Discrete Structures: Miscellaneous Topics

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The Wisdom of Crowds

How Many Marbles are in the Jar? Math, statistics, and Wisdom of crowds.

https://www.youtube.com/watch?v=WZtCC-h2pQU&t=174s

The concept.

https://www.youtube.com/watch?v=Zv80ibV-0FM
What is the Average of Numbers?

Notations

- Let $x_1, x_2, \ldots, x_n$ be $n$ positive real numbers.
- Denote their average by $\text{ave}(x_1, x_2, \ldots, x_n)$.
What is the Average of Numbers?

Intuition: Five “Natural” Requirements

1. The average is not smaller the minimum number and is not larger the maximum number:
   \[\min \{x_1, x_2, \ldots, x_n\} \leq \text{ave}(x_1, x_2, \ldots, x_n) \leq \max \{x_1, x_2, \ldots, x_n\}\]

2. The average is equal to both the minimum and the maximum numbers iff all the numbers are equal:
   \[x_1 = \cdots = x_n \iff \text{avg}(x_1, \ldots, x_n) = \min \{x_1, \ldots, x_n\} = \max \{x_1, \ldots, x_n\}\]

3. The average is a monotonic increasing continuous function with each number:
   \[\text{avg}(x_1, \ldots, x_i + \Delta, \ldots, x_n) > \text{avg}(x_1, \ldots, x_i, \ldots, x_n)\]

4. The average is independent of the order of the numbers (symmetry):
   \[\text{avg}(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) = \text{avg}(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n)\]

5. Multiplying each number by a positive constant \(c\) also multiplies the average by \(c\):
   \[\text{avg}(cx_1, cx_2, \ldots, cx_n) = c \cdot \text{avg}(x_1, x_2, \ldots, x_n)\]
What is the Average of Numbers?

Median

- The Median of \( \{ x_1, x_2, \ldots, x_n \} \) obeys the above requirements.
- But it is a different story!
Four Possible Definitions

**Arithmetic Mean**

\[ A(x_1, x_2, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i = \frac{x_1 + x_2 + \cdots + x_n}{n} \]

**Geometric Mean**

\[ G(x_1, x_2, \ldots, x_n) = \left( \prod_{i=1}^{n} x_i \right)^{\frac{1}{n}} = (x_1 \cdot x_2 \cdots x_n)^{1/n} \]

**Harmonic Mean**

\[ H(x_1, x_2, \ldots, x_n) = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}} = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}} \]

**Root Mean Square**

\[ R(x_1, x_2, \ldots, x_n) = \sqrt{\frac{1}{n} \sum_{i=1}^{n} x_i^2} = \sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}} \]
Example

https://www.youtube.com/watch?v=cGU66Rhoby8
Proposition

The Geometric Mean of 2 positive real numbers is the Geometric Mean of the Arithmetic Mean and Harmonic Mean of these 2 numbers.

Proof

\[ G(x, y) = G(A(x, y), H(x, y)) \]

\[ \sqrt{xy} = \sqrt{\frac{x + y}{2} \cdot \frac{2xy}{x + y}} \]

\[ \sqrt{xy} = \sqrt{\frac{x + y}{2} \cdot \frac{2}{\frac{1}{x} + \frac{1}{y}}} \]

\[ G(x, y) = \sqrt{A(x, y) \cdot H(x, y)} \]

\[ G(x, y) = G(A(x, y), H(x, y)) \]
Theorem

For positive real numbers $x \geq y$:

$$y \leq H(x, y) \leq G(x, y) \leq A(x, y) \leq R(x, y) \leq x$$

Equality holds if and only if $x = y$.

The Theorem with Variables

$$y \leq \frac{2}{\frac{1}{x} + \frac{1}{y}} \leq \sqrt{xy} \leq \frac{x + y}{2} \leq \sqrt{\frac{x^2 + y^2}{2}} \leq x$$
Proof for $x = y$

**Harmonic Mean**

\[ H(x, y) = \frac{2}{\frac{1}{x} + \frac{1}{y}} = \frac{2}{\frac{2}{x}} = x \]

**Geometric Mean**

\[ G(x, y) = \sqrt{xy} = \sqrt{x^2} = x \]

**Arithmetic Mean**

\[ A(x, y) = \frac{x + y}{2} = \frac{2x}{2} = x \]

