Discrete Structures: Induction

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**The Principle of Induction**

**The principle**

- Let $P_n$ be a statement about all positive integers $n = 1, 2, 3, \ldots$
- If the following hold:
  - **Induction base:** $P_1$ is true
  - **Inductive step:** For all integers $k \geq 1$, if $P_k$ is true then $P_{k+1}$ is true
- Then $P_n$ is true for all integers $n \geq 1$
- The assumption “$P_k$ is true” is the **induction hypothesis**

**Cartoons**

Why Induction Works?

“Justification” with the Well-Ordering Principle

- Assume that there exists $j \geq 2$ such that $P_j$ is false
- Let $S$ be the set of all integers $h \geq 1$ for which $P_h$ is false
  - $S$ is a non empty set that can contain infinite number of integers
- Let $k + 1$ be the minimum integer in $S$
  - The Well-Ordering Principle
- $k \geq 1$ since by the induction base $P_1$ is true
- $P_k$ is true and $P_{k+1}$ is false by the minimality of $k + 1$
- A contradiction to the inductive step
Generalizations

Other Induction Bases
- For any $m \geq 0$ the induction base could be $P_m$ instead of $P_1$
- In this case, the induction is applied to $n = m, m + 1, \ldots$

Strong Induction
- The induction base is that $P_1, \ldots, P_m$ are true for some $m \geq 1$
- The induction hypothesis is that $P_1, P_2, \ldots, P_k$ are true for some $k \geq m$
- The inductive step is that $P_{k+1}$ is implied by a non-empty subset of statements from the set $\{P_1, P_2, \ldots, P_k\}$
Notations

The induction variable

- The **inductive step** could be that $P_{n+1}$ is implied by $P_n$ and then $P_n$ is the **induction hypothesis**
- The **inductive step** could be that $P_n$ is implied by $P_{n-1}$ and then $P_{n-1}$ is the **induction hypothesis**
Some Online Resources

- An introductory video in less than 4 minutes:
  [https://www.youtube.com/watch?v=bePpPFos0kE](https://www.youtube.com/watch?v=bePpPFos0kE)

- Introduction in 15 minutes:
  [https://www.youtube.com/watch?v=ruBnYcLzVlU](https://www.youtube.com/watch?v=ruBnYcLzVlU)

- Sum of the first $n$ integers in 7 minutes:
  [https://www.youtube.com/watch?v=dMn5w4_ztSw&feature=youtu.be](https://www.youtube.com/watch?v=dMn5w4_ztSw&feature=youtu.be)

- Sum of the first $n$ odd integers in 10 minutes:
  [https://www.youtube.com/watch?v=twA6vZgX_U4](https://www.youtube.com/watch?v=twA6vZgX_U4)

- Sum of first $n$ integers of the form $5k - 1$ in 6 minutes:
  [https://www.youtube.com/watch?v=IFqna5F0kW8](https://www.youtube.com/watch?v=IFqna5F0kW8)

- $6^n + 4$ is divisible by 5 in 6 minutes:
  [https://youtu.be/MpjkLf7lfRA](https://youtu.be/MpjkLf7lfRA)

- Introduction in 8 minutes (from 11:25 to 20:04):
  [https://youtu.be/0UgID8C9RvE?list=PLZzHxk_TPOStgPtqRZ6KzmkUQBQ8TSWVX](https://youtu.be/0UgID8C9RvE?list=PLZzHxk_TPOStgPtqRZ6KzmkUQBQ8TSWVX)
Sum of First $n$ Positive Integers

An identity

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + (n - 1) + n = \frac{n(n + 1)}{2}$$

An equivalent identity

$$\sum_{i=1}^{n-1} i = 1 + 2 + \cdots + (n - 2) + (n - 1) = \frac{(n - 1)n}{2}$$
Correctness for Small $n$

$$1 = 1 = \frac{1 \cdot 2}{2}$$

$$1 + 2 = 3 = \frac{2 \cdot 3}{2}$$

$$1 + 2 + 3 = 6 = \frac{3 \cdot 4}{2}$$

$$1 + 2 + 3 + 4 = 10 = \frac{4 \cdot 5}{2}$$

$$1 + 2 + 3 + 4 + 5 = 15 = \frac{5 \cdot 6}{2}$$

$$1 + 2 + 3 + 4 + 5 + 6 = 21 = \frac{6 \cdot 7}{2}$$

$$1 + 2 + 3 + 4 + 5 + 6 + 7 = 28 = \frac{7 \cdot 8}{2}$$

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36 = \frac{8 \cdot 9}{2}$$
Proof By Induction

Notations

- \( L(n) = \sum_{i=1}^{n} i = 1 + 2 + \cdots + (n - 1) + n \)
- \( R(n) = \frac{n(n+1)}{2} \)

The induction base: \( n = 1 \)

- \( L(1) = R(1) \), because \( L(1) = 1 \) and \( R(1) = \frac{1 \cdot 2}{2} = 1 \)

The induction hypothesis: \( L(k) = R(k) \) for \( k \geq 1 \)

\[
\sum_{i=1}^{k} i = 1 + 2 + \cdots + (k - 1) + k = \frac{k(k + 1)}{2}
\]
The inductive step: $L(k + 1) = R(k + 1)$ for $k \geq 1$

\[
\begin{align*}
L(k + 1) & = 1 + 2 + \cdots + k + (k + 1) \\
& = L(k) + (k + 1) \\
& = R(k) + (k + 1) \\
& = \frac{k(k + 1)}{2} + (k + 1) \\
& = \frac{k(k + 1)}{2} + \frac{2(k + 1)}{2} \\
& = \frac{k(k + 1) + 2(k + 1)}{2} \\
& = \frac{(k + 2)(k + 1)}{2} \\
& = \frac{(k + 1)((k + 1) + 1)}{2} \\
& = R(k + 1)
\end{align*}
\]
Another Proof

Idea

- Prove that $2L(n) = 2R(n)$ implying $L(n) = R(n)$

Example

\[
2L(4) = 2(1 + 2 + 3 + 4)
\]
\[
= (1 + 2 + 3 + 4) + (4 + 3 + 2 + 1)
\]
\[
= (1 + 4) + (2 + 3) + (3 + 2) + (4 + 1)
\]
\[
= 4 \cdot 5
\]
\[
= 2 \frac{4 \cdot 5}{2}
\]
\[
= 2R(4)
\]
Another Proof

The General Case

\[
2L(n) = (1 + 2 + \cdots + (n - 1) + n) + (n + (n - 1) + \cdots + 2 + 1) \\
= (1 + n) + (2 + (n - 1)) + \cdots + ((n - 1) + 2) + (n + 1) \\
= (n + 1) + (n + 1) + \cdots + (n + 1) + (n + 1) \\
= n(n + 1) \\
= 2 \frac{n(n + 1)}{2} \\
= 2R(n)
\]

A proof without words

https://i.stack.imgur.com/yerzW.png
**Identity**

\[
\sum_{i=1}^{n} 2i = 2 + 4 + \cdots + 2(n - 1) + 2n = n(n + 1)
\]

