frac{n((n-1)+(n+1))}{2} = \frac{n \cdot 2n}{2} = n^2
\]
Alternating Sum of First $2n$ Triangular Numbers

Identity for integer $n \geq 1$

$$\sum_{k=1}^{2n} (-1)^k T_k = T_{2n} - T_{2n-1} + T_{2n-2} - T_{2n-3} + \cdots + T_2 - T_1 = 2T_n$$

Small values of $n$

- $3 - 1 = 2 = 2T_1$
- $10 - 6 + 3 - 1 = 6 = 2T_2$
- $21 - 15 + 10 - 6 + 3 - 1 = 12 = 2T_3$
- $36 - 28 + 21 - 15 + 10 - 6 + 3 - 1 = 20 = 2T_4$

Visual proof

https://www.youtube.com/watch?v=Z_FNVP5eJrI
Alternating Sum of First $2n$ Triangular Numbers

Identity for integer $n \geq 1$

$$T_{2n} - T_{2n-1} + T_{2n-2} - T_{2n-3} + \cdots + T_2 - T_1 = 2T_n$$

Proof

$$(T_{2n} - T_{2n-1}) + (T_{2n-2} - T_{2n-3}) + \cdots + (T_2 - T_1) = 2n + (2n - 2) + \cdots + 2$$

$= 2(n + (n - 1) + \cdots + 1)$

$= 2T_n$

$= 2 \cdot \frac{n(n + 1)}{2}$

$= n(n + 1)$
Alternating Sum of First $2n - 1$ Triangular Numbers

Identity for integer $n \geq 1$

$$\sum_{k=1}^{2n-1} (-1)^k T_k = T_{2n-1} - T_{2n-2} + T_{2n-3} - T_{2n-4} + \cdots + T_3 - T_2 + T_1 = n^2$$

Small values of $n$

- $1 = 1 = 1^2$
- $6 - 3 + 1 = 4 = 2^2$
- $15 - 10 + 6 - 3 + 1 = 9 = 3^2$
- $28 - 21 + 15 - 10 + 6 - 3 + 1 = 16 = 4^2$

Visual proof

https://www.youtube.com/watch?v=dRa3ItqEZwM
Identity for integer $n \geq 1$

$$T_{2n-1} - T_{2n-2} + T_{2n-3} - T_{2n-4} + \cdots + T_3 - T_2 + T_1 = n^2$$

Proof

$$(T_{2n-1} - T_{2n-2}) + (T_{2n-3} - T_{2n-2}) + \cdots + (T_3 - T_2) + T_1 = (2n - 1) + (2n - 3) + \cdots + 3 + 1$$

$$= ((2n - 2) + 1) + ((2n - 4) + 1) + (2 + 1) + 1$$

$$= ((2n - 2) + (2n - 4) + \cdots + 2) + n$$

$$= 2((n - 1) + (n - 2) + \cdots + 1) + n$$

$$= 2 \cdot \frac{(n - 1)n}{2} + n$$

$$= (n - 1)n + n$$

$$= n^2 - n + n$$

$$= n^2$$
A Triangular Number Recurrence

Identity for integer $n \geq 1$

$$T_{n+1} = \frac{n+2}{n} T_n$$

Proof sketch

$$nT_{n+1} = n\left(\frac{(n+1)(n+2)}{2}\right) = (n+2)\frac{n(n+1)}{2} = (n+2)T_n$$

Examples

- $\frac{3+2}{3} \cdot T_3 = \frac{5}{3} \cdot 6 = \frac{5\cdot6}{3} = 10 = T_4$
- $\frac{4+2}{4} \cdot T_4 = \frac{6}{4} \cdot 10 = \frac{6\cdot10}{4} = 15 = T_5$

Visual Proof

https://www.youtube.com/watch?v=69TvfNA0Lxc
Identity for integer $n \geq 1$

$$\sum_{i=1}^{n} i^3 = 1 + 8 + 27 + \cdots + (n - 1)^3 + n^3$$

$$= T_n^2$$

$$= \left( \frac{n + 1}{2} \right)^2$$

$$= \left( \frac{n(n + 1)}{2} \right)^2$$

$$= (1 + 2 + 3 + \cdots + (n - 1) + n)^2$$

Visual Proofs

- **Figure:** [https://i.stack.imgur.com/XHc4q.png](https://i.stack.imgur.com/XHc4q.png)
- **Animation 1:** [https://www.youtube.com/watch?v=YQLicI8R4Gs](https://www.youtube.com/watch?v=YQLicI8R4Gs)
- **Animation 2:** [https://www.youtube.com/watch?v=Ye9OPNqV9FA](https://www.youtube.com/watch?v=Ye9OPNqV9FA)
- **Animation 3:** [https://www.youtube.com/watch?v=NxOcT_VKQR0](https://www.youtube.com/watch?v=NxOcT_VKQR0)
Sum of First $n$ Cubes

