Discrete Structures: Induction

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The Principle of Induction

The principle

Let $P_n$ be a statement about all positive integers $n = 1, 2, 3, \ldots$

If the following hold:

- **Induction base:** $P_1$ is true
- **Inductive step:** For all integers $n > 1$, if $P_{n-1}$ is true then $P_n$ is true

Then $P_n$ is true for all integers $n \geq 1$

The assumption “$P_{n-1}$ is true” is the induction hypothesis

Cartoons


Introductory Video

- https://www.youtube.com/watch?v=bePpPFos0kE
Why Induction Works?

“Justification” with the Well-Ordering Principle

1. Assume that there exists $h \geq 1$ such that $P_h$ is false.
2. Let $S$ be the set of all integers $k \geq 1$ for which $P_k$ is false.
   - $S$ is a non-empty set that can contain infinite number of integers.
3. Let $n$ be the minimum integer in $S$.
   - The Well-Ordering Principle.
4. $n > 1$ since by the induction base $P_1$ is true.
5. $P_{n-1}$ is true and $P_n$ is false by the minimality of $n$.
6. A contradiction to the inductive step.
Generalizations

Other Induction Bases
- For any \( m \geq 0 \) the **induction base** could be \( P_m \) instead of \( P_1 \)
- In this case, the induction is applied to \( n = m, m + 1, \ldots \)

Strong Induction
- The **induction base** is that \( P_1, \ldots, P_m \) are true for some \( m \geq 1 \)
- The **induction hypothesis** is that \( P_1, P_2, \ldots, P_{n-1} \) are true
- The **inductive step** is that \( P_n \) is implied by a non-empty subset of statements from the set \( \{P_1, P_2, \ldots, P_{n-1}\} \)
The induction variable

- The **inductive step** could be that \( P_{n+1} \) is implied by \( P_n \) and then \( P_n \) is the **induction hypothesis**

- The **inductive step** could be that \( P_{k+1} \) is implied by \( P_k \) or that \( P_k \) is implied by \( P_{k-1} \) although the statement is \( P_n \)
Some Online Resources

- **Introduction in 8 minutes (from 11:25 to 20:04):**
  
  https://youtu.be/0UgID8C9RvE?list=PLZzHxk_TPOStgPtqRZ6KzmkUQBQ8TSWVX

- **Introduction in 15 minutes:**
  
  https://www.youtube.com/watch?v=ruBnYcLzVLU

- **Sum of the first \( n \) integers in 7 minutes:**
  
  https://www.youtube.com/watch?v=dMn5w4_ztSw&feature=youtu.be

- **Sum of the first \( n \) odd integers in 10 minutes:**
  
  https://www.youtube.com/watch?v=twA6vZgX_U4

- **Sum of first \( n \) integers of the form \( 5k - 1 \) in 6 minutes:**
  
  https://www.youtube.com/watch?v=IFqna5F0kW8

- **\( 6^n + 4 \) is divisible by 5 in 6 minutes:**
  
  https://youtu.be/MpjKLf7lfRA
Sum Manipulation Rules

For integers $\ell \leq u$

- $\sum_{i=\ell}^u c a_i = c \sum_{i=\ell}^u a_i$
- $\sum_{i=\ell}^u (a_i + b_i) = \sum_{i=\ell}^u a_i + \sum_{i=\ell}^u b_i$
- $\sum_{i=\ell}^u (a_i - b_i) = \sum_{i=\ell}^u a_i - \sum_{i=\ell}^u b_i$
- $(\sum_{i=\ell}^u a_i) \times (\sum_{j=\ell}^u b_j) = \sum_{\ell \leq i, j \leq u} (a_i \cdot b_j)$

For integers $\ell < u$

- $\sum_{i=\ell}^u a_i = \sum_{i=\ell}^m a_i + \sum_{i=m+1}^u a_i$ for $\ell \leq m \leq u$
- $\sum_{i=\ell+1}^u (a_i - a_{i-1}) = a_u - a_{\ell}$
Sum of First $n$ Positive Integers

**An identity**

\[ \sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2} = \binom{n+1}{2} \]

**A proof without words**

https://i.stack.imgur.com/yerzW.png

**An equivalent identity**

\[ \sum_{i=1}^{n-1} i = 1 + 2 + \cdots + n - 1 = \frac{(n-1)n}{2} = \binom{n}{2} \]

**A proof with animation**

https://mathoverflow.net/questions/8846/proofs-without-words?page=1&tab=votes#tab-top
Correctness for Small $n$

\[
1 = 1 = \frac{1 \cdot 2}{2}
\]
\[
1 + 2 = 3 = \frac{2 \cdot 3}{2}
\]
\[
1 + 2 + 3 = 6 = \frac{3 \cdot 4}{2}
\]
\[
1 + 2 + 3 + 4 = 10 = \frac{4 \cdot 5}{2}
\]
\[
1 + 2 + 3 + 4 + 5 = 15 = \frac{5 \cdot 6}{2}
\]
\[
1 + 2 + 3 + 4 + 5 + 6 = 21 = \frac{6 \cdot 7}{2}
\]
\[
1 + 2 + 3 + 4 + 5 + 6 + 7 = 28 = \frac{7 \cdot 8}{2}
\]
\[
1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36 = \frac{8 \cdot 9}{2}
\]
Proof By Induction

Notations

- \( L(n) = \sum_{i=1}^{n} i \)
- \( R(n) = \frac{n(n+1)}{2} \)

The induction base: \( n = 1 \)

- \( L(1) = R(1) \), because \( L(1) = 1 \) and \( R(1) = \frac{1\cdot2}{2} = 1 \)

The induction hypothesis: \( L(n - 1) = R(n - 1) \) for \( n > 1 \)

\[
\sum_{i=1}^{n-1} i = 1 + 2 + \cdots + (n - 1) = \frac{(n - 1)n}{2}
\]
The inductive step: \( L(n) = R(n) \) for \( n > 1 \)

\[
L(n) = 1 + 2 + \cdots + (n - 1) + n = L(n - 1) + n = R(n - 1) + n = \frac{(n - 1)n}{2} + n
\]

\[
= \frac{(n - 1)n}{2} + \frac{2n}{2} = \frac{(n - 1)n + 2n}{2}
\]

\[
= \frac{(n - 1)n + 2n}{2}
\]

\[
= \frac{(n + 1)n}{2}
\]

\[
= \frac{n(n + 1)}{2} = R(n)
\]
Another Proof

Idea

- Prove that $2L(n) = 2R(n)$ implying $L(n) = R(n)$

Example

$$2L(4) = 2(1 + 2 + 3 + 4)$$
$$= (1 + 2 + 3 + 4) + (4 + 3 + 2 + 1)$$
$$= (1 + 4) + (2 + 3) + (3 + 2) + (4 + 1)$$
$$= 4 \cdot 5$$
$$= 2^\frac{4 \cdot 5}{2}$$
$$= 2R(4)$$
Another Proof