**Proof**

\[
\sum_{i=1}^{n} 2i = 2 + 4 + \cdots + 2(n - 1) + 2n
\]

\[
= 2(1 + 2 + \cdots + (n - 1) + n)
\]

\[
= 2 \cdot \frac{n(n + 1)}{2}
\]

\[
= n(n + 1)
\]
Sum of First $n$ Odd Positive Integers

**Identity**

$$\sum_{i=1}^{n} (2i - 1) = 1 + 3 + 5 + \cdots + (2n - 3) + (2n - 1) = n^2$$

**Correctness for small $n$**

- $1 = 1 = 1^2$
- $1 + 3 = 4 = 2^2$
- $1 + 3 + 5 = 9 = 3^2$
- $1 + 3 + 5 + 7 = 16 = 4^2$
- $1 + 3 + 5 + 7 + 9 = 25 = 5^2$
- $1 + 3 + 5 + 7 + 9 + 11 = 36 = 6^2$
- $1 + 3 + 5 + 7 + 9 + 11 + 13 = 49 = 7^2$
- $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 = 64 = 8^2$
- $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 = 81 = 9^2$
- $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 = 100 = 10^2$
Sum of First $n$ Odd Positive Integers

Identity

$$\sum_{i=1}^{n} (2i - 1) = 1 + 3 + \cdots + (2n - 3) + (2n - 1) = n^2$$

Proof by reduction

$$\sum_{i=1}^{n} (2i - 1) = \sum_{i=1}^{n} (2i) - \sum_{i=1}^{n} 1$$

$$= n(n + 1) - n$$

$$= n^2 + n - n$$

$$= n^2$$

Visual proofs

- https://www.youtube.com/watch?v=IJ0EQCkJCTc
- https://www.youtube.com/watch?v=ZeEOgbLo0Rg
Proof By Induction

Notations
- \( L(n) = \sum_{i=1}^{n} (2i - 1) = 1 + 3 + \cdots + (2n - 3) + (2n - 1) \)
- \( R(n) = n^2 \)

The induction base: \( n = 1 \)
- \( L(1) = R(1) \), because \( L(1) = 1 \) and \( R(1) = 1^2 = 1 \)

The induction hypothesis: \( L(k) = R(k) \) for \( k \geq 1 \)

\[
\sum_{i=1}^{k} (2i - 1) = 1 + 3 + \cdots + (2k - 3) + (2k - 1) = k^2
\]
Proof By Induction

The inductive step: $L(k + 1) = R(k + 1)$ for $k \geq 1$

\[
L(k + 1) = 1 + 3 + \cdots + (2k - 1) + (2k + 1)
\]
\[
= L(k) + (2k + 1)
\]
\[
= R(k) + (2k + 1)
\]
\[
= k^2 + (2k + 1)
\]
\[
= (k + 1)^2
\]
\[
= R(k + 1)
\]
Sum of the First $2n$ Odd Positive Integers

Identity

The sum of the first $n$ odd integers is $1/3$ the sum of the next $n$ odd integers:

$$\frac{\sum_{i=1}^{n} (2i-1)}{\sum_{i=n+1}^{2n} (2i-1)} = \frac{1 + 3 + \cdots + (2n-1)}{(2n+1) + (2n+3) + \cdots + (4n-1)} = \frac{1}{3}$$

Proof

$$\sum_{i=n+1}^{2n} (2i-1) = \sum_{i=1}^{2n} (2i-1) - \sum_{i=1}^{n} (2i-1)$$

$$= (2n)^2 - n^2 = 4n^2 - n^2 = 3n^2$$

$$= 3 \sum_{i=1}^{n} (2i - 1)$$

Visual Proofs

https://youtu.be/MmOTqrtbtFQ?list=PLZh9gzIvXQUbr38YfIlul9j7_54hXZy_

https://www.youtube.com/watch?v=fTBvVeURb3Q
Arithmetic Progressions

Definition

- A sequence $a_1, a_2, \ldots, a_n$ is an **arithmetic progression** if $a_i - a_{i-1} = d$ for all $2 \leq i \leq n$ for some real number $d$.

Key observations

- **Observation 1:** $a_i = a_1 + (i - 1)d$ for $1 \leq i \leq n$
- **Observation 2:** $a_i = a_n - (n - i)d$ for $1 \leq i \leq n$

Theorem

\[
\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_{n-1} + a_n = \frac{n(a_1 + a_n)}{2}
\]

\[
\frac{\sum_{i=1}^{n} a_i}{n} = \frac{a_1 + a_2 + \cdots + a_{n-1} + a_n}{n} = \frac{a_1 + a_n}{2}
\]
Arithmetic Progressions

The theorem in words version I
- The sum of all the \( n \) numbers in an arithmetic progression of length \( n \) is the average between the first and the last numbers multiplied by \( n \).

The theorem in words version II
- The average of all the \( n \) numbers in an arithmetic progression of length \( n \) is the average between the first and the last numbers.

Remark
- The definition and the theorem work for real numbers and negative numbers. For simplicity we assume that all the numbers in the sequence are positive integers.
Arithmetic Progressions: \( a_1 = 5, \ d = 3, \) and \( n = 11 \)

**Sequence**

5, 8, 11, 14, 17, 20, 23, 26, 29, 32, 35

**Observation 1**

\[
a_4 = a_1 + (4 - 1)d = 5 + 3 \cdot 3 = 5 + 9 = 14
\]

**Observation 2**

\[
a_7 = a_{11} - (11 - 7)d = 35 - 4 \cdot 3 = 35 - 12 = 23
\]

**Sum of all numbers**

\[
5 + 8 + 11 + 14 + 17 + 20 + 23 + 26 + 29 + 32 + 35 = 220
\]

**Average of all numbers**

\[
\frac{220}{11} = 20
\]

**Average of the first and the last numbers**

\[
\frac{(5 + 35)}{2} = \frac{40}{2} = 20
\]
Arithmetic Progressions

**Theorem**

\[
\sum_{i=1}^{n} a_i = \frac{n(a_1 + a_n)}{2}
\]

**Notation**

- Define \( S_n = a_1 + a_2 + \cdots + a_{n-1} + a_n \)

**Direct proof**

\[
\begin{align*}
S_n & = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_1 + (n-2)d) + (a_1 + (n-1)d) \\
S_n & = a_n + (a_n - d) + (a_n - 2d) + \cdots + (a_n - (n-2)d) + (a_n - (n-1)d) \\
2S_n & = n(a_1 + a_n) \\
S_n & = \frac{n(a_1 + a_n)}{2}
\end{align*}
\]
Proof By Induction

Notations

- $L(n) = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_{n-1} + a_n$
- $R(n) = \frac{n(a_1 + a_n)}{2}$

The induction base: $n = 1$

- $L(1) = R(1)$, because $L(1) = a_1$ and $R(1) = \frac{1 \cdot (a_1 + a_1)}{2} = a_1$

The induction hypothesis: $L(k) = R(k)$ for $k \geq 1$

$$\sum_{i=1}^{k} a_i = \frac{k(a_1 + a_k)}{2}$$
Proof By Induction

The inductive step: \( L(k + 1) = R(k + 1) \) for \( k \geq 1 \)

\[
L(k + 1) = \frac{k(a_1 + a_k)}{2} + a_{k+1}
\]

\[
= \frac{ka_1}{2} + \frac{ka_k}{2} + \frac{2a_{k+1}}{2}
\]

\[
= \frac{ka_1 + a_k}{2} + \frac{2a_{k+1} + (k - 1)a_k}{2}
\]

\[
= \frac{ka_1 + (a_1 + (k - 1)d)}{2} + \frac{2a_{k+1} + (k - 1)(a_{k+1} - d)}{2}
\]

\[
= \frac{(k + 1)a_1 + (k - 1)d}{2} + \frac{(k + 1)a_{k+1} - (k - 1)d}{2}
\]

\[
= \frac{(k + 1)(a_1 + a_{k+1})}{2}
\]

\[
= R(k + 1)
\]
Theorem

- $2^n > n^2$ for any integer $n \geq 5$

Why $n \geq 5$?

