Discrete Structures: Introduction to Proofs

Amotz Bar-Noy

Department of Computer and Information Science
Brooklyn College
What is a Proof?

Five out of many definitions

16 words: The cogency of evidence that compels acceptance by the mind of a truth or a fact.

18 words: A formal series of statements showing that if one thing is true something else necessarily follows from it.

20 words: A chain of reasoning using rules of inference, ultimately based on a set of axioms, that lead to a conclusion.

24 words: The process or an instance of establishing the validity of a statement especially by derivation from other statements in accordance with principles of reasoning.

39 words: A sequence of statements, each of which is either validly derived from those preceding it or is an axiom or assumption, and the final member of which, the conclusion, is the statement of which the truth is thereby established.
Proofs from the Book

A story

- Paul Erdős, although an atheist, spoke of “The Book”, an imaginary book in which God had written down the best and most elegant proofs for mathematical theorems.

- He said, “You don’t have to believe in God, but you should believe in The Book”.

- He accused God of keeping the most elegant mathematical proofs to itself.

- When he saw a particularly beautiful mathematical proof he would exclaim: “This one is from The Book”.

Proofs and End of Proofs

Some history

- [https://www.youtube.com/watch?v=S0DSM-EkQE8](https://www.youtube.com/watch?v=S0DSM-EkQE8)

End of Proof

- Q.E.D. or ■ or □
- Latin: Quod Erat Demonstrandum
- English: Which Was to be Demonstrated

Videos

- What is QED? [https://www.youtube.com/watch?v=U6FmQP7fQw8](https://www.youtube.com/watch?v=U6FmQP7fQw8)
- Quite Easily Done: [https://www.youtube.com/watch?v=oTkQD65XMwg](https://www.youtube.com/watch?v=oTkQD65XMwg)
Some Major Techniques

Direct proofs: by construction and/or by exhaustion

Proof by contradiction

Proof by contrapositive

Proof by induction
Some Major Techniques

Direct proofs: by construction and/or by exhaustion
- **Goal:** prove that $Q$ is **TRUE**.
- Start with a **TRUE** $P$.
- Demonstrate that $Q$ **must** follow from $P$.

Proof by contradiction

Proof by contrapositive

Proof by induction
Some Major Techniques

Direct proofs: by construction and/or by exhaustion

Proof by contradiction
- **Goal:** prove that $Q$ is **TRUE**.
- Assume that $Q$ is **FALSE**.
- Demonstrate a **contradiction**.

Proof by contrapositive

Proof by induction
Some Major Techniques

Direct proofs: by construction and/or by exhaustion

Proof by contradiction

Goal: prove that $P$ is TRUE implies that $Q$ is TRUE.

Assume that $Q$ is FALSE.

Demonstrate that $P$ is FALSE must follow from $Q$ is FALSE.

Observe that ($P$ is TRUE implies $Q$ is TRUE) if-and-only-if ($Q$ is FALSE implies $P$ is FALSE).

Proof by induction
Some Major Techniques

Direct proofs: by construction and/or by exhaustion

Proof by contradiction

Proof by contrapositive

Proof by induction

- **Goal:** prove $Q$ is **TRUE** as a function of some ordered set $S$.
- **Basis:** show $Q$ is **TRUE** for a specific initial element $k \in S$.
- **Inductive Hypothesis:** Assume $Q$ is **TRUE** for some element $n \in S$ such that $n \geq k$.
- **Inductive step:** Demonstrate $Q$ is **TRUE** for the element $n + 1$.
- **Conclude:** $Q$ is **TRUE** for all elements in $S$ that are greater than or equal to $k$. 
Resources

Lectures about logic & proofs from “Introduction to Higher Math”

- Problem Solving:
  www.youtube.com/watch?v=CMWFmjlB8v0&list=PLZzHxk_TPOStgPtqRZ6KzmkUQBQ8TSWVX&index=1

- Introduction to Proofs:
  www.youtube.com/watch?v=_x650JU8Uq4&index=2&list=PLZzHxk_TPOStgPtqRZ6KzmkUQBQ8TSWVX

- Propositional Logic:
  www.youtube.com/watch?v=3kzhDsSzKCU&index=3&list=PLZzHxk_TPOStgPtqRZ6KzmkUQBQ8TSWVX

- Proof Techniques:
  www.youtube.com/watch?v=WSb9q4Rj2Bg&index=4&list=PLZzHxk_TPOStgPtqRZ6KzmkUQBQ8TSWVX

Lectures about proofs from TrevTutor

- Direct Proofs:
  https://www.youtube.com/watch?v=YFZzLQN5qOU&feature=youtu.be

- Proof by Case:
  https://www.youtube.com/watch?v=YDKBEOuMuy8&feature=youtu.be

- Proof by Contraposition:
  https://www.youtube.com/watch?v=X-hJ7krLBn0&feature=youtu.be

- Proof by Contradiction:
  https://www.youtube.com/watch?v=sRDwsfNDXak&feature=youtu.be
Resources

A text book


* Section 1.7: “Introduction to Proofs” (pages 80–90).
* Section 1.8: “Proof Methods and Strategy” (pages 92–107).