Examples with Small $n$

1

$1 + 8$

$1 + 8 + 27$

$1 + 8 + 27 + 64$

$1 + 8 + 27 + 64 + 125$

$1 + 8 + 27 + 64 + 125 + 216$

$1 + 8 + 27 + 64 + 125 + 216 + 343$

$= 1 = 1^2 = T_1^2$

$= 9 = 3^2 = T_2^2$

$= 36 = 6^2 = T_3^2$

$= 100 = 10^2 = T_4^2$

$= 225 = 15^2 = T_5^2$

$= 441 = 21^2 = T_6^2$

$= 784 = 28^2 = T_7^2$
Sum of First $n$ Cubes

Identity for integer $n \geq 1$

$$\sum_{i=1}^{n} i^3 = T_n^2$$

Proof sketch

- The proof is based on the following identity for $i^3$
  $$T_i^2 - T_{i-1}^2 = (T_i + T_{i-1})(T_i - T_{i-1}) = i^2 \cdot i = i^3$$

- As a result
  $$\sum_{i=1}^{n} i^3 = \sum_{i=1}^{n} (T_i^2 - T_{i-1}^2) = (T_n^2 - T_{n-1}^2) + (T_{n-1}^2 - T_{n-2}^2) + \cdots + (T_1^2 - T_0^2) = T_n^2$$
The Triangular Number of a Sum

Identity for integers \( n \geq 1 \) and \( k \geq 1 \)

\[
T_{n+k} = T_n + T_k + n \cdot k
\]

Examples

- \( T_7 = T_3 + T_4 + 3 \cdot 4 = 6 + 10 + 12 = 28 \)
- \( T_9 = T_2 + T_7 + 2 \cdot 7 = 3 + 28 + 14 = 45 \)

Proof

\[
T_{n+k} = \frac{(n+k)(n+k+1)}{2} = \frac{n(n+1) + k(k+1) + nk + kn}{2} = \frac{n(n+1)}{2} + \frac{k(k+1)}{2} + \frac{nk + kn}{2} = T_n + T_k + nk
\]
The Triangular Number of a Product

Identity for integers $n > 1$ and $k > 1$

$$T_{n \cdot k} = T_n \cdot T_k + T_{n-1} \cdot T_{k-1}$$

Examples

- $T_8 = T_2 \cdot T_4 + T_1 \cdot T_3 = 3 \cdot 10 + 1 \cdot 6 = 30 + 6 = 36$
- $T_9 = T_3 \cdot T_3 + T_2 \cdot T_2 = 6 \cdot 6 + 3 \cdot 3 = 36 + 9 = 45$
- $T_{12} = T_4 \cdot T_3 + T_3 \cdot T_2 = 10 \cdot 6 + 6 \cdot 3 = 60 + 18 = 78$

Visual proof

https://www.youtube.com/watch?v=p9hIwxLmCFk
The Triangular Number of a Product

Identity for integers $n > 1$ and $k > 1$

$$T_{n \cdot k} = T_n \cdot T_k + T_{n-1} \cdot T_{k-1}$$

Proof

$$T_{nk} = \frac{nk(nk + 1)}{2} = \frac{n^2k^2 + nk}{2} = \frac{2n^2k^2 + 2nk}{4}$$

$$= \frac{2n^2k^2 + (n^2k - n^2k) + (nk^2 - nk^2) + 2nk}{4}$$

$$= \frac{(n^2k^2 + n^2k + nk^2 + nk) + (n^2k^2 - n^2k - nk^2 + nk)}{4}$$

$$= \frac{n^2k^2 + n^2k + nk^2 + nk}{4} + \frac{n^2k^2 - n^2k - nk^2 + nk}{4}$$

$$= \frac{(n^2 + n)(k^2 + k)}{4} + \frac{(n^2 - n)(k^2 - k)}{4}$$

$$= \frac{(n(n + 1))(k(k + 1))}{4} + \frac{((n - 1)n)((k - 1)k)}{4}$$

$$= \frac{n(n + 1)}{2} \cdot \frac{k(k + 1)}{2} + \frac{(n - 1)n}{2} \cdot \frac{(k - 1)k}{2}$$

$$= T_n T_k + T_{n-1} T_{k-1}$$
Odd Squares as Difference of Triangular Numbers

Identity for integer \( n \geq 1 \)

\[
(2n + 1)^2 = T_{3n+1} - T_n
\]