- $2^1 = 2 > 1 = 1^2$
- $2^2 = 4 = 4 = 2^2$
- $2^3 = 8 < 9 = 3^2$
- $2^4 = 16 = 16 = 4^2$
- $2^5 = 32 > 25 = 5^2$
- $2^6 = 64 > 36 = 6^2$
- $2^7 = 128 > 49 = 7^2$
$2^n > n^2$: Proof By Induction

The induction base for $n = 5$

- $2^5 = 32 > 25 = 5^2$

The induction hypothesis for $k \geq 5$

- Assume that $2^k > k^2$

The inductive step for $k \geq 5$: prove that $2^{k+1} > (k + 1)^2$

\[
2^{k+1} = 2 \cdot 2^k \\
> 2k^2 \quad (* \text{ the induction hypothesis } *) \\
= k^2 + k^2 \\
> k^2 + 3k \quad (* \text{ because } k > 3 *) \\
> k^2 + 2k + 1 \quad (* \text{ because } k > 1 *) \\
= (k + 1)^2
\]
A Divisibility Theorem: Proof By Induction

Theorem

\[ n(n + 1)(n + 2) \text{ is divisible by 6 for } n \geq 1 \]

The induction base: for \( n = 1, 2, 3, 4, 5 \)

\[ 
\begin{align*}
1 \cdot 2 \cdot 3 &= 6 & = 1 \cdot 6 \\
2 \cdot 3 \cdot 4 &= 24 & = 4 \cdot 6 \\
3 \cdot 4 \cdot 5 &= 60 & = 10 \cdot 6 \\
4 \cdot 5 \cdot 6 &= 120 & = 20 \cdot 6 \\
5 \cdot 6 \cdot 7 &= 210 & = 35 \cdot 6
\end{align*}
\]

The induction hypothesis for \( k \geq 1 \)

\[ 
\begin{itemize}
\item Assume that \( k(k + 1)(k + 2) \) is divisible by 6
\item That is, \( k(k + 1)(k + 2) = 6q \) for an integer \( q \)
\end{itemize}
\]
A Divisibility Theorem: Proof By Induction

The inductive step for $k \geq 1$

\[
(k + 1)(k + 2)(k + 3) = k(k + 1)(k + 2) + 3(k + 1)(k + 2) \quad (\text{* algebra *})
\]
\[
= 6q + 3(k + 1)(k + 2) \quad (\text{* induction hypothesis *})
\]
\[
= 6q + 6\frac{(k + 1)(k + 2)}{2} \quad (\text{* algebra *})
\]
\[
= 6q + 6p \quad (\text{* either } k + 1 \text{ or } k + 2 \text{ is even *})
\]
\[
= 6(q + p) \quad (\text{* Q.E.D. *})
\]
A Divisibility Theorem: Second Proof

**Theorem**

- $n(n + 1)(n + 2)$ is divisible by 6 for $n \geq 1$

**Proof**

- $n$, $n + 1$, and $n + 2$ are three consecutive integers
- One of them must be divisible by 3
- One (could be the same integer) must be even and therefore is divisible by 2
- Therefore, the product of the three integers must be divisible by $6 = 3 \cdot 2$
Another Divisibility Theorem

**Theorem**

- $n(n + 1)(n + 2)$ is divisible by 24 for an even $n \geq 2$

**Small Values of $n$**

- $2 \cdot 3 \cdot 4 = 24 = 1 \cdot 24$
- $4 \cdot 5 \cdot 6 = 120 = 5 \cdot 24$
- $6 \cdot 7 \cdot 8 = 336 = 14 \cdot 24$

**Proof**

- $n$, $n + 1$, and $n + 2$ are three consecutive integers
- One of them must be divisible by 3
- $n$ and $n + 2$ are two consecutive even integers
- One of them must be divisible by 4 while the other is divisible by 2
- Therefore, the product of the three integers must be divisible by $24 = 3 \cdot 4 \cdot 2$
A Set of Size $n \geq 0$ Has $2^n$ Subsets

Proof
- By induction on the size of the set

The induction base for $n = 0$ and $n = 1$
- The only subset of the empty set is the empty set and $2^0 = 1$
- The empty set and the entire set are the only subsets of a set of size 1 and $2^1 = 2$

The induction hypothesis for $k \geq 1$
- Any set of size $k$ has $2^k$ subsets

Notations
- Let $S = \{s_1, s_2, \ldots, s_k, s_{k+1}\}$ be a set of size $k + 1$
- Let $S' = \{s_1, s_2, \ldots, s_k\}$ be the subset of $S$ containing all of its members except $s_{k+1}$
A Set of Size \( n \geq 0 \) Has \( 2^n \) Subsets

The inductive step for \( k \geq 1 \): prove that \( S \) has \( 2^{k+1} \) subsets

- By the induction hypothesis, \( S' \) has \( 2^k \) subsets all of them are also subsets of \( S \)
- Let \( R \) be a subset of \( S \) that is not a subset of \( S' \)
  - It follows that \( s_{k+1} \in R \) and that \( R' = R \setminus \{s_{k+1}\} \) is a subset of \( S' \)
- Let \( R' \) be a subset of \( S' \)
  - Then, \( R = R' \cup \{s_{k+1}\} \) is a subset of \( S \) that is not a subset of \( S' \)

The above two arguments establishes a one-to-one mapping from the set of all the subsets that contain \( s_{k+1} \) to the set of all the subsets that do not contain \( s_{k+1} \)

Therefore, there are also \( 2^k \) subsets of \( S \) that contain \( s_{k+1} \)

Since a subset of \( S \) either contains \( s_{k+1} \) or does not contain \( s_{k+1} \), it follows that the number of subsets of \( S \) is \( 2^k + 2^k = 2^{k+1} \)
Example: \( S = \{C, R, B, G, M\} \)

Matching the 16 subsets without \( M \) to the 16 subsets with \( M \):

\[
\begin{align*}
\emptyset & \longleftrightarrow \{M\} & \{R, B\} & \longleftrightarrow & \{R, B, M\} \\
\{C\} & \longleftrightarrow \{C, M\} & \{R, G\} & \longleftrightarrow & \{R, G, M\} \\
\{R\} & \longleftrightarrow \{R, M\} & \{B, G\} & \longleftrightarrow & \{B, G, M\} \\
\{B\} & \longleftrightarrow \{B, M\} & \{C, R, B\} & \longleftrightarrow & \{C, R, B, M\} \\
\{G\} & \longleftrightarrow \{G, M\} & \{C, R, G\} & \longleftrightarrow & \{C, R, G, M\} \\
\{C, R\} & \longleftrightarrow \{C, R, M\} & \{C, B, G\} & \longleftrightarrow & \{C, B, G, M\} \\
\{C, B\} & \longleftrightarrow \{C, B, M\} & \{R, B, G\} & \longleftrightarrow & \{R, B, G, M\} \\
\{C, G\} & \longleftrightarrow \{C, G, M\} & \{C, R, B, G\} & \longleftrightarrow & \{C, R, B, G, M\}
\end{align*}
\]
Geometric Progressions