**Root Mean Square**

\[ R(x, y) = \sqrt{\frac{x^2 + y^2}{2}} = \sqrt{\frac{2x^2}{2}} = x \]
Geometric Mean $\leq$ Arithmetic Mean

$$4xy = (x + y)^2 - (x - y)^2 \leq (x + y)^2$$

$$xy \leq \frac{(x + y)^2}{2^2}$$

$$xy \leq \left(\frac{x + y}{2}\right)^2$$

$$\sqrt{xy} \leq \frac{x + y}{2}$$
Harmonic Mean ≤ Geometric Mean

\[
4xy = (x + y)^2 - (x - y)^2 \leq (x + y)^2
\]

\[
\frac{4x^2 y^2}{xy} \leq (x + y)^2
\]

\[
\frac{4x^2 y^2}{(x + y)^2} \leq xy
\]

\[
\frac{2xy}{x + y} \leq \sqrt{xy}
\]

\[
\frac{2}{\frac{1}{x} + \frac{1}{y}} \leq \sqrt{xy}
\]
Arithmetic Mean ≤ Root Square Mean

\[
0 \leq (x - y)^2 \\
2xy \leq x^2 + y^2 \\
(x + y)^2 = x^2 + y^2 + 2xy \leq 2x^2 + 2y^2 \\
\frac{(x + y)^2}{4} \leq \frac{x^2 + y^2}{2} \\
\frac{x + y}{2} \leq \sqrt{\frac{x^2 + y^2}{2}}
\]
Geometric Proofs

\[ \sqrt{xy} \leq \frac{x+y}{2} \]

https://www.youtube.com/watch?v=dUhleJQ6QV4

A Full Geometric Proof

The HM-GM-AM-RMS general Theorem

**Theorem**

For $n$ positive real numbers $x_1, x_2, \ldots, x_n$:

\[
\min \{x_1, x_2, \ldots, x_n\} \leq H(x_1, x_2, \ldots, x_n) \leq G(x_1, x_2, \ldots, x_n) \leq A(x_1, x_2, \ldots, x_n) \leq R(x_1, x_2, \ldots, x_n) \leq \max \{x_1, x_2, \ldots, x_n\}
\]

Equality holds if and only if $x_1 = \cdots = x_n$. 
Corollary

The sum of \( n \) positive real numbers multiplied by the sum of their \( n \) reciprocals is greater equal to \( n^2 \).

\[
A(x_1, x_2, \ldots, x_n) \geq H(x_1, x_2, \ldots, x_n) \\
\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}
\]

\[
(x_1 + x_2 + \cdots + x_n) \left( \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \right) \geq n^2
\]

The Corollary Positive \( x \) and \( y \)

\[
4 \leq (x + y) \left( \frac{1}{x} + \frac{1}{y} \right)
\]
The Zeno Paradoxes

- The dichotomy paradox:

- Achilles and the Tortoise I:
  - https://www.youtube.com/watch?v=skM37PcZmWE

- Achilles and the Tortoise II:
  - https://www.youtube.com/watch?v=3vN1f2zGLaE

- Achilles and the Tortoise and beyond:
  - https://www.youtube.com/watch?v=Nctw5f6XPF4
Infinity and Beyond

How Big is Infinity

https://ed.ted.com/lessons/how-big-is-infinity

Infinity is bigger than you think!

https://www.youtube.com/watch?v=elvOZm0d4H0

The Infinite Hotel

https://www.youtube.com/watch?v=Uj3_KqkI9Zo
More about The Infinite Hotel

Empty Hotel with Infinite Number of Guests

- Infinite number of buses each with infinite number of passengers arrive at 11:00 then 11:30 then 11:45 and so on.
- With each arrival the occupant of room $i$ moves to room $2i$ and the new guests get the odd number rooms.
- At 12:00 the owner of the hotel finds all the rooms to be empty!

Infinite IDs

- If each person gets an infinite binary ID.
- Then there exists a person that is not in the hotel.
- Its ID is different in its $i$’s coordinate from the $i$’s coordinate of the guest of room $i$. 
$1 - 1 + 1 - 1 + 1 - \cdots = ???$
$1 - 1 + 1 - 1 + 1 - \cdots = \ ???$

**0 = 1?**

https://www.youtube.com/watch?v=aOxMTgCORhA

**Manipulating the sequence $S = 1 - 1 + 1 - 1 + 1 + \cdots$**

- $S = (1 - 1) + (1 - 1) + \cdots = 0$
- $S = 1 + (-1 + 1) + (-1 + 1) + \cdots = 1$
- $S = n + (1 - 1) + (1 - 1) + \cdots = n$

$$S = 1_1 - 1_2 + 1_3 - 1_4 + 1_5 - 1_6 + \cdots + 1_{2n-1} - 1_{2n} + \cdots$$
$$= (1_1 + 1_3 + \cdots + 1_{2n-1}) + (1_{2n+1} - 1_2) + (1_{2n+3} - 1_4) + \cdots$$
$$= n + 0 + 0 + \cdots$$
$$= n$$
The Harmonic Numbers and the Harmonic Series

Definition

\[ H_n = \sum_{i=1}^{n} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n-1} + \frac{1}{n} \]

Theorem

\[ H_n = \ln(n) + \gamma + \varepsilon_n < \ln(n) + 1 \]

for \( \gamma = 0.577 \ldots \) (Euler’s constant) and \( \varepsilon_n \approx \frac{1}{2n} \).