Two articles about proofs

Basic Proof Techniques:
https://www.cse.wustl.edu/~cytron/547Pages/f14/IntroToProofs_Final.pdf

Mathematical proof (Wikipedia):
https://en.wikipedia.org/wiki/Mathematical_proof
A “Trivial” Proof

Theorem
- \( n^2 \geq n \) for all integers \( n \)

Examples
- \( 0^2 = 0 \geq 0 \)
- \( (-1)^2 = 1^2 = 1 \geq 1 > -1 \)
- \( (-2)^2 = 2^2 = 4 > 2 > -2 \)

Proof
- Prove by case analysis:
  - \( n^2 > 0 > n \) for \( n \leq -1 \)
  - \( n^2 = 0^2 = 0 \geq 0 = n \) for \( n = 0 \)
  - \( n \geq 1 \Rightarrow n \cdot n \geq n \cdot 1 \Rightarrow n^2 \geq n \) for \( n \geq 1 \)
- Q.E.D.
A “Non-Trivial” Proof

**Theorem**

- $n^2$ is the maximum possible product of two positive integers $h$ and $k$ such that $h + k = 2n$

**Remark**

- Without loss of generality assume that $h \leq k$ because otherwise $h$ and $k$ may be exchanged

**Example: $n = 5$**

- For $h = 1$ and $k = 9$ the product is $9 = 1 \times 9$
- For $h = 2$ and $k = 8$ the product is $16 = 2 \times 8$
- For $h = 3$ and $k = 7$ the product is $21 = 3 \times 7$
- For $h = 4$ and $k = 6$ the product is $24 = 4 \times 6$
- For $h = 5$ and $k = 5$ the product is $25 = 5 \times 5$
A “Non-Trivial” Proof

Theorem

- $n^2$ is the maximum possible product of two positive integers $h \leq k$ such that $h + k = 2n$

Proof

- Observe that $k \geq n$ because $k \geq h$ and $k \leq 2n - 1$ since $h \geq 1$
- Assign $k = n + i$ for $0 \leq i \leq n - 1$
- Since $h + k = 2n$, it follows that $h = n - i$
- Therefore the product $h \cdot k$ is
  \[(n - i)(n + i) = n^2 - i^2 \leq n^2\]
- The last inequality follows because $i^2$ is never negative
- Q.E.D.
Theorem

\[ \left\lfloor \frac{m}{2} \right\rfloor \times \left\lceil \frac{m}{2} \right\rceil \] is the maximum possible product of two positive integers \( h \) and \( k \) such that \( h + k = m \)

Remark

Without loss of generality assume that \( h \leq k \) because otherwise \( h \) and \( k \) may be exchanged.

Observation

For an even \( m \) this theorem is reduced to the previous theorem.

Substitute \( m \) with \( 2n \) implies that \( \left\lfloor \frac{m}{2} \right\rfloor = \left\lceil \frac{m}{2} \right\rceil = n \) and therefore \( \left\lfloor \frac{m}{2} \right\rfloor \times \left\lceil \frac{m}{2} \right\rceil = n^2 \)
A More General Theorem

Example: $m = 10$
- For $h = 1$ and $k = 9$ the product is $9 = 1 \times 9$
- For $h = 2$ and $k = 8$ the product is $16 = 2 \times 8$
- For $h = 3$ and $k = 7$ the product is $21 = 3 \times 7$
- For $h = 4$ and $k = 6$ the product is $24 = 4 \times 6$
- For $h = 5$ and $k = 5$ the product is $25 = 5 \times 5$

Example: $m = 11$
- For $h = 1$ and $k = 10$ the product is $10 = 1 \times 10$
- For $h = 2$ and $k = 9$ the product is $18 = 2 \times 9$
- For $h = 3$ and $k = 8$ the product is $24 = 3 \times 8$
- For $h = 4$ and $k = 7$ the product is $28 = 4 \times 7$
- For $h = 5$ and $k = 6$ the product is $30 = 5 \times 6$
Theorem
Let \( a, b, \) and \( c \) be positive integers. If \( a \) divides \( b \) and \( a \) divides \( c \), then \( a \) divides the sum \( b + c \)

Proof
- \( a \) divides \( b \) implies that \( b = a \cdot k \) for some integer \( k \)
- \( a \) divides \( c \) implies that \( c = a \cdot \ell \) for some integer \( \ell \)
- \( b + c = a \cdot k + a \cdot \ell = a \cdot (k + \ell) \)
- Therefore, \( a \) divides \( b + c \) because \( k + \ell \) is an integer
- Q.E.D.