Definition
- A sequence \( a_1, a_2, \ldots, a_n \) is a geometric progression with a common positive ratio \( q > 0 \) if \( a_i = qa_{i-1} \) for all \( 2 \leq i \leq n \)

Simplifying assumptions
- Set \( a_1 = q \) and as a result the sequence becomes \( q^1, q^2, \ldots, q^n \)
- Add \( a_0 = 1 = q^0 \) to the beginning of the sequence and as a result the sequence becomes \( q^0, q^1, q^2, \ldots, q^n \)

Theorem
For a real number \( q > 0 \) and \( q \neq 1 \)

\[
\sum_{i=0}^{n-1} q^i = 1 + q + \cdots + q^{n-2} + q^{n-1} = \frac{q^n - 1}{q - 1}
\]
Proof By Induction

Notations

- \( L(n) = 1 + q + \cdots + q^{n-2} + q^{n-1} \)
- \( R(n) = \frac{q^{n-1}}{q-1} \)

The Induction base: \( n = 1 \)

- \( L(1) = R(1), \) because \( L(1) = 1 \) and \( R(1) = \frac{q^1 - 1}{q-1} = 1 \)

The induction hypothesis: \( L(k) = R(k) \) for \( k \geq 1 \)

\[
\sum_{i=0}^{k-1} q^i = 1 + q + \cdots + q^{k-2} + q^{k-1} = \frac{q^k - 1}{q-1}
\]
Proof By Induction

The inductive step: \( L(k + 1) = R(k + 1) \) for \( k \geq 1 \)

\[
L(k + 1) = 1 + q + \cdots + q^{k-1} + q^k
\]
\[
= L(k) + q^k
\]
\[
= R(k) + q^k
\]
\[
= \frac{q^k - 1}{q - 1} + q^k
\]
\[
= \frac{(q^k - 1) + ((q - 1)q^k)}{q - 1}
\]
\[
= \frac{(q^k - 1) + (q^{k+1} - q^k)}{q - 1}
\]
\[
= \frac{q^{k+1} - 1}{q - 1}
\]
\[
= R(k + 1)
\]
Another proof

Theorem
For a real number \( q > 0 \) and \( q \neq 1 \)

\[
\sum_{i=0}^{n-1} q^i = 1 + q + \cdots + q^{n-2} + q^{n-1} = \frac{q^n - 1}{q - 1}
\]

Proof

\[
(q - 1) \sum_{i=0}^{n-1} q^i = q \sum_{i=0}^{n-1} q^i - \sum_{i=0}^{n-1} q^i
\]

\[
= (q + q^2 + \cdots + q^{n-1} + q^n) - (1 + q + \cdots + q^{n-2} + q^{n-1})
\]

\[
= q^n - 1
\]
Corollary

For a real number \( q > 0 \) and \( q \neq 1 \)

\[
\sum_{i=1}^{n-1} q^i = q + \cdots + q^{n-2} + q^{n-1} = \frac{q^n - q}{q - 1}
\]

Proof

\[
\begin{align*}
\sum_{i=1}^{n-1} q^i &= \sum_{i=0}^{n-1} q^i - 1 \\
&= \frac{q^n - 1}{q - 1} - \frac{q - 1}{q - 1} \\
&= \frac{q^n - q}{q - 1}
\end{align*}
\]
Geometric Progressions with $q = 2$

Identity

$$\sum_{i=0}^{n-1} 2^i = 1 + 2 + 4 + \cdots + 2^{n-1}$$

$$= \frac{2^n - 1}{2 - 1} = 2^n - 1$$

Small $n$

$$1 = 1 = 2^1 - 1$$
$$1 + 2 = 3 = 2^2 - 1$$
$$1 + 2 + 4 = 7 = 2^3 - 1$$
$$1 + 2 + 4 + 8 = 15 = 2^4 - 1$$
$$1 + 2 + 4 + 8 + 16 = 31 = 2^5 - 1$$
$$1 + 2 + 4 + 8 + 16 + 32 = 63 = 2^6 - 1$$
Geometric Progressions with $q = 3$

Identity

$$\sum_{i=0}^{n-1} 3^i = 1 + 3 + 9 + \cdots + 3^{n-1}$$

$$= \frac{3^n - 1}{3 - 1} = \frac{3^n - 1}{2}$$

Small $n$

$$1 = 1 = \frac{3^1 - 1}{2} = \frac{3 - 1}{2}$$

$$1 + 3 = 4 = \frac{3^2 - 1}{2} = \frac{9 - 1}{2}$$

$$1 + 3 + 9 = 13 = \frac{3^3 - 1}{2} = \frac{27 - 1}{2}$$

$$1 + 3 + 9 + 27 = 40 = \frac{3^4 - 1}{2} = \frac{81 - 1}{2}$$

$$1 + 3 + 9 + 27 + 81 = 121 = \frac{3^5 - 1}{2} = \frac{243 - 1}{2}$$
Geometric Progressions Visual Proofs

$q = 3$
- https://www.youtube.com/watch?v=9IAm75UY2U8

$q = 4$
- https://www.youtube.com/watch?v=yTpzDEDP090&list=PLZh9gzIvXQUsgw8W5TUVDtF0q4JEJ3iaw

$q = 7$
- https://www.youtube.com/watch?v=lwIdJxSfUz4&list=PLZh9gzIvXQUsgw8W5TUVDtF0q4JEJ3iaw

$q = 8$
- https://www.youtube.com/watch?v=vcO5pa7iZOU

$q = 9$
- https://www.youtube.com/watch?v=Ch7GFdsc9pQ
Geometric Progressions for Large $q$

**First approximation: large $q$**

\[
\sum_{i=0}^{n-1} q^i = \frac{q^n - 1}{q - 1} \\
= \frac{q^n}{q - 1} - \frac{1}{q - 1} \\
\approx \frac{q^n}{q - 1}
\]

**Second approximation: very large $q$**

\[
\sum_{i=0}^{n-1} q^i \approx \frac{q^n}{q - 1} \approx \frac{q^n}{q} = q^{n-1}
\]
Another Version of the Identity for the Sum

**Theorem**
For a real number \( q > 0 \) and \( q \neq 1 \)

\[
\sum_{i=0}^{n-1} q^i = 1 + q + \cdots + q^{n-1} = \frac{1 - q^n}{1 - q}
\]

**Proof**

\[
\sum_{i=0}^{n-1} q^i = \frac{q^n - 1}{q - 1} = \frac{(-1)(q^n - 1)}{(-1)(q - 1)} = \frac{1 - q^n}{1 - q}
\]
Another Version of the Identity for the Sum

**Corollary**

For a real number $q > 0$ and $q \neq 1$

$$\sum_{i=1}^{n-1} q^i = q + q^2 + \cdots + q^{n-1} = \frac{q - q^n}{1 - q}$$

**Proof**

$$\sum_{i=1}^{n-1} q^i = \frac{q^n - q}{q - 1}$$

$$= \frac{(-1)(q^n - q)}{(-1)(q - 1)}$$

$$= \frac{q - q^n}{1 - q}$$
Which Identity To Use?