Examples

1. \((2 \cdot 3 + 1)^2 = 7^2 = 49 = 55 - 6 = T(10) - T(3)\)
2. \((2 \cdot 4 + 1)^2 = 9^2 = 81 = 91 - 10 = T(13) - T(4)\)
3. \((2 \cdot 5 + 1)^2 = 11^2 = 121 = 136 - 15 = T(16) - T(5)\)

Visual Proof

https://www.youtube.com/watch?v=hP5ExUA5P8A
Identity for integer $n \geq 1$

$$(2n + 1)^2 = T_{3n+1} - T_n$$

Proof

$$T_{3n+1} - T_n = \frac{(3n + 1)(3n + 2)}{2} - \frac{n(n + 1)}{2}$$

$$= \frac{9n^2 + 9n + 2}{2} - \frac{n^2 + n}{2}$$

$$= \frac{8n^2 + 8n + 2}{2}$$

$$= 4n^2 + 2n + 1$$

$$= (2n + 1)^2$$
Sums of Powers of 9 are Triangular Numbers

Identity for integer $n \geq 1$

$$\sum_{i=0}^{n-1} 9^i = \frac{9^n - 1}{8} = T\sum_{i=0}^{n-1} 3^i$$

Examples

- $1 + 9 = 10 = T_4 = T_{1+3}$
- $1 + 9 + 81 = 91 = T_{13} = T_{1+3+9}$

Visual Proof

https://www.youtube.com/watch?v=Ch7GFdsc9pQ
Sum of the Reciprocals of All Triangular Numbers

Identity and proof

\[
\sum_{n=1}^{\infty} \frac{1}{T_n} = \frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \frac{1}{21} + \cdots
\]

\[
= \sum_{n=1}^{\infty} \frac{1}{n(n+1)}
\]

\[
= 2 \sum_{n=1}^{\infty} \frac{1}{n(n+1)}
\]

\[
= 2 \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right)
\]

\[
= 2 \left( \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots \right)
\]

\[
= 2
\]

Visual proof

[Video link: https://www.youtube.com/watch?v=I0juicS3SIc&t=119s]
Tetrahedral Numbers

Definition
- The tetrahedral number $T_e_n$ counts objects arranged in a four equilateral triangular faces pyramid in which each triangular face has $T_n$ objects

Illustrations
- https://www.geeksforgeeks.org/tetrahedral-numbers/

Animation

Recursive definition
$$T_e_n = \begin{cases} 1 & \text{for } n = 1 \\ T_e_{n-1} + T_n & \text{for } n > 1 \end{cases}$$
Tetrahedral Numbers

Closed-form expression

\[ T_n = \sum_{i=1}^{n} T_i \]
\[ = T_n + T_{n-1} + T_{n-2} + \cdots + T_2 + T_1 \]
\[ = \sum_{i=1}^{n} \binom{n+1}{2} \]
\[ = \binom{n+2}{3} \]
\[ = \frac{n(n+1)(n+2)}{6} \]

Visual proof

https://www.youtube.com/watch?v=N0ETyJ5K6j0
Tetrahedral Numbers

Proof sketch

\[
\begin{align*}
\binom{n+2}{3} &= \binom{n+1}{2} + \binom{n+1}{3} \\
&= \binom{n+1}{2} + \binom{n}{2} + \binom{n}{3} \\
&= \binom{n+1}{2} + \binom{n}{2} + \binom{n-1}{2} + \binom{n-1}{3} \\
&\quad \vdots \\
&= \binom{n+1}{2} + \binom{n}{2} + \binom{n-1}{2} + \cdots + \binom{3}{2} + \binom{3}{3} \\
&= \binom{n+1}{2} + \binom{n}{2} + \binom{n-1}{2} + \cdots + \binom{3}{2} + \binom{2}{2} \\
&= \sum_{i=1}^{n} \binom{n+1}{2}
\end{align*}
\]
Tetrahedral Numbers in Pascal’s Triangle