The General Case

\[ 2L(n) = (1 + 2 + \cdots + (n - 1) + n) + (n + (n - 1) + \cdots + 2 + 1) \]
\[ = (1 + n) + (2 + (n - 1)) + \cdots + ((n - 1) + 2) + (n + 1) \]
\[ = (n + 1) + (n + 1) + \cdots + (n + 1) + (n + 1) \]
\[ = n(n + 1) \]
\[ = 2 \frac{n(n + 1)}{2} \]
\[ = 2R(n) \]
Sum of First $n$ Multiples of a Positive Integer $d \geq 2$

Identity

$$\sum_{i=1}^{n} (i \cdot d) = d + 2d + \cdots + nd = \frac{n(n+1)}{2} d$$

Proof by reduction

$$\sum_{i=1}^{n} (i \cdot d) = d + 2d + \cdots + nd$$
$$= (1 + 2 + \cdots + n)d$$
$$= \frac{n(n+1)}{2} d$$
Proof By Induction

Notations

- \( L(n) = \sum_{i=1}^{n} (i \cdot d) \)
- \( R(n) = \frac{n(n+1)}{2} \cdot d \)

The Induction base: \( n = 1 \)

- \( L(1) = R(1) \), because \( L(1) = d \) and \( R(1) = \frac{1\cdot2}{2} d = d \)

The induction hypothesis: \( L(n-1) = R(n-1) \) for \( n > 1 \)

\[
\sum_{i=1}^{n-1} (i \cdot d) = d + 2d + \cdots + (n-1)d = \frac{(n-1)n}{2} d
\]
Proof By Induction

The inductive step: \( L(n) = R(n) \) for \( n > 1 \)

\[
L(n) = d + 2d + \cdots + (n-1)d + nd
= L(n-1) + nd
= R(n-1) + nd
= \frac{(n-1)n}{2}d + nd
= \frac{(n-1)n}{2}d + \frac{2n}{2}d
= \frac{(n-1)n + 2n}{2}d
= \frac{(n+1)n}{2}d
= \frac{n(n+1)}{2}d
= R(n)
\]
Sum of First $n$ Even Positive Integers

Identity

$$\sum_{i=1}^{n} 2i = 2 + 4 + \cdots + 2n = n(n + 1)$$

Proof by reduction

- Replace $d$ with 2 in the identity for the sum of the first $n$ multiples of an integer $d$

$$\sum_{i=1}^{n} (i \cdot 2) = 2 + 4 + \cdots + 2n = \frac{n(n + 1)}{2} \cdot 2 = n(n + 1)$$
Sum of First $n$ Odd Positive Integers

**Identity**

$$\sum_{i=1}^{n} (2i - 1) = 1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

**Correctness for small $n$**

- $1 = 1^2$
- $1 + 3 = 4 = 2^2$
- $1 + 3 + 5 = 9 = 3^2$
- $1 + 3 + 5 + 7 = 16 = 4^2$
- $1 + 3 + 5 + 7 + 9 = 25 = 5^2$
- $1 + 3 + 5 + 7 + 9 + 11 = 36 = 6^2$
- $1 + 3 + 5 + 7 + 9 + 11 + 13 = 49 = 7^2$

**A proof without words**

https://www2.math.upenn.edu/~deturck/probsolv/podpix/grid.gif
Sum of First $n$ Odd Positive Integers

**Identity**

$$\sum_{i=1}^{n} (2i - 1) = 1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

**Proof by reduction**

$$\sum_{i=1}^{n} (2i - 1) = \sum_{i=1}^{n} (2i) - \sum_{i=1}^{n} 1$$

$$= n(n + 1) - n$$

$$= n^2 + n - n$$

$$= n^2$$
Proof By Induction

Notations

- \( L(n) = \sum_{i=1}^{n} (2i - 1) \)
- \( R(n) = n^2 \)

The induction base: \( n = 1 \)

- \( L(1) = R(1) \), because \( L(1) = 1 \) and \( R(1) = 1^2 = 1 \)

The induction hypothesis: \( L(n-1) = R(n-1) \) for \( n > 1 \)

\[
\sum_{i=1}^{n-1} (2i - 1) = 1 + 3 + \cdots + (2n - 3) = (n - 1)^2
\]
The inductive step: \( L(n) = R(n) \) for \( n > 1 \)

\[
L(n) = 1 + 3 + \cdots + (2n - 3) + (2n - 1)
\]
\[
= L(n - 1) + (2n - 1)
\]
\[
= R(n - 1) + (2n - 1)
\]
\[
= (n - 1)^2 + (2n - 1)
\]
\[
= (n^2 - 2n + 1) + (2n - 1)
\]
\[
= n^2
\]
\[
= R(n)
\]
Sum of First $n$ Positive Integers

Even $n$

$$
\sum_{i=1}^{n} i = \sum_{i=1}^{n/2} (2i) + \sum_{i=1}^{n/2} (2i - 1)
$$

$$
= \frac{n}{2} \left( \frac{n}{2} + 1 \right) + \left( \frac{n}{2} \right)^2
$$

$$
= \frac{n^2}{4} + \frac{n}{2} + \frac{n^2}{4}
$$

$$
= \frac{n^2}{2} + \frac{n}{2}
$$

$$
= \frac{n^2 + n}{2}
$$

$$
= \frac{n(n + 1)}{2}
$$
**Sum of First $n$ Positive Integers**

**Odd $n$**

\[
\sum_{i=1}^{n} i = \sum_{i=1}^{(n-1)/2} (2i) + \sum_{i=1}^{(n+1)/2} (2i - 1)
\]

\[
\begin{align*}
\text{Odd } n & = \frac{n-1}{2} \left( \frac{n-1}{2} + 1 \right) + \left( \frac{n+1}{2} \right)^2 \\
& = \left( \frac{n-1}{2} \right) \left( \frac{n+1}{2} \right) + \left( \frac{n+1}{2} \right)^2 \\
& = \left( \frac{n-1}{2} + \frac{n+1}{2} \right) \left( \frac{n+1}{2} \right) \\
& = n \left( \frac{n+1}{2} \right) \\
& = \frac{n(n+1)}{2}
\end{align*}
\]
Arithmetic Progressions

Definition

- A sequence $a_1, a_2, \ldots, a_n$ is an arithmetic progression if there exists $d$ such that $a_i - a_{i-1} = d$ for all $2 \leq i \leq n$

Key observations

- $a_i = a_1 + (i - 1)d$ for $1 \leq i \leq n$
- $a_i = a_n - (n - i)d$ for $1 \leq i \leq n$

Theorem

\[ \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n = \frac{n(a_1 + a_n)}{2} \]

\[ \frac{\sum_{i=1}^{n} a_i}{n} = \frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{a_1 + a_n}{2} \]
Arithmetic Progressions

The theorem in words version I

- The sum of all the $n$ numbers in an arithmetic progression of length $n$ is the average between the first and the last numbers multiplied by $n$.