The two identities

\[
\sum_{i=0}^{n-1} q^i = 1 + q + \cdots + q^{n-1} = \frac{q^n - 1}{q - 1} \tag{1}
\]

\[
\sum_{i=0}^{n-1} q^i = 1 + q + \cdots + q^{n-1} = \frac{1 - q^n}{1 - q} \tag{2}
\]

Avoid negative numbers

- Use the first when \( q > 1 \) so both the numerator and the denominator are positive
- Use the second when \( q < 1 \) so both the numerator and the denominator are positive
Geometric Progressions with \( q = \frac{1}{2} \)

Identity

\[
\sum_{i=0}^{n-1} \left( \frac{1}{2} \right)^i = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}
\]

\[
= \frac{1 - \left( \frac{1}{2} \right)^n}{1 - \frac{1}{2}}
\]

\[
= 2 \left( 1 - \left( \frac{1}{2} \right)^n \right) = 2 - \frac{1}{2^{n-1}}
\]

Small \( n \)

\[
\begin{align*}
1 &= 2 - \frac{1}{1} = 1 \\
1 + \frac{1}{2} &= 2 - \frac{1}{2} = \frac{3}{2} \\
1 + \frac{1}{2} + \frac{1}{4} &= 2 - \frac{1}{4} = \frac{7}{4} \\
1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} &= 2 - \frac{1}{8} = \frac{15}{8}
\end{align*}
\]
Geometric Progressions with $q = \frac{2}{3}$

Identity

$$\sum_{i=0}^{n-1} \left( \frac{2}{3} \right)^i = 1 + \frac{2}{3} + \frac{4}{9} + \cdots + \frac{2^{n-1}}{3^{n-1}}$$

$$= \frac{1 - \left( \frac{2}{3} \right)^n}{1 - \frac{2}{3}}$$

$$= 3 \left( 1 - \left( \frac{2}{3} \right)^n \right)$$

$$= 3 - \frac{2^n}{3^{n-1}}$$
Geometric Progressions with $q = \frac{k-1}{k}$

Identity

\[
\sum_{i=0}^{n-1} \left(\frac{k-1}{k}\right)^i = 1 + \frac{k-1}{k} + \frac{(k-1)^2}{k^2} + \cdots + \frac{(k-1)^{n-1}}{k^{n-1}}
\]

\[
= \frac{1 - \left(\frac{k-1}{k}\right)^n}{1 - \frac{k-1}{k}}
\]

\[
= k \left( 1 - \left(\frac{k-1}{k}\right)^n \right)
\]

\[
= k - \frac{(k-1)^n}{k^{n-1}}
\]
Geometric Progressions with $q = \frac{1}{2}$

Identity

$$\sum_{i=1}^{n-1} \left( \frac{1}{2} \right)^i = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}$$

$$= \frac{\frac{1}{2} - \left( \frac{1}{2} \right)^n}{1 - \frac{1}{2}}$$

$$= 2 \left( \frac{1}{2} - \left( \frac{1}{2} \right)^n \right) = 1 - \frac{1}{2^{n-1}}$$

Small numbers

\[
\begin{align*}
\frac{1}{2} &= 1 - \frac{1}{2} = \frac{1}{2} \\
\frac{1}{2} + \frac{1}{4} &= 1 - \frac{1}{4} = \frac{3}{4} \\
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} &= 1 - \frac{1}{8} = \frac{7}{8} \\
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} &= 1 - \frac{1}{16} = \frac{15}{16}
\end{align*}
\]
Geometric Progressions with $q = \frac{1}{3}$

**Identity**

$$\sum_{i=1}^{n-1} \left(\frac{1}{3}\right)^i = \frac{1}{3} + \frac{1}{9} + \cdots + \frac{1}{3^{n-1}}$$

$$= \frac{1}{3} - \left(\frac{1}{3}\right)^n$$

$$= \frac{3}{2} \left(\frac{1}{3} - \left(\frac{1}{3}\right)^n\right)$$

$$= \frac{1}{2} - \frac{1}{3^{n-1}}$$
Geometric Progressions with $q = \frac{1}{k}$

Identity

$$\sum_{i=1}^{n-1} \left(\frac{1}{k}\right)^i = \frac{1}{k} + \frac{1}{k^2} + \cdots + \frac{1}{k^{n-1}}$$

$$= \frac{\frac{1}{k} - \left(\frac{1}{k}\right)^n}{1 - \frac{1}{k}}$$

$$= \frac{k}{k-1} \left(\frac{1}{k} - \left(\frac{1}{k}\right)^n\right)$$

$$= \frac{1}{k - 1} - \frac{1}{k^{n-1}}$$
Infinite Geometric Progressions with $0 < q < 1$

**Theorem**

\[
\sum_{i=0}^{\infty} q^i = 1 + q + q^2 + \cdots = \frac{1}{1 - q}
\]

\[
\sum_{i=1}^{\infty} q^i = q + q^2 + q^3 + \cdots = \frac{q}{1 - q}
\]

**Proof sketch**

- $q^n \to 0$ when $n \to \infty$ and therefore $q^\infty = 0$

\[
\sum_{i=0}^{\infty} q^i = \frac{1 - q^\infty}{1 - q} = \frac{1 - 0}{1 - q} = \frac{1}{1 - q}
\]

\[
\sum_{i=1}^{\infty} q^i = \frac{q - q^\infty}{1 - q} = \frac{q - 0}{1 - q} = \frac{q}{1 - q}
\]
Another proof

Theorem

For a real number $0 < q < 1$

$$\sum_{i=0}^{\infty} q^i = 1 + q + q^2 + \cdots = \frac{1}{1 - q}$$

Proof

$$(1 - q) \sum_{i=0}^{\infty} q^i = \sum_{i=0}^{\infty} q^i - q \sum_{i=0}^{\infty} q^i$$

$$= (1 + q + q^2 + \cdots) - (q + q^2 + q^3 + \cdots)$$

$$= 1$$

Application

https://www.youtube.com/watch?v=3cNdM7W0VlQ
Infinite Geometric Progressions with $q = \frac{k-1}{k}$

**Small $k$**

- $\sum_{i=0}^{\infty} \left( \frac{1}{2} \right)^i = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1-\frac{1}{2}} = 2$
- $\sum_{i=0}^{\infty} \left( \frac{2}{3} \right)^i = 1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots = \frac{1}{1-\frac{2}{3}} = 3$
- $\sum_{i=0}^{\infty} \left( \frac{3}{4} \right)^i = 1 + \frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \cdots = \frac{1}{1-\frac{3}{4}} = 4$