The Harmonic Series and an Ant on a Rubber Band

https://www.youtube.com/watch?v=Dgcoa2yAUfw
The Block Stacking Problem

The Problem

https://www.youtube.com/watch?v=ndmaiH4CnXk

Animation

https://www.youtube.com/watch?v=nTbV3QxGKGg

Demonstration with cards

https://www.youtube.com/watch?v=zP5UX221MVo
Different Ways to Multiply and Divide

Multiplying with a Parabola:

- A geometric variation of $xy = \left(\frac{x+y}{2}\right)^2 - \left(\frac{x-y}{2}\right)^2$.

  https://www.youtube.com/watch?v=eLR3zKxpcso&feature=youtu.be

How to divide by 7?

- Interesting but not necessarily faster ways to divide by 7.

  https://www.youtube.com/watch?v=6fW6ntXUdLY

Slide Rule

- https://www.youtube.com/watch?v=TgQNuDgyjGQ
- https://www.youtube.com/watch?v=waiprjueVpQ


- https://www.youtube.com/watch?v=LmVt_KiyDC4&feature=youtu.be
- https://tapintoteenminds.com/japanese-multiplication/
Playing With Numbers

The 3x + 1 Problem

- Start with any positive number greater than 1.
- As long as the number is greater than 1: Divide it by 2 if it is even or multiply it by 3 and add 1 if it is odd.
- Experimentally the process always ends with 1.

https://www.youtube.com/watch?annotation_id=annotation_1125270707&feature=iv&index=3&list=UUjwOWaOX-c-NeLnj_YGiNEg&src_vid=odfhFSnLaaw&v=m4CjXk_b8zo

The Ducci Challenge

- Start with 4 positive integers arranged as a circle.
- As long as one of the number is positive: replace the 4 numbers with their 4 non-negative differences.
- The process always ends up with 4 zeros.

https://www.youtube.com/watch?v=SivHMQDnbWM
Puzzles

- Crossing a Bridge.
  - [Link](https://ed.ted.com/lessons/can-you-solve-the-bridge-riddle-alex-gendler)

- Ants on a log.
  - [Link](https://www.youtube.com/watch?v=TDp-dGnaxU4&feature=youtu.be)

- The game of Nim.
  - [Link](https://ed.ted.com/lessons/can-you-solve-the-rogue-ai-riddle-dan-finkel)

- Cups and robbers.
  - [Link](https://ed.ted.com/lessons/can-you-solve-the-seven-planets-riddle-edwin-f-meyer)

- Autobiographical numbers.
  - [Link](https://ed.ted.com/lessons/can-you-solve-the-leonardo-da-vinci-riddle-tanya-khovanova)
6 “Classic” Relatively Easy Puzzles

https://www.youtube.com/watch?v=CCV1YVyEI74

Some “Classic” Puzzles

https://www.youtube.com/watch?v=HCp_eN6JSac

“Interview” Puzzles

https://www.youtube.com/watch?v=JzyrbnF0aZQ
https://www.youtube.com/watch?v=yxGf3Knon2I
https://www.youtube.com/watch?v=N4wBTXcx9eI
https://www.youtube.com/watch?v=NgH5MAqOtoo
Illusions

- The vanishing Leprechaun.
  - https://www.youtube.com/watch?v=9Pt8Ot3tZMI

- The vanishing line.
  - https://www.youtube.com/watch?v=3LOCUgquM7M
Geometry Problems

The Art Gallery Problem
https://www.youtube.com/watch?annotation_id=annotation_3737919039&feature=iv&src_vid=m4CjXk_b8zo&v=OERYaFGBPbM

Viviani’s Theorem
https://www.youtube.com/watch?v=uf2ChRpFTZk&feature=youtu.be

The Area of a Square Inside a Square:
https://www.youtube.com/watch?v=nTdqScjZ6kk
\[ \pi = 3.14 \ldots \]

Introduction to \( \pi \)
- https://www.youtube.com/watch?v=txAf7GvaWjM

The Infinite Life of \( \pi \)
- https://www.youtube.com/watch?v=9a5vHXsUvUw

A Brief History of \( \pi \)
- https://www.youtube.com/watch?v=1-JAx3nUwms

Approximating \( \pi \)
- https://www.youtube.com/watch?v=ELetCV_wX_c&feature=youtu.be

How Many Digits of \( \pi \) Are Useful?
- https://www.youtube.com/watch?v=v4eAjcHKlos
\[ \pi = 3.14 \ldots \]

Archimedes’ Method for Approximating \( \pi \):
- https://www.youtube.com/watch?v=zUVx0TQaxME
- https://www.youtube.com/watch?v=DLZMZ-CT7YU

The Area of a Circle is \( \pi r^2 \)
- Converting the circle into a \( r \times 2\pi r \) right triangle:
  - http://animated-mathematics.net/calculus-proof-circle-area.html
The Five Platonic Solids

- Definitions
  - https://www.youtube.com/watch?v=hBATuVPpa3U

- Producing them from an arrangement of circles in the plane:
  - https://www.youtube.com/watch?v=osg4LggbqI

- Animated explanation of relations between polyhedrons:
  - https://www.youtube.com/watch?v=bUw44jvP0T4