The theorem in words version II

- The average of all the $n$ numbers in an arithmetic progression of length $n$ is the average between the first and the last numbers.

Remark

- The definition and the theorem work for real numbers and negative numbers. For simplicity we assume that all the numbers in the sequence are positive integers.
**Arithmetic Progressions:** \( a_1 = 5, \ d = 3, \text{ and } n = 11 \)

<table>
<thead>
<tr>
<th>Sequence</th>
<th>5, 8, 11, 14, 17, 20, 23, 26, 29, 32, 35</th>
</tr>
</thead>
<tbody>
<tr>
<td>Observation 1</td>
<td>[ a_4 = a_1 + (4 - 1)d = 5 + 3 \cdot 3 = 5 + 9 = 14 ]</td>
</tr>
<tr>
<td>Observation 2</td>
<td>[ a_7 = a_{11} - (11 - 7)d = 35 - 4 \cdot 3 = 35 - 12 = 23 ]</td>
</tr>
<tr>
<td>Sum of all numbers</td>
<td>[ 5 + 8 + 11 + 14 + 17 + 20 + 23 + 26 + 29 + 32 + 35 = 220 ]</td>
</tr>
<tr>
<td>Average of all numbers</td>
<td>[ 220/11 = 20 ]</td>
</tr>
<tr>
<td>Average of the first and the last numbers</td>
<td>[ (5 + 35)/2 = 40/2 = 20 ]</td>
</tr>
</tbody>
</table>
Arithmetic Progressions

Theorem

\[ \sum_{i=1}^{n} a_i = \frac{n(a_1 + a_n)}{2} \]

Notation

- Define \( S_n = a_1 + a_2 + \cdots + a_n \)

Direct proof

- \( S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_1 + (n-2)d) + (a_1 + (n-1)d) \)
- \( S_n = a_n + (a_n - d) + (a_n - 2d) + \cdots + (a_n - (n-2)d) + (a_n - (n-1)d) \)
- \( 2S_n = n(a_1 + a_n) \)
- \( S_n = \frac{n(a_1 + a_n)}{2} \)
Proof By Induction

Notations

- \( L(n) = \sum_{i=1}^{n} a_i \)
- \( R(n) = \frac{n(a_1 + a_n)}{2} \)

The induction base: \( n = 1 \)

- \( L(1) = R(1) \), because \( L(1) = a_1 \) and \( R(1) = \frac{a_1 + a_1}{2} = a_1 \)

The induction hypothesis: \( L(n - 1) = R(n - 1) \) for \( n > 1 \)

\[
\sum_{i=1}^{n-1} a_i = \frac{(n - 1)(a_1 + a_{n-1})}{2}
\]
Proof By Induction

The inductive step: $L(n) = R(n)$ for $n > 1$

\[
L(n) = a_1 + a_2 + \cdots + a_{n-1} + a_n \\
= L(n-1) + a_n \\
= R(n-1) + a_n \\
= \frac{(n-1)(a_1 + a_{n-1})}{2} + a_n \\
= \frac{(n-1)a_1}{2} + \frac{(n-1)a_{n-1}}{2} + \frac{2a_n}{2} \\
= \frac{(n-1)a_1 + a_{n-1}}{2} + \frac{2a_n + (n-2)a_{n-1}}{2} \\
= \frac{(n-1)a_1 + (a_1 + (n-2)d)}{2} + \frac{2a_n + (n-2)(a_n - d)}{2} \\
= \frac{na_1 + (n-2)d}{2} + \frac{na_n - (n-2)d}{2} \\
= \frac{n(a_1 + a_n)}{2} \\
= R(n)
\]
Number of Subsets: Proof by Induction

**Theorem**
- A set of size $n \geq 0$ has $2^n$ subsets

**The induction base for $n = 0$ and $n = 1$**
- The only subset of the empty set is the empty set and $2^0 = 1$
- The empty set and the entire set are the only subsets of a set of size 1 and $2^1 = 2$

**The induction hypothesis for $n > 1$**
- Any set of size $n - 1$ has $2^{n-1}$ subsets

**Notations**
- Let $S = \{ s_1, s_2, \ldots, s_n \}$ be a set of size $n$
- Let $S' = \{ s_1, s_2, \ldots, s_{n-1} \}$
Number of Subsets: Proof by Induction

The inductive step for $n > 1$

- By the induction hypothesis, $S'$ has $2^{n-1}$ subsets all of them are also subsets of $S$

- Let $R$ be a subset of $S$ that is not a subset of $S'$
  - It follows that $s_n \in R$ and that $R' = R \setminus \{s_n\}$ is a subset of $S'$

- Let $R'$ be a subset of $S'$
  - Then, $R = R' \cup \{s_n\}$ is a subset of $S$ that is not a subset of $S'$

This is a one-to-one mapping from the set of all the subsets that contain $s_n$ to the set of all the subsets that do not contain $s_n$

- Therefore, there are also $2^{n-1}$ subsets of $S$ that contain $s_n$

- Since a subset of $S$ either contains $s_n$ or does not contain $s_n$, it follows that the number of subsets of $S$ is $2^{n-1} + 2^{n-1} = 2^n$
**Example:** \( S = \{C, R, B, G, M\} \)

**Matching the 16 subsets without** \( M \) **to the 16 subsets with** \( M \)

<table>
<thead>
<tr>
<th>Subset</th>
<th>Corresponding Subset</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \emptyset )</td>
<td>{M}</td>
</tr>
<tr>
<td>{C}</td>
<td>{C, M}</td>
</tr>
<tr>
<td>{R}</td>
<td>{R, M}</td>
</tr>
<tr>
<td>{B}</td>
<td>{B, M}</td>
</tr>
<tr>
<td>{G}</td>
<td>{G, M}</td>
</tr>
<tr>
<td>{C, R}</td>
<td>{C, R, M}</td>
</tr>
<tr>
<td>{C, B}</td>
<td>{C, B, M}</td>
</tr>
<tr>
<td>{C, G}</td>
<td>{C, G, M}</td>
</tr>
<tr>
<td>{R, B}</td>
<td>{R, B, M}</td>
</tr>
<tr>
<td>{R, G}</td>
<td>{R, G, M}</td>
</tr>
<tr>
<td>{B, G}</td>
<td>{B, G, M}</td>
</tr>
<tr>
<td>{C, R, B}</td>
<td>{C, R, B, M}</td>
</tr>
<tr>
<td>{C, R, G}</td>
<td>{C, R, G, M}</td>
</tr>
<tr>
<td>{C, B, G}</td>
<td>{C, B, G, M}</td>
</tr>
<tr>
<td>{R, B, G}</td>
<td>{R, B, G, M}</td>
</tr>
<tr>
<td>{C, R, B, G}</td>
<td>{C, R, B, G, M}</td>
</tr>
</tbody>
</table>
$2^n$ vs. $n^2$