**The general case**

$$\sum_{i=0}^{\infty} \left( \frac{k-1}{k} \right)^i = 1 + \frac{k-1}{k} + \frac{(k-1)^2}{k^2} + \frac{(k-1)^3}{k^3} + \cdots$$

$$= \frac{1}{1 - \frac{k-1}{k}} = \frac{1}{\frac{1}{k}} = k$$
Infinite Geometric Progressions with \( q = \frac{1}{k} \)

**Small \( k \)**

1. \( \sum_{i=1}^{\infty} \left( \frac{1}{2} \right)^i = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1 \)
2. \( \sum_{i=1}^{\infty} \left( \frac{1}{3} \right)^i = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots = \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{1}{2} \)
3. \( \sum_{i=1}^{\infty} \left( \frac{1}{4} \right)^i = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3} \)
4. \( \sum_{i=1}^{\infty} \left( \frac{1}{5} \right)^i = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots = \frac{\frac{1}{5}}{1 - \frac{1}{5}} = \frac{1}{4} \)
5. \( \sum_{i=1}^{\infty} \left( \frac{1}{6} \right)^i = \frac{1}{6} + \frac{1}{36} + \frac{1}{216} + \cdots = \frac{\frac{1}{6}}{1 - \frac{1}{6}} = \frac{1}{5} \)

**The general case**

\( \sum_{i=1}^{\infty} \left( \frac{1}{k} \right)^i = \frac{1}{k} + \frac{1}{k^2} + \frac{1}{k^3} + \cdots = \frac{\frac{1}{k}}{1 - \frac{1}{k}} = \frac{1}{k-1} \)
Infinite Geometric Progressions with $q = \frac{1}{k}$

**Visual Proofs**

- $q = 1/2$: https://www.youtube.com/watch?v=saJJGN1sfn8
- $q = 1/3$:
  - https://www.youtube.com/watch?v=vfEDDI3vfHU
  - https://www.youtube.com/watch?v=RmTzmnRkqss
- $q = 1/4$: https://www.youtube.com/watch?v=8ilxj5ORwUw
- $q = 1/5$:
  - https://www.youtube.com/watch?v=yp7afEXYeC4
  - https://www.youtube.com/watch?v=IguRXWNwrn8&t=47s
- $q = 1/7$: https://www.youtube.com/watch?v=6wgCoIzsaA8
- $q = 1/9$: https://www.youtube.com/watch?v=C4t_ps3VKvI
- $q = 1/2, 1/3, \ldots, 1/9$: https://www.youtube.com/watch?v=JteQEN1XPyC
Infinite Geometric Progressions with $q = \frac{k}{2k+1}$

**Small $k$**

- $\sum_{i=1}^{\infty} \left(\frac{1}{3}\right)^i = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots = \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{1}{2}$
- $\sum_{i=1}^{\infty} \left(\frac{2}{5}\right)^i = \frac{2}{5} + \frac{4}{25} + \frac{8}{125} + \cdots = \frac{\frac{2}{5}}{1 - \frac{2}{5}} = \frac{2}{3}$
- $\sum_{i=1}^{\infty} \left(\frac{3}{7}\right)^i = \frac{3}{7} + \frac{9}{49} + \frac{27}{243} + \cdots = \frac{\frac{3}{7}}{1 - \frac{3}{7}} = \frac{3}{4}$

**The general case**

- $\sum_{i=1}^{\infty} \left(\frac{k}{2k+1}\right)^i = \frac{k}{2k+1} + \frac{k^2}{(2k+1)^2} + \frac{k^3}{(2k+1)^3} + \cdots = \frac{\frac{k}{2k+1}}{1 - \frac{k}{2k+1}} = \frac{k}{k+1}$

**A visual proof**

- $q = \frac{4}{9}$: [https://www.youtube.com/watch?v=woKvh51KPl4](https://www.youtube.com/watch?v=woKvh51KPl4)
Sum of Powers of First $n$ Integers

Small exponents

\[\sum_{i=1}^{n} i^0 = 1 + 1 + \cdots + 1 = n \approx \frac{1}{1} n^1\]

\[\sum_{i=1}^{n} i^1 = 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \approx \frac{1}{2} n^2\]

\[\sum_{i=1}^{n} i^2 = 1 + 4 + 9 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \approx \frac{1}{3} n^3\]

\[\sum_{i=1}^{n} i^3 = 1 + 8 + 27 + \cdots + n^3 = \frac{n^2(n+1)^2}{4} \approx \frac{1}{4} n^4\]

\[\vdots\]

\[\sum_{i=1}^{n} i^k = 1^k + 2^k + \cdots + n^k \approx \frac{1}{k+1} n^{k+1}\]
Sum of First \( n \) Squares

Identity

\[
\sum_{i=1}^{n} i^2 = 1 + 4 + 9 + \cdots + n^2 = \frac{n(n + 1)(2n + 1)}{6}
\]

Visual proofs

- Proof 1: [https://www.youtube.com/watch?v=-tJhH_k2LaM](https://www.youtube.com/watch?v=-tJhH_k2LaM)
- Proof 2: [https://www.youtube.com/watch?v=UqVmocdLFGc](https://www.youtube.com/watch?v=UqVmocdLFGc)
- Proof 3: [https://www.youtube.com/watch?v=WidzHiUFWNA](https://www.youtube.com/watch?v=WidzHiUFWNA)

Proof by induction

- [https://www.youtube.com/watch?v=OI-nSvpZTpE](https://www.youtube.com/watch?v=OI-nSvpZTpE)

Another identity with double summation

\[
\sum_{i=1}^{n} i^2 = \sum_{i=1}^{n} \sum_{j=i}^{n} j
\]

A visual proof: [https://www.youtube.com/watch?v=Q-frL0Oot2m4](https://www.youtube.com/watch?v=Q-frL0Oot2m4)
Sum of First $n$ Squares

Correctness for Small $n$

<table>
<thead>
<tr>
<th>$n$</th>
<th>Sum</th>
<th>$n(n+1)(2n+1)/6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$1$</td>
<td>$1 = 1 = \frac{1\cdot2\cdot3}{6} = \frac{6}{6}$</td>
</tr>
<tr>
<td>$2$</td>
<td>$1+4$</td>
<td>$5 = 2\cdot3\cdot5 = \frac{30}{6}$</td>
</tr>
<tr>
<td>$3$</td>
<td>$1+4+9$</td>
<td>$14 = 3\cdot4\cdot7 = \frac{84}{6}$</td>
</tr>
<tr>
<td>$4$</td>
<td>$1+4+9+16$</td>
<td>$30 = 4\cdot5\cdot9 = \frac{180}{6}$</td>
</tr>
<tr>
<td>$5$</td>
<td>$1+4+9+16+25$</td>
<td>$55 = 5\cdot6\cdot11 = \frac{330}{6}$</td>
</tr>
<tr>
<td>$6$</td>
<td>$1+4+9+16+25+36$</td>
<td>$91 = 6\cdot7\cdot13 = \frac{546}{6}$</td>
</tr>
<tr>
<td>$7$</td>
<td>$1+4+9+16+25+36+49$</td>
<td>$140 = 7\cdot8\cdot15 = \frac{840}{6}$</td>
</tr>
</tbody>
</table>
Proof By Induction

Notations
- \( L(n) = 1 + 4 + 9 + \cdots + (n - 1)^2 + n^2 \)
- \( R(n) = \frac{n(n+1)(2n+1)}{6} \)