The Math of “The 12 Days Of Christmas”

https://www.youtube.com/watch?v=fC8W4s6N9HQ

Amotz Bar-Noy (Brooklyn College)
Tetrahedral Numbers as Sum of Products

**Theorem for** \( n \geq 1 \)

\[
\binom{n+2}{3} = \sum_{h=1}^{n} h(n+1-h)
\]

**The binomial side**

\[
\binom{n+2}{3} = \frac{(n+2) \cdot (n+1) \cdot n}{3 \cdot 2 \cdot 1} = \frac{n^3 + 3n^2 + 2n}{6}
\]

**The sum of products side**

\[
1 \cdot n + 2 \cdot (n-1) + 3 \cdot (n-2) + \cdots + h(n+1-h) + \cdots + (n-2) \cdot 3 + (n-1) \cdot 2 + n \cdot 1
\]

**Visual proof**

https://www.youtube.com/watch?v=pucFDbdEyuE&t=3s
Tetrahedral Numbers as Sum of Products

Examples

\[
\binom{4}{3} = \frac{4 \cdot 3 \cdot 2}{3 \cdot 2 \cdot 1} = 4 \\
= 1 \cdot 2 + 2 \cdot 1 = 2 + 2 = 4
\]

\[
\binom{5}{3} = \frac{5 \cdot 4 \cdot 3}{3 \cdot 2 \cdot 1} = 10 \\
= 1 \cdot 3 + 2 \cdot 2 + 3 \cdot 1 = 3 + 4 + 3 = 10
\]

\[
\binom{6}{3} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20 \\
= 1 \cdot 4 + 2 \cdot 3 + 3 \cdot 2 + 4 \cdot 1 = 4 + 6 + 6 + 4 = 20
\]

\[
\binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35 \\
= 1 \cdot 5 + 2 \cdot 4 + 3 \cdot 3 + 4 \cdot 2 + 5 \cdot 1 = 5 + 8 + 9 + 8 + 5 = 35
\]
Combinatorial proof: \( \binom{n+2}{3} = \sum_{h=1}^{n} h(n+1-h) \)

- There are \( \binom{n+2}{3} \) ordered triplets \((i < j < k)\) in the set \(\{1, \ldots, n+2\}\)
- Fix the middle index \(j\)
- \(j\) is neither 1 nor \(n+2\) and therefore \(2 \leq j \leq n+1\)
- There are \(j - 1\) ways to select \(i \in \{1, 2, \ldots, j-1\}\) and there are \(n+2-j\) ways to select \(k \in \{j+1, j+2, \ldots, n+2\}\)
- Therefore, the number of triplets \((i, j, k)\) with \(j\) as the middle index is \((j - 1)(n + 2 - j)\)
- The total number of triplets is \(\sum_{j=2}^{n+1} (j - 1)(n + 2 - j)\)
- Replacing \(j\) with \(h + 1\) implies that this number is \(\sum_{h=1}^{n} h(n + 1 - h)\)
Tetrahedral Numbers as Sum of Products

The combinatorial proof using pseudocodes

- Pseudocode I: The value of $c$ at the end is
  \[
  \binom{n+2}{3} = \frac{(n+2)(n+1)n}{6}
  \]
- Pseudocode II: The value of $c$ is
  \[
  \sum_{j=2}^{n+1} (j-1)(n+2-j)
  \]

**Pseudocode I**

```plaintext
function f(n) (* integer n ≥ 1 *)
c = 0
for i = 1 to n do
    for j = i + 1 to n + 1 do
        for k = j + 1 to n + 2 do
            c := c + 1
```

**Pseudocode II**

```plaintext
function f(n) (* integer n ≥ 1 *)
c = 0
for j = 2 to n + 1 do
    for i = 1 to j - 1 do
        for k = j + 1 to n + 2 do
            c := c + 1
```
Example $n = 3$: lexicographic order vs. proof order

(123)  (123)
(124)  (124)
(125)  (125)
(134)  (134)
(135)  (135)
(145)  (234)
(234)  (235)
(235)  (145)
(245)  (245)
(345)  (345)
Tetrahedral Numbers as Sum of Products