**Theorem**

- $2^n > n^2$ for any integer $n \geq 5$

**Why $n \geq 5$?**

<table>
<thead>
<tr>
<th>$n$</th>
<th>$2^n$</th>
<th>$n^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>16</td>
</tr>
<tr>
<td>5</td>
<td>32</td>
<td>25</td>
</tr>
<tr>
<td>6</td>
<td>64</td>
<td>36</td>
</tr>
<tr>
<td>7</td>
<td>128</td>
<td>49</td>
</tr>
</tbody>
</table>

$2^1 = 2 > 1 = 1^2$

$2^2 = 4 = 4 = 2^2$

$2^3 = 8 < 9 = 3^2$

$2^4 = 16 = 16 = 4^2$

$2^5 = 32 > 25 = 5^2$

$2^6 = 64 > 36 = 6^2$

$2^7 = 128 > 49 = 7^2$
$2^n > n^2$: Proof By Induction

The induction base for $n = 5$

- $2^5 = 32 > 25 = 5^2$

The induction hypothesis for $n > 5$

- Assume that $2^{n-1} > (n - 1)^2$

The inductive step for $n > 5$

\[
2^n = 2 \cdot 2^{n-1} \\
> 2(n - 1)^2 \quad (\ast \text{ the induction hypothesis } \ast) \\
= 2n^2 - 4n + 2 \\
> 2n^2 - 4n \\
> 2n^2 - n^2 \quad (\ast \text{ because } n^2 > 4n \text{ for } n > 4 \ast) \\
= n^2
\]
A Divisibility Theorem: Proof By Induction

**Theorem**

- \( n(n + 1)(n + 2) \) is divisible by 6 for \( n \geq 1 \)

**The induction base: for** \( n = 1, 2, 3, 4, 5 \)

\[
\begin{align*}
1 \cdot 2 \cdot 3 &= 6 &= 1 \cdot 6 \\
2 \cdot 3 \cdot 4 &= 24 &= 4 \cdot 6 \\
3 \cdot 4 \cdot 5 &= 60 &= 10 \cdot 6 \\
4 \cdot 5 \cdot 6 &= 120 &= 20 \cdot 6 \\
5 \cdot 6 \cdot 7 &= 210 &= 35 \cdot 6
\end{align*}
\]

**The induction hypothesis for** \( n > 1 \)

- Assume that \((n - 1)n(n + 1)\) is divisible by 6
- That is, \((n - 1)n(n + 1) = 6q\) for an integer \(q\)
A Divisibility Theorem: Proof By Induction

The inductive step for \( n > 1 \)

\[
\begin{align*}
n(n + 1)(n + 2) &= (n - 1)n(n + 1) + 3n(n + 1) \quad \text{(*) algebra *)} \\
&= 6q + 3n(n + 1) \quad \text{(*) induction hypothesis *)} \\
&= 6q + 6\frac{n(n + 1)}{2} \quad \text{(*) algebra *)} \\
&= 6q + 6\binom{n}{2} \quad \text{(*) definition *)} \\
&= 6q + 6p \quad \text{(*) } p = \binom{n}{2} \text{ is an integer *)} \\
&= 6(q + p) \quad \text{(*) Q.E.D. *)}
\end{align*}
\]
Theorem

\[ n(n + 1)(n + 2) \text{ is divisible by 6 for } n \geq 1 \]

Proof

- \( n, n + 1, \) and \( n + 2 \) are three consecutive integers
- One of them must be divisible by 3
- One (could be the same integer) must be even and therefore is divisible by 2
- Therefore, the product of the three integers must be divisible by \( 6 = 3 \cdot 2 \)
A Divisibility Theorem: A Third Proof

Theorem

- \( n(n+1)(n+2) \) is divisible by 6 for \( n \geq 1 \)

Proof

- For \( n \geq 1 \), the number of subsets of size 3 of a set of size \( n+2 \) is
  \[
  \binom{n+2}{3} = \frac{(n+2)(n+1)n}{6}
  \]
- This number must be an integer
- Therefore \( n(n+1)(n+2) \) is divisible by 6
Another Divisibility Theorem

**Theorem**
- $n(n + 1)(n + 2)$ is divisible by 24 for an even $n \geq 2$

**Small Values of $n$**

- $2 \cdot 3 \cdot 4 = 24 = 1 \cdot 24$
- $4 \cdot 5 \cdot 6 = 120 = 5 \cdot 24$
- $6 \cdot 7 \cdot 8 = 336 = 14 \cdot 24$

**Proof**
- $n$, $n + 1$, and $n + 2$ are three consecutive integers
- One of them must be divisible by 3
- $n$ and $n + 2$ are two consecutive even integers
- One of them must be divisible by 4 while the other is divisible by 2
- Therefore, the product of the three integers must be divisible by 24
  $$24 = 3 \cdot 4 \cdot 2$$
Geometric Progressions

Definition

A sequence \( a_1, a_2, \ldots, a_n \) is a geometric progression with a common positive ratio \( q > 0 \) if \( a_i = qa_{i-1} \) for all \( 2 \leq i \leq n \).

Simplifying assumptions

- \( a_1 = q \) and as a result the sequence becomes \( q^1, q^2, \ldots, q^n \)
- Add \( a_0 = 1 = q^0 \) to the beginning of the sequence and as a result the sequence becomes \( q^0, q^1, q^2, \ldots, q^n \)

Theorem

For a real number \( q > 0 \)

\[
\sum_{i=0}^{n-1} q^i = 1 + q + \cdots + q^{n-1} = \frac{q^n - 1}{q - 1}
\]
Proof By Induction

Notations

- $L(n) = 1 + q + \cdots + q^{n-1}$
- $R(n) = \frac{q^{n-1}}{q-1}$

The Induction base: $n = 1$

- $L(1) = R(1)$, because $L(1) = 1$ and $R(1) = \frac{q^1-1}{q-1} = 1$

The induction hypothesis: $L(n - 1) = R(n - 1)$ for $n > 1$

$$\sum_{i=0}^{n-2} q^i = 1 + q + \cdots + q^{n-2} = \frac{q^{n-1} - 1}{q-1}$$
The inductive step: \( L(n) = R(n) \) for \( n > 1 \)

\[
L(n) = 1 + q + \cdots + q^{n-2} + q^{n-1} \\
= L(n-1) + q^{n-1} \\
= R(n-1) + q^{n-1} \\
= \frac{q^{n-1} - 1}{q - 1} + q^{n-1} \\
= \frac{(q^{n-1} - 1) + ((q - 1)q^{n-1})}{q - 1} \\
= \frac{(q^{n-1} - 1) + (q^n - q^{n-1})}{q - 1} \\
= \frac{q^n - 1}{q - 1} \\
= R(n)
\]
Another proof