The induction base: \( n = 1 \)
- \( L(1) = R(1) \), because \( L(1) = 1^2 = 1 \) and \( R(1) = \frac{1\cdot2\cdot3}{6} = 1 \)

The induction hypothesis: \( L(k) = R(k) \) for \( k \geq 1 \)
- \( \sum_{i=1}^{k} i^2 = 1 + 4 + 9 + \cdots + (k - 1)^2 + k^2 = \frac{k(k + 1)(2k + 1)}{6} \)
Proof By Induction

The inductive step: \( L(k + 1) = R(k + 1) \) for \( k \geq 1 \)

\[
\begin{align*}
L(k + 1) &= 1 + 4 + 9 + \cdots + k^2 + (k + 1)^2 \\
&= L(k) + (k + 1)^2 \\
&= R(k) + (k + 1)^2 \\
&= \frac{k(k + 1)(2k + 1)}{6} + (k + 1)^2 \\
&= \frac{(2k^3 + 3k^2 + k) + (6k^2 + 12k + 6)}{6} \\
&= \frac{2k^3 + 9k^2 + 13k + 6}{6} \\
&= \frac{(k + 1)(k + 2)(2k + 3)}{6} \\
&= \frac{(k + 1)((k + 1) + 1)(2(k + 1) + 1)}{6} \\
&= R(k + 1)
\end{align*}
\]
Proof By Induction

The inductive step: \( L(k + 1) = R(k + 1) \) for \( k \geq 1 \)

\[
L(k + 1) = 1 + 4 + 9 + \cdots + k^2 + (k + 1)^2 \\
= L(k) + (k + 1)^2 \\
= R(k) + (k + 1)^2 \\
= \frac{k(k + 1)(2k + 1)}{6} + (k + 1)^2 \\
= \frac{(k^2 + k)(2k + 1)}{6} + \frac{6(k + 1)^2}{6} \\
= \frac{(2k^3 + 3k^2 + k) + (6k^2 + 12k + 6)}{6} \\
= \frac{2k^3 + 9k^2 + 13k + 6}{6}
\]

\[
R(k + 1) = \frac{(k + 1)((k + 1) + 1)(2(k + 1) + 1)}{6} \\
= \frac{(k + 1)(k + 2)(2k + 3)}{6} \\
= \frac{(k^2 + 3k + 2)(2k + 3)}{6} \\
= \frac{2k^3 + 9k^2 + 13k + 6}{6}
\]
Sum of First $n$ Cubes

Identity

$$\sum_{i=1}^{n} i^3 = 1 + 8 + 27 + \cdots + (n - 1)^3 + n^3$$

$$= \frac{n^2(n + 1)^2}{4}$$

$$= \left( \frac{n(n + 1)}{2} \right)^2$$

$$= (1 + 2 + 3 + \cdots + (n - 1) + n)^2$$

Visual Proofs

- Proof 1: https://www.youtube.com/watch?v=YQLicI8R4Gs
- Proof 2: https://www.youtube.com/watch?v=Ye9OPNqV9FA
- Proof 3: https://www.youtube.com/watch?v=Nx0cT.VKQR0
## Sum of First $n$ Cubes

### Correctness for Small $n$

<table>
<thead>
<tr>
<th>$1 + 8 + 27 + \ldots$</th>
<th>Value</th>
<th>Formula</th>
<th>Simplified</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$1$</td>
<td>$1^2 \cdot 2^2$</td>
<td>$\frac{4}{4}$</td>
</tr>
<tr>
<td>$1 + 8$</td>
<td>$9$</td>
<td>$2^2 \cdot 3^2$</td>
<td>$\frac{36}{4}$</td>
</tr>
<tr>
<td>$1 + 8 + 27$</td>
<td>$36$</td>
<td>$3^2 \cdot 4^2$</td>
<td>$\frac{144}{4}$</td>
</tr>
<tr>
<td>$1 + 8 + 27 + 64$</td>
<td>$100$</td>
<td>$4^2 \cdot 5^2$</td>
<td>$\frac{400}{4}$</td>
</tr>
<tr>
<td>$1 + 8 + 27 + 64 + 125$</td>
<td>$225$</td>
<td>$5^2 \cdot 6^2$</td>
<td>$\frac{900}{4}$</td>
</tr>
<tr>
<td>$1 + 8 + 27 + 64 + 125 + 216$</td>
<td>$441$</td>
<td>$6^2 \cdot 7^2$</td>
<td>$\frac{1764}{4}$</td>
</tr>
<tr>
<td>$1 + 8 + 27 + 64 + 125 + 216 + 343$</td>
<td>$784$</td>
<td>$7^2 \cdot 8^2$</td>
<td>$\frac{3136}{4}$</td>
</tr>
</tbody>
</table>
Proof By Induction

Notations

- \( L(n) = 1 + 8 + 27 + \cdots + (n - 1)^3 + n^3 \)
- \( R(n) = \frac{n^2(n+1)^2}{4} \)

The induction base: \( n = 1 \)

- \( L(1) = R(1) \), because \( L(1) = 1^3 = 1 \) and \( R(1) = \frac{12 \cdot 2^2}{4} = 1 \)

The induction hypothesis: \( L(k) = R(k) \) for \( k \geq 1 \)

\[
\sum_{i=1}^{k} i^3 = 1 + 8 + 27 + \cdots + (k - 1)^3 + k^3 = \frac{k^2(k + 1)^2}{4}
\]
Proof By Induction

The inductive step: \( L(k + 1) = R(k + 1) \) for \( k \geq 1 \)

\[
L(k + 1) = 1 + 8 + 27 + \cdots + k^3 + (k + 1)^3
= L(k) + (k + 1)^3
= R(k) + (k + 1)^3
= \frac{k^2(k + 1)^2}{4} + (k + 1)^3
= \frac{k^2(k + 1)^2 + 4(k + 1)^3}{4}
= \frac{(k + 1)^2(k^2 + 4k + 4)}{4}
= \frac{(k + 1)^2(k + 2)^2}{4}
= R(k + 1)
\]
Another Identity

Identity

\[
\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}
\]

Correctness for Small \( n \)

\[
\frac{1}{2} = \frac{1}{2} = 1 - \frac{1}{2}
\]

\[
\frac{1}{3} + \frac{1}{4} = \frac{7}{12} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}
\]

\[
\frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{37}{60} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6}
\]

\[
\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = \frac{533}{840} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8}
\]
Proof By Induction

Notations

- \( L(n) = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \)
- \( R(n) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} \)

The induction base: \( n = 1 \)

- \( L(1) = R(1) \), because \( L(1) = \frac{1}{2} \) and \( R(1) = 1 - \frac{1}{2} = \frac{1}{2} \)

The induction hypothesis: \( L(k) = R(k) \) for \( k \geq 1 \)

\[
\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{2k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2k-1} - \frac{1}{2k}
\]
Proof By Induction