Example $n = 4$: lexicographic order vs. proof order

\[
\begin{array}{cccc}
(123) & (123) & (234) & (145) \\
(124) & (124) & (235) & (146) \\
(125) & (125) & (236) & (245) \\
(126) & (126) & (245) & (246) \\
(134) & (134) & (246) & (346) \\
(135) & (135) & (256) & (346) \\
(136) & (136) & (345) & (156) \\
(145) & (234) & (346) & (256) \\
(146) & (235) & (356) & (356) \\
(156) & (236) & (456) & (456)
\end{array}
\]
Sum of Squares of Binomial Coefficients

Theorem

\[
\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2
\]

Example \( n = 1 \)

\[
\binom{2}{1} = \frac{2}{1} = 2
\]

\[
\binom{2}{1} = \sum_{k=0}^{1} \binom{1}{k}^2
\]

\[
= \binom{1}{0}^2 + \binom{1}{1}^2
\]

\[
= 1^2 + 1^2 = 1 + 1 = 2
\]
Sum of Squares of Binomial Coefficients

Theorem

\[ \binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2 \]

Example \( n = 2 \)

\[
\binom{4}{2} = \frac{4 \cdot 3}{2 \cdot 1} = 6
\]

\[
\binom{4}{2} = \sum_{k=0}^{2} \binom{2}{k}^2
\]

\[
= \binom{2}{0}^2 + \binom{2}{1}^2 + \binom{2}{2}^2
\]

\[
= 1^2 + 2^2 + 1^2 = 1 + 4 + 1 = 6
\]
Sum of Squares of Binomial Coefficients

**Theorem**

\[
\binom{2n}{n} = \sum_{k=0}^{n} \left( \binom{n}{k} \right)^2
\]

**Example** \( n = 3 \)

\[
\binom{6}{3} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20
\]

\[
\binom{6}{3} = \sum_{k=0}^{3} \left( \binom{3}{k} \right)^2
\]

\[
= \binom{3}{0}^2 + \binom{3}{1}^2 + \binom{3}{2}^2 + \binom{3}{3}^2
\]

\[
= 1^2 + 3^2 + 3^2 + 1^2 = 1 + 9 + 9 + 1 = 20
\]
Sum of Squares of Binomial Coefficients

Theorem

\[
\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2
\]

Example \( n = 4 \)

\[
\binom{4}{4} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = 70
\]

\[
\binom{4}{4} = \sum_{k=0}^{4} \binom{4}{k}^2
\]

\[
= \binom{4}{0}^2 + \binom{4}{1}^2 + \binom{4}{2}^2 + \binom{4}{3}^2 + \binom{4}{4}^2
\]

\[
= 1^2 + 4^2 + 6^2 + 4^2 + 1^2 = 1 + 16 + 36 + 16 + 1 = 70
\]
Sum of Squares of Binomial Coefficients

Combinatorial proof: \( \binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2 \)

- There are \( \binom{2n}{n} \) ways to select \( n \) numbers from the set \( S = \{1, 2, \ldots, 2n\} \).
- Partition the set \( S \) into two disjoint sets \( L = \{1, 2, \ldots, n\} \) and \( R = \{n+1, n+2, \ldots, 2n\} \).
- Every selection of \( n \) numbers from \( S \) is a selection of \( k \) numbers from \( L \) and \( n-k \) numbers from \( R \) for some \( 0 \leq k \leq n \).
- For a given \( k \), there are \( f(n, k) = \binom{n}{k} \binom{n}{n-k} \) such selections.
- By the symmetry of the binomial coefficient, \( f(n, k) = \binom{n}{k}^2 \).
- Sum \( f(n, k) \) for all \( 0 \leq k \leq n \) to get all the selections:

\[
\binom{2n}{n} = \sum_{k=0}^{n} f(n, k) = \sum_{k=0}^{n} \binom{n}{k}^2
\]
### Example 2n = 6: lexicographic order vs. proof order

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</table>
**Sum of Products**

**Identity I**

\[
\binom{r+s}{h} = \sum_{k=0}^{h} \binom{r}{k} \binom{s}{h-k} \quad \text{for given } h \leq r \text{ and } h \leq s
\]

\[
\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2 \quad \text{for } h = r = s = n
\]

**Identity II**

\[
\binom{n+1}{k+1} = \sum_{m=j}^{n-k+j} \binom{m}{j} \binom{n-m}{k-j} \quad \text{for given } 0 \leq j \leq k \leq n
\]

\[
\binom{n+1}{k+1} = \sum_{m=k}^{n} \binom{m}{k} \quad \text{for } j = k
\]