Theorem
For a real number $q > 0$

$$
\sum_{i=0}^{n-1} q^i = 1 + q + \cdots + q^{n-1} = \frac{q^n - 1}{q - 1}
$$

Proof

$$(q - 1) \left( \sum_{i=0}^{n-1} q^i \right) = q \left( \sum_{i=0}^{n-1} q^i \right) - \left( \sum_{i=0}^{n-1} q^i \right)
= (q + q^2 + \cdots + q^n) - (1 + q + \cdots + q^{n-1})
= q^n - 1$$
Geometric Progressions

Corollary

For a real number $q > 0$

$$\sum_{i=1}^{n-1} q^i = q + \cdots + q^{n-1} = \frac{q^n - q}{q - 1}$$

Proof

$$\sum_{i=1}^{n-1} q^i = \left( \sum_{i=0}^{n-1} q^i \right) - 1$$

$$= \frac{q^n - 1}{q - 1} - \frac{q - 1}{q - 1}$$

$$= \frac{q^n - q}{q - 1}$$
Geometric Progressions with $q = 2$

Identity

$$\sum_{i=0}^{n-1} 2^i = 1 + 2 + 4 + \cdots + 2^{n-1}$$

$$= \frac{2^n - 1}{2 - 1}$$

$$= 2^n - 1$$

Small numbers

$$1 = 1 = 2^1 - 1$$

$$1 + 2 = 3 = 2^2 - 1$$

$$1 + 2 + 4 = 7 = 2^3 - 1$$

$$1 + 2 + 4 + 8 = 15 = 2^4 - 1$$

$$1 + 2 + 4 + 8 + 16 = 31 = 2^5 - 1$$

$$1 + 2 + 4 + 8 + 16 + 32 = 63 = 2^6 - 1$$
Geometric Progressions with $q = 3$

**Identity**

\[
\sum_{i=0}^{n-1} 3^i = 1 + 3 + 9 + \cdots + 3^{n-1}
\]

\[
= \frac{3^n - 1}{3 - 1} = \frac{3^n - 1}{2}
\]

**Small numbers**

\[
1 = 1 = \frac{3^1 - 1}{2} = \frac{3 - 1}{2}
\]

\[
1 + 3 = 4 = \frac{3^2 - 1}{2} = \frac{9 - 1}{2}
\]

\[
1 + 3 + 9 = 13 = \frac{3^3 - 1}{2} = \frac{27 - 1}{2}
\]

\[
1 + 3 + 9 + 27 = 40 = \frac{3^4 - 1}{2} = \frac{81 - 1}{2}
\]
Geometric Progressions for Large \( q \)

**Approximation**

\[
\sum_{i=0}^{n-1} q^i = 1 + q + q^2 + \cdots + q^{n-1} = \frac{q^n - 1}{q - 1} \approx \frac{q^n}{q - 1}
\]
Another Version of the Identity for the Sum

**Theorem**
For a real number $q > 0$

\[
\sum_{i=0}^{n-1} q^i = 1 + q + \cdots + q^{n-1} = \frac{1 - q^n}{1 - q}
\]

**Proof**

\[
\sum_{i=0}^{n-1} q^i = \frac{q^n - 1}{q - 1} = \frac{(-1)(q^n - 1)}{(-1)(q - 1)} = \frac{1 - q^n}{1 - q}
\]
Which Identity To Use?

The two identities

\[
\sum_{i=0}^{n-1} q^i = 1 + q + \cdots + q^{n-1} = \frac{q^n - 1}{q - 1} \tag{1}
\]

\[
\sum_{i=0}^{n-1} q^i = 1 + q + \cdots + q^{n-1} = \frac{1 - q^n}{1 - q} \tag{2}
\]

Avoid negative numbers

- Use the first when \( q > 1 \) so both the numerator and the denominator are positive
- Use the second when \( q < 1 \) so both the numerator and the denominator are positive
Geometric Progressions with $q = \frac{1}{2}$

Identity

\[ \sum_{i=0}^{n-1} \left( \frac{1}{2} \right)^i = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} \]

\[ = \frac{1 - \left( \frac{1}{2} \right)^n}{1 - \frac{1}{2}} \]

\[ = 2 \left( 1 - \left( \frac{1}{2} \right)^n \right) = 2 - \frac{1}{2^{n-1}} \]

Small numbers

\[ 1 = 1 = 2 - \frac{1}{1} \]

\[ 1 + \frac{1}{2} = \frac{3}{2} = 2 - \frac{1}{2} \]

\[ 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4} = 2 - \frac{1}{4} \]

\[ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8} = 2 - \frac{1}{8} \]
Geometric Progressions with $q = \frac{2}{3}$

Identity

$$\sum_{i=0}^{n-1} \left( \frac{2}{3} \right)^i = 1 + \frac{2}{3} + \frac{4}{9} + \cdots + \frac{2^{n-1}}{3^{n-1}}$$

$$= \frac{1 - \left( \frac{2}{3} \right)^n}{1 - \frac{2}{3}}$$

$$= 3 \left( 1 - \left( \frac{2}{3} \right)^n \right)$$

$$= 3 - \frac{2^n}{3^{n-1}}$$
Identity

\[ \sum_{i=0}^{n-1} \left( \frac{k - 1}{k} \right)^i = 1 + \frac{k - 1}{k} + \frac{(k - 1)^2}{k^2} + \cdots + \frac{(k - 1)^{n-1}}{k^{n-1}} \]

\[ = \frac{1 - \left( \frac{k - 1}{k} \right)^n}{1 - \frac{k - 1}{k}} \]

\[ = k \left( 1 - \left( \frac{k - 1}{k} \right)^n \right) \]

\[ = k - \frac{(k - 1)^n}{k^{n-1}} \]
Geometric Progressions with $q = \frac{1}{3}$

**Identity**

\[
\sum_{i=0}^{n-1} \left(\frac{1}{3}\right)^i = 1 + \frac{1}{3} + \frac{1}{9} + \cdots + \frac{1}{3^{n-1}} = \frac{1 - \left(\frac{1}{3}\right)^n}{1 - \frac{1}{3}} = \frac{3}{2} \left(1 - \left(\frac{1}{3}\right)^n\right)
\]
Geometric Progressions with $q = \frac{1}{k}$