The inductive step: \( L(k + 1) = R(k + 1) \) for \( k \geq 1 \)

\[
\begin{align*}
L(k + 1) &= \frac{1}{k + 2} + \frac{1}{k + 3} + \cdots + \frac{1}{2k} + \frac{1}{2k + 1} + \frac{1}{2k + 2} \\
&= \frac{1}{k + 1} + \frac{1}{k + 2} + \cdots + \frac{1}{2k} + \frac{1}{2k + 1} + \frac{1}{2k + 2} - \frac{1}{k + 1} \\
&= L(k) + \frac{1}{2k + 1} + \left( \frac{1}{2k + 2} - \frac{1}{k + 1} \right) \\
&= L(k) + \frac{1}{2k + 1} - \frac{1}{2k + 2} \\
&= R(k) + \frac{1}{2k + 1} - \frac{1}{2k + 2} \\
&= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2k - 1} - \frac{1}{2k} + \frac{1}{2k + 1} - \frac{1}{2k + 2} \\
&= R(k + 1)
\end{align*}
\]
Prime Factorization

**Theorem**

Every positive integer \( n \geq 2 \) is a power of a prime number or the product of powers of prime numbers

**Proof by Induction**

- **Induction base:** \( 2 = 2^1 \) is a power of a prime
- **Induction hypothesis:** Assume every positive integer less than \( n \) is a prime number or a product of powers of prime numbers
- **Inductive step:**
  - If \( n \) is a prime, then \( n = n^1 \) is a power of a prime
  - Otherwise, \( n = m \cdot h \) is a product of two numbers \( m < n \) and \( h < n \)
  - By the induction hypothesis, both \( m \) and \( h \) are power of prime numbers or products of prime numbers
  - Therefore, \( n = m \cdot h \) is also a power of a prime number or a product of powers of prime numbers
Prime Factorization

Example I
- \( 90 = 15 \cdot 6 = (3 \cdot 5)(2 \cdot 3) \)
- Therefore by induction, \( 90 = 2 \cdot 3^2 \cdot 5 \)

Example II
- \( 216 = 12 \cdot 18 = (2^2 \cdot 3)(2 \cdot 3^2) \)
- Therefore by induction, \( 216 = 2^3 \cdot 3^3 \)

Example III
- \( 128 = 8 \cdot 16 = 2^3 \cdot 2^4 \)
- Therefore by induction, \( 128 = 2^7 \)

Online resource
- Strong induction: prime factorization and another example
  https://www.youtube.com/watch?v=g9YSizeBwgo&t=317s
A Chocolate Bar Problem

Problem

- A chocolate bar consisting of $n \geq 1$ unit squares is arranged as an $m \times h$ rectangular grid ($n = m \cdot h$)
- The goal is to split the bar into $n$ individual unit squares by breaking along the lines
- It is not allowed to break more than one rectangular piece at a time (e.g., by piling them together).
- What is the required number of breaks?

Online resource

- https://www.youtube.com/watch?v=yftf3fs9k6s
A Chocolate Bar Problem

Claim
For $0 \leq k \leq n - 1$, after $k$ breaks, there are $k + 1$ pieces

Proof by induction sketch
- By induction on the number of breaks
- After $k = 0$ breaks there is one piece and indeed $1 = 0 + 1$
- Before the $k$th break, by the induction hypothesis, there were $k = (k - 1) + 1$ pieces
- After the $k$th break there are $k + 1$ pieces because the break replaces one of the pieces with two pieces

Corollary
- After $n - 1$ breaks there are $n$ pieces. That is, the bar was split into $n$ individuals unit squares
More Examples

Three problems

- L-shape tiles of size 3 can tile any square of size $2^n \times 2^n$ small squares with any missing square
- Number of steps needed to solve the tower of Hanoi problem
- Any partition of the circle with chords can be face-colored with two colors

Online resource

- https://www.youtube.com/watch?v=5Hn8vUE3cBQ

Remark

- All proofs imply an algorithm!
A False Divisibility Claim

Claim

- \( n^3 - n + 1 \) is divisible by 3

Wrong for small values of \( n \)

\[
\begin{align*}
1^3 - 1 + 1 &= 1 &= 3 \cdot 0 + 1 \\
2^3 - 2 + 1 &= 7 &= 3 \cdot 2 + 1 \\
3^3 - 3 + 1 &= 25 &= 3 \cdot 8 + 1 \\
4^3 - 4 + 1 &= 61 &= 3 \cdot 20 + 1
\end{align*}
\]
Proof By Induction

The induction base
- Skip the base case!

The induction hypothesis for \( k \)
- Assume that \( k^3 - k + 1 \) is divisible by 3

The inductive step for \( k + 1 \)

\[
(k + 1)^3 - (k + 1) + 1 = k^3 + 3k^2 + 3k + 1 - k - 1 + 1
\]
\[
= (k^3 - k + 1) + (3k^2 + 3k)
\]
\[
= 3q + 3(k^2 + k) \quad (\text{\asterisk the induction hypothesis \asterisk})
\]
\[
= 3(q + k^2 + k) \quad (\text{\asterisk Q.E.D. \asterisk})
\]
**Correct Claim**

**Theorem**
- \( n^3 - n \) is divisible by 6

**Small values of \( n \)**

\[
\begin{align*}
1^3 - 1 &= 0 = 6 \cdot 0 \\
2^3 - 2 &= 6 = 6 \cdot 1 \\
3^3 - 3 &= 24 = 6 \cdot 4 \\
4^3 - 4 &= 60 = 6 \cdot 10
\end{align*}
\]

**Proof**
- \( n^3 - n = n(n^2 - 1) = (n - 1)n(n + 1) \)
- That is, \( n^3 - n \) is a product of three consecutive integers
- One of them must be divisible by 3
- One (could be the same integer) must be even
- Therefore, the product of the three integers must be divisible by 6
**Proof by induction for** \( n \geq 0 \)

- Clearly, if \( n = 0 \), then \( 6n = 0 \)
- Let \( n > 0 \) and assume that \( 6k = 0 \) for all \( 0 \leq k < n \)
- Let \( n = h + m \) for integers \( 0 \leq h < n \) and \( 0 \leq m < n \)
- By the **strong** induction hypothesis, \( 6h = 0 \) and \( 6m = 0 \).
- Therefore \( 6n = 6(h + m) = 6h + 6m = 0 + 0 = 0 \)
- Q.E.D.

**Where is the Error?**

- The proof fails for \( n = 1 \)
- \( 1 \) cannot be expressed as a sum of two non-negative integers that are smaller than 1
All Horses in the World are of the Same Color

**Proof by induction on the number of horses**
- The base of the induction is that if there is one horse, then it is trivially the same color as itself.
- Suppose that there are \( n \) horses, numbered 1 through \( n \).
- By the induction hypothesis, the \( n - 1 \) horses 1 through \( n - 1 \) are all of the same color.
- Assume this color is black. In particular, horse 2 is black.
- This means that the \( n - 1 \) horses 2 through \( n \) must be black by the induction hypothesis.
- Therefore, all of the horses 1 through \( n \) are of the same color.

**Where is the Error?**
- Proof fails for \( n = 2 \) in which horse 2 may be of a different color.

**Online resources**
- [https://www.youtube.com/watch?v=sCUg5DNCETI](https://www.youtube.com/watch?v=sCUg5DNCETI)
- [https://en.wikipedia.org/wiki/All_horses_are_the_same_color](https://en.wikipedia.org/wiki/All_horses_are_the_same_color)