**Identity**

\[
\sum_{i=0}^{n-1} \left(\frac{1}{k}\right)^i = 1 + \frac{1}{k} + \frac{1}{k^2} + \cdots + \frac{1}{k^{n-1}}
\]

\[
= \frac{1 - \left(\frac{1}{k}\right)^n}{1 - \frac{1}{k}}
\]

\[
= \frac{k}{k - 1} \left(1 - \left(\frac{1}{k}\right)^n\right)
\]
Infinite Geometric Progressions with $0 < q < 1$

**Theorem**

$$
\sum_{i=0}^{\infty} q^i = 1 + q + q^2 + \cdots = \frac{1}{1 - q}
$$

**Proof sketch**

- If $0 < q < 1$ then $q^n \to 0$ when $n \to \infty$
- Therefore, $q^\infty = 0$ for $0 < q < 1$
- This implies that

$$
\sum_{i=0}^{\infty} q^i = \frac{1 - q^\infty}{1 - q} = \frac{1 - 0}{1 - q} = \frac{1}{1 - q}
$$
Another proof

**Theorem**

For a real number $0 < q < 1$,

$$\sum_{i=0}^{\infty} q^i = 1 + q + q^2 + \cdots = \frac{1}{1-q}$$

**Proof**

$$(1 - q) \left( \sum_{i=0}^{\infty} q^i \right) = \left( \sum_{i=0}^{\infty} q^i \right) - q \left( \sum_{i=0}^{\infty} q^i \right)$$

$$= \left( 1 + q + q^2 + \cdots \right) - \left( q + q^2 + q^3 + \cdots \right)$$

$$= 1$$
Infinite Geometric Progressions with $q < 1$

**Special Cases**

1. \[\sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1 - \frac{1}{2}} = 2\]
2. \[\sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^i = 1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots = \frac{1}{1 - \frac{2}{3}} = 3\]
3. \[\sum_{i=0}^{\infty} \left(\frac{k-1}{k}\right)^i = 1 + \frac{k-1}{k} + \frac{(k-1)^2}{k^2} + \frac{(k-1)^3}{k^3} + \cdots = \frac{1}{1 - \frac{k-1}{k}} = k\]
4. \[\sum_{i=0}^{\infty} \left(\frac{1}{3}\right)^i = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}\]
5. \[\sum_{i=0}^{\infty} \left(\frac{1}{k}\right)^i = 1 + \frac{1}{k} + \frac{1}{k^2} + \frac{1}{k^3} + \cdots = \frac{1}{1 - \frac{1}{k}} = \frac{k}{k-1}\]
Sum of Powers of First $n$ Integers

Small exponents

- $\sum_{i=1}^{n} i^0 = 1 + 1 + \cdots + 1 = n \approx \frac{1}{1} n^1$
- $\sum_{i=1}^{n} i^1 = 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \approx \frac{1}{2} n^2$
- $\sum_{i=1}^{n} i^2 = 1 + 4 + 9 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \approx \frac{1}{3} n^3$
- $\sum_{i=1}^{n} i^3 = 1 + 8 + 27 + \cdots + n^3 = \frac{n^2(n+1)^2}{4} \approx \frac{1}{4} n^4$
- $\sum_{i=1}^{n} i^k = 1^k + 2^k + \cdots + n^k \approx \frac{1}{k+1} n^{k+1}$
Sum of Powers of First $n$ Integers

Small exponents

\[
\sum_{i=1}^{n} i^1 = \left(\frac{1}{2}\right)(n^2 + n)
\]

\[
\sum_{i=1}^{n} i^2 = \left(\frac{1}{6}\right)(2n^3 + 3n^2 + n)
\]

\[
\sum_{i=1}^{n} i^3 = \left(\frac{1}{4}\right)(n^4 + 2n^3 + n^2)
\]

\[
\sum_{i=1}^{n} i^4 = \left(\frac{1}{30}\right)(6n^5 + 15n^4 + 10n^3 - n)
\]

\[
\sum_{i=1}^{n} i^5 = \left(\frac{1}{12}\right)(2n^6 + 6n^5 + 5n^4 - n^2)
\]

\[
\sum_{i=1}^{n} i^6 = \left(\frac{1}{42}\right)(6n^7 + 21n^6 + 21n^5 - 7n^3 + n)
\]
Sum of Powers of First $n$ Integers

**Small exponents**

\[
\begin{align*}
\sum_{i=1}^{n} i^1 &= \frac{n(n+1)}{2} \\
\sum_{i=1}^{n} i^2 &= \frac{n(n+1)(2n+1)}{6} \\
\sum_{i=1}^{n} i^3 &= \frac{n^2(n+1)^2}{4} \\
\sum_{i=1}^{n} i^4 &= \frac{n(2n+1)(3n^2 + 3n - 1)}{30} \\
\sum_{i=1}^{n} i^5 &= \frac{n^2(n+1)^2(2n^2 + 2n - 1)}{12} \\
\sum_{i=1}^{n} i^6 &= \frac{n(n+1)(2n+1)(3n^4 + 6n^3 - 3n + 1)}{42}
\end{align*}
\]
Sum of First $n$ Squares

Identity

$$\sum_{i=1}^{n} i^2 = 1 + 4 + 9 + \cdots + n^2 = \frac{n(n + 1)(2n + 1)}{6}$$

Proof without words

[Image: https://i.stack.imgur.com/UaRQC.png]
Sum of First $n$ Squares

Correctness for Small $n$

\[
egin{align*}
1 &= 1 &= rac{1 \cdot 2 \cdot 3}{6} &= \frac{6}{6} \\
1 + 4 &= 5 &= rac{2 \cdot 3 \cdot 5}{6} &= \frac{30}{6} \\
1 + 4 + 9 &= 14 &= rac{3 \cdot 4 \cdot 7}{6} &= \frac{84}{6} \\
1 + 4 + 9 + 16 &= 30 &= rac{4 \cdot 5 \cdot 9}{6} &= \frac{180}{6} \\
1 + 4 + 9 + 16 + 25 &= 55 &= rac{5 \cdot 6 \cdot 11}{6} &= \frac{330}{6} \\
1 + 4 + 9 + 16 + 25 + 36 &= 91 &= rac{6 \cdot 7 \cdot 13}{6} &= \frac{546}{6}
\end{align*}
\]
Proof By Induction

Notations

- \( L(n) = \sum_{i=1}^{n} i^2 \)
- \( R(n) = \frac{n(n+1)(2n+1)}{6} \)

The induction base: \( n = 1 \)

\( L(1) = R(1), \) because \( L(1) = 1^2 = 1 \) and \( R(1) = \frac{1 \cdot 2 \cdot 3}{6} = 1 \)

The induction hypothesis: \( L(n - 1) = R(n - 1) \) for \( n > 1 \)

\[
\sum_{i=1}^{n-1} i^2 = 1 + 4 + \cdots + (n - 1)^2 = \frac{(n - 1)n(2n - 1)}{6}
\]
Proof By Induction

The inductive step: $L(n) = R(n)$ for $n > 1$

\[
L(n) = 1 + 4 + \cdots + (n - 1)^2 + n^2 = L(n - 1) + n^2 = R(n - 1) + n^2 = \frac{(n - 1)n(2n - 1)}{6} + n^2 = \frac{(2n^3 - n^2 - 2n^2 + n) + 6n^2}{6} = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(n + 1)(2n + 1)}{6} = R(n)
\]
Sum of First \( n \) Cubes

Identity

\[
\sum_{i=1}^{n} i^3 = 1 + 8 + 27 + \cdots + n^3
\]

\[
= \frac{n^2(n+1)^2}{4}
\]

\[
= \left( \frac{n(n+1)}{2} \right)^2
\]

\[
= (1 + 2 + 3 + \cdots + n)^2
\]

\[
= \left( \frac{n+1}{2} \right)^2
\]

Proof without words

https://i.stack.imgur.com/XHc4q.png
Sum of First $n$ Cubes

Correctness for Small $n$

\begin{align*}
1 &= 1 = \frac{1^2 \cdot 2^2}{4} = \frac{4}{4} \\
1 + 8 &= 9 = \frac{2^2 \cdot 3^2}{4} = \frac{36}{4} \\
1 + 8 + 27 &= 36 = \frac{3^2 \cdot 4^2}{4} = \frac{144}{4} \\
1 + 8 + 27 + 64 &= 100 = \frac{4^2 \cdot 5^2}{4} = \frac{400}{4} \\
1 + 8 + 27 + 64 + 125 &= 225 = \frac{5^2 \cdot 6^2}{4} = \frac{900}{4} \\
1 + 8 + 27 + 64 + 125 + 216 &= 441 = \frac{6^2 \cdot 7^2}{4} = \frac{1764}{4} \\
1 + 8 + 27 + 64 + 125 + 216 + 343 &= 784 = \frac{7^2 \cdot 8^2}{4} = \frac{3136}{4}
\end{align*}
Proof By Induction

Notations

- \( L(n) = 1 + 8 + 27 + \cdots + n^3 \)
- \( R(n) = \frac{n^2(n+1)^2}{4} \)

The induction base: \( n = 1 \)

- \( L(1) = R(1) \), because \( L(1) = 1^3 = 1 \) and \( R(1) = \frac{1^2 \cdot 2^2}{4} = 1 \)

The induction hypothesis: \( L(n - 1) = R(n - 1) \) for \( n > 1 \)

\[
\sum_{i=1}^{n-1} i^3 = 1 + 8 + \cdots + (n-1)^3 = \frac{(n-1)^2 n^2}{4}
\]
Proof By Induction

The inductive step: \( L(n) = R(n) \) for \( n > 1 \)

\[
L(n) = 1 + 8 + \cdots + (n - 1)^3 + n^3 \\
= L(n - 1) + n^3 \\
= R(n - 1) + n^3 \\
= \frac{(n - 1)^2 n^2}{4} + n^3 \\
= \frac{(n^4 - 2n^3 + n^2) + 4n^3}{4} \\
= \frac{n^4 + 2n^3 + n^2}{4} \\
= \frac{n^2(n + 1)^2}{4} \\
= R(n)
\]
Another Identity

Identity

\[
\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}
\]

Correctness for Small \( n \)

\[
\begin{align*}
\frac{1}{2} &= \frac{1}{2} = 1 - \frac{1}{2} \\
\frac{1}{3} + \frac{1}{4} &= \frac{7}{12} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \\
\frac{1}{4} + \frac{1}{5} + \frac{1}{6} &= \frac{37}{60} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \\
\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} &= \frac{533}{840} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8}
\end{align*}
\]
Proof By Induction

Notations

- \( L(n) = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \)
- \( R(n) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} \)

The induction base: \( n = 1 \)

- \( L(1) = R(1) \), because \( L(1) = \frac{1}{2} \) and \( R(1) = 1 - \frac{1}{2} = \frac{1}{2} \)

The induction hypothesis: \( L(n - 1) = R(n - 1) \) for \( n > 1 \)

\[
\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-2} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-3} - \frac{1}{2n-2}
\]
Proof By Induction

The inductive step: \( L(n) = R(n) \) for \( n > 1 \)

\[
L(n) = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-2} + \frac{1}{2n-1} + \frac{1}{2n}
\]

\[
= L(n-1) + \frac{1}{2n-1} + \frac{1}{2n} - \frac{1}{n}
\]

\[
= L(n-1) + \frac{1}{2n-1} - \frac{1}{2n}
\]

\[
= R(n-1) + \frac{1}{2n-1} - \frac{1}{2n}
\]

\[
= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-3} - \frac{1}{2n-2} + \frac{1}{2n-1} - \frac{1}{2n}
\]

\[
= R(n)
\]
Prime Factorization

Theorem

- Every positive integer $n \geq 2$ is a power of a prime number or the product of powers of prime numbers

Proof by Induction

- **Induction base:** $2 = 2^1$ is a power of a prime
- **Induction hypothesis:** Assume every positive integer less than $n$ is a prime number or a product of powers of prime numbers
- **Inductive step:**
  - If $n$ is a prime, then $n = n^1$ is a power of a prime
  - Otherwise, $n = k \cdot h$ is a product of two numbers $k < n$ and $h < n$
  - By the induction hypothesis, both $k$ and $h$ are power of prime numbers or products of prime numbers
  - Therefore, $n = k \cdot h$ is also a power of a prime number or a product of powers of prime numbers
Prime Factorization

Example I

- $90 = 15 \cdot 6 = (3 \cdot 5)(2 \cdot 3)$
- Therefore by induction, $90 = 2 \cdot 3^2 \cdot 5$

Example II

- $216 = 12 \cdot 18 = (2^2 \cdot 3)(2 \cdot 3^2)$
- Therefore by induction, $216 = 2^3 \cdot 3^3$

Example III

- $128 = 8 \cdot 16 = 2^3 \cdot 2^4$
- Therefore by induction, $128 = 2^7$
A Chain of Links Problem

Problem

- How many chain-partitions are required to split a chain of $n$ links into $n$ individual links?
- In each partition, a chain of size $n$ is replaced by two unbroken chains of size $k$ and $m$ such that $k + m = n$
- Trivially, this can be done with $n - 1$ partitions by partitioning a single link in each step
- Is it possible to split the chain with less than $n - 1$ partitions?

Example: Splitting a 5-Chain

- Partition 1: Replace the 5-chain with a 3-chain and a 2-chain
- Partition 3: Replace the 2-chain with 2 links
- Partition 3: Replace the 3-chain with a 2-chain and a link
- Partition 4: Replace the 2-chain with 2 links
A Chain of Links Problem

Answer
- \( f(n) = n - 1 \) is the number of partitions needed to split up a chain of \( n \) links

Proof by induction sketch
- **Induction base:** \( f(1) = 0 \) because no partitions are required when \( n = 1 \)
- **Induction hypothesis:** Assume that \( f(h) = h - 1 \) for all \( 1 \leq h < n \)
- **Inductive step:** Any partition of the \( n \)-chain splits it into two chains of sizes \( 1 \leq k < n \) and \( 1 \leq m < n \) where \( n = k + m \)

\[
f(n) = 1 + f(k) + f(m)
\]
\[
= 1 + (k - 1) + (m - 1)
\]
\[
= (k + m) - 1
\]
\[
= n - 1
\]
Problem

- A chocolate bar consisting of \( n \geq 1 \) unit squares is arranged as an \( m \times k \) rectangular grid (\( n = m \cdot k \))

- The goal is to split the bar into \( n \) individual unit squares by breaking along the lines

- It is not allowed to break more than one rectangular piece at a time (e.g., by piling them together).

- What is the required number of breaks?

Online resource

- https://www.youtube.com/watch?v=yftf3fs9k6s
A Chocolate Bar Problem

Claim

For $0 \leq h \leq n - 1$, after $h$ breaks, there are $h + 1$ pieces.

Proof by induction sketch

- By induction on the number of breaks.
- After $h = 0$ breaks there is one piece and indeed $1 = 0 + 1$.
- Before the $h^{th}$ break, by the induction hypothesis, there were $h = (h - 1) + 1$ pieces.
- After the $h^{th}$ break there are $h + 1$ pieces because the break replaces one of the pieces with two pieces.

Corollary

After $n - 1$ breaks there are $n$ pieces. That is, the bar was split into $n$ individuals unit squares.
More Examples

Three problems

- L-shape tiles of size 3 can tile any square of size $2^n \times 2^n$ small squares with any missing square
- Number of steps needed to solve the tower of Hanoi problem
- Any partition of the circle with chords can be face-colored with two colors

Online resource

- https://www.youtube.com/watch?v=5Hn8vUE3cBQ

Remark

- All proofs imply an algorithm!
A False Divisibility Claim

Claim

\( n^3 - n + 1 \) is divisible by 3

Small Values of \( n \)

\[
\begin{align*}
1^3 - 1 + 1 &= 1 = 6 \cdot 0 + 1 \\
2^3 - 2 + 1 &= 7 = 6 \cdot 1 + 1 \\
3^3 - 3 + 1 &= 25 = 6 \cdot 4 + 1 \\
4^3 - 4 + 1 &= 61 = 6 \cdot 10 + 1
\end{align*}
\]
Proof By Induction

The induction base
- Skip the base case!

The induction hypothesis
- Assume that \((n - 1)^3 - (n - 1) + 1 = 3q\) is divisible by 3

The inductive step

\[
\begin{align*}
n^3 - n + 1 &= n^3 + (3n^2 - 3n^2) + (3n - 3n) - n + (2 - 1) \\
&= ((n^3 - 3n^2 + 3n - 1) - n + 2) + (3n^2 - 3n) \\
&= ((n - 1)^3 - (n - 1) + 1) + 3(n^2 - n) \\
&= 3q + 3(n^2 - n) \\
&= 3(q + n^2 - n) \\
&= 3q + 3(n^2 - n) \\
&= 3(q + n^2 - n)
\end{align*}
\]

\(\ast\) Q.E.D. \(\ast\)
Correct Claim

Theorem

- $n^3 - n$ is divisible by 6

Small Values of $n$

1. $1^3 - 1 = 0 = 6 \cdot 0$
2. $2^3 - 2 = 6 = 6 \cdot 1$
3. $3^3 - 3 = 24 = 6 \cdot 4$
4. $4^3 - 4 = 60 = 6 \cdot 10$

Proof

- $n^3 - n = n(n^2 - 1) = (n - 1)n(n + 1)$
- That is, $n^3 - n$ is a product of three consecutive integers
- One of them must be divisible by 3
- One (could be the same integer) must be even
- Therefore, the product of the three integers must be divisible by 6
Proof by induction for $n \geq 0$

- Clearly, if $n = 0$, then $6n = 0$
- Let $n > 0$ and assume that $6i = 0$ for all $0 \leq i < n$
- Let $n = k + h$ for integers $0 \leq k < n$ and $0 \leq h < n$
- By the induction hypothesis, $6k = 0$ and $6h = 0$. Therefore

$$6n = 6(k + h) = 6k + 6h = 0 + 0 = 0$$
Proof by induction for \( n \geq 0 \)

- Clearly, if \( n = 0 \), then \( 6n = 0 \)
- Let \( n > 0 \) and assume that \( 6i = 0 \) for all \( 0 \leq i < n \)
- Let \( n = k + h \) for integers \( 0 \leq k < n \) and \( 0 \leq h < n \)
- By the induction hypothesis, \( 6k = 0 \) and \( 6h = 0 \). Therefore
  \[
  6n = 6(k + h) = 6k + 6h = 0 + 0 = 0
  \]

Where is the Error?

- The proof fails for \( n = 1 \)
- 1 cannot be expressed as a sum of two non-negative integers that are smaller than 1
All Horses in the World are of the Same Color

Proof by induction on the number of horses

- The base of the induction is that if there is one horse, then it is trivially the same color as itself.
- Suppose that there are \( n \) horses, numbered 1 through \( n \).
- By the induction hypothesis, the \( n - 1 \) horses 1 through \( n - 1 \) are all of the same color.
- Assume this color is black. In particular, horse 2 is black.
- This means that the \( n - 1 \) horses 2 through \( n \) must be black by the induction hypothesis.
- Therefore, all of the horses 1 through \( n \) are of the same color.

Where is the Error?
Proof fails for \( n = 2 \) in which horse 2 may be of a different color.
All Horses in the World are of the Same Color

Proof by induction on the number of horses

- The base of the induction is that if there is one horse, then it is trivially the same color as itself.
- Suppose that there are $n$ horses, numbered 1 through $n$.
- By the induction hypothesis, the $n - 1$ horses 1 through $n - 1$ are all of the same color.
- Assume this color is black. In particular, horse 2 is black.
- This means that the $n - 1$ horses 2 through $n$ must be black by the induction hypothesis.
- Therefore, all of the horses 1 through $n$ are of the same color.

Where is the Error?

- Proof fails for $n = 2$ in which horse 2 may be of a different color.