Visual proof
https://www.youtube.com/watch?v=BKS0KIq6kTY&t=4s
A Proof With a “Non-Trivial” Visualizations

Theorem
- A positive number plus its reciprocal is always $\geq 2$
- $x + \frac{1}{x} \geq 2$ for any positive number $x$

Discussion
- Trivial for $x \geq 2$ because $x + \frac{1}{x} \geq x \geq 2$
- Equality for $x = 1$ because $x + \frac{1}{x} = 1 + \frac{1}{1} = 1 + 1 = 2$
- Trivial for $x \leq \frac{1}{2}$ because $x + \frac{1}{x} \geq \frac{1}{x} \geq \frac{1}{1/2} \geq 2$
- What about the range $\frac{1}{2} < x < 2$?
Proof

\[
(x - 1)^2 \geq 0 \quad (\ast \text{squares are always nonnegative} \ast)
\]

\[
x^2 - 2x + 1 \geq 0 \quad (\ast \text{algebra} \ast)
\]

\[
x^2 + 1 \geq 2x \quad (\ast \text{algebra} \ast)
\]

\[
\frac{x^2+1}{x} \geq 2 \quad (\ast \text{algebra} \ast)
\]

\[
x + \frac{1}{x} \geq 2 \quad (\ast \text{algebra} \ast)
\]

Visual proofs for \(x \geq 1\)

- https://www.youtube.com/watch?v=zBUK8C6wSqs
- https://www.youtube.com/watch?v=IghOHBl0Do8
- https://www.youtube.com/watch?v=BkQvnriVblY
The Arithmetic Mean vs. the Geometric Mean

**Theorem**
- Let $x > 0$ and $y > 0$ be two positive real numbers
- Let $\frac{x+y}{2}$ be the **arithmetic mean** of $x$ and $y$
- Let $\sqrt{xy}$ be the **geometric mean** of $x$ and $y$
- Then
  \[
  \frac{x + y}{2} \geq \sqrt{xy}
  \]

**Remarks**
- The inequality is strict ("\(>\)" and not "\(\geq\)") when $x \neq y$
- When $x = y$, both means are equal to $x = y$
The Arithmetic Mean vs. the Geometric Mean

Examples

\[
\frac{8+2}{2} = \frac{10}{2} = 5 > 4 = \sqrt{16} = \sqrt{8 \cdot 2}
\]

\[
\frac{7+5}{2} = \frac{12}{2} = 6 > 5.916 \approx \sqrt{35} = \sqrt{7 \cdot 5}
\]

\[
\frac{9+9}{2} = \frac{18}{2} = 9 = 9 = \sqrt{81} = \sqrt{9 \cdot 9}
\]

Corollary: \( y = \frac{1}{x} \)

- Arithmetic mean (AM): \( \frac{x + \frac{1}{x}}{2} \)
- Geometric mean (GM): \( \sqrt{x \cdot \frac{1}{x}} = \sqrt{1} = 1 \)
- \( AM \geq GM: \frac{x + \frac{1}{x}}{2} \geq 1 \implies x + \frac{1}{x} \geq 2 \)
The Arithmetic Mean vs. the Geometric Mean

Theorem

- Let \( x > 0 \) and \( y > 0 \) be two positive real numbers
- Then \( \frac{x+y}{2} \geq \sqrt{xy} \)

Proof

\[
\begin{align*}
(x - y)^2 & \geq 0 \\
x^2 - 2xy + y^2 & \geq 0 \\
x^2 + 2xy + y^2 & \geq 4xy \\
(x + y)^2 & \geq 4xy \\
\sqrt{(x + y)^2} & \geq \sqrt{4xy} \\
x + y & \geq 2\sqrt{xy} \\
\frac{x + y}{2} & \geq \sqrt{xy}
\end{align*}
\]
The Arithmetic Mean vs. the Geometric Mean

Visual proofs

- **The algebraic proof**
  - [https://www.youtube.com/watch?v=62G9fak1vyk](https://www.youtube.com/watch?v=62G9fak1vyk)
  - [https://www.youtube.com/watch?v=02yM5LqMWUs](https://www.youtube.com/watch?v=02yM5LqMWUs)

- **The traditional geometric proof**
  - [https://www.youtube.com/watch?v=dUhleJQ6QV4](https://www.youtube.com/watch?v=dUhleJQ6QV4)
  - [https://www.youtube.com/watch?v=Pt8IX_Q5U1A&t=5s](https://www.youtube.com/watch?v=Pt8IX_Q5U1A&t=5s)

- **Additional proofs**
  - [https://www.youtube.com/watch?v=IJGwvvQdfgg](https://www.youtube.com/watch?v=IJGwvvQdfgg)
  - [https://www.youtube.com/watch?v=jNx05gXSb1E](https://www.youtube.com/watch?v=jNx05gXSb1E)
  - [https://www.youtube.com/watch?v=HFGSsPbYc-s](https://www.youtube.com/watch?v=HFGSsPbYc-s)
  - [https://www.youtube.com/watch?v=SEPkdPWKd00](https://www.youtube.com/watch?v=SEPkdPWKd00)
Theorem

Let $x > 0$ and $y > 0$ be two positive real numbers
Let $\sqrt{(x^2 + y^2)/2}$ be the root mean square of $x$ and $y$
Let $(x + y)/2$ be the arithmetic mean of $x$ and $y$
Let $\sqrt{xy}$ be the geometric mean of $x$ and $y$
Let $2/(1/x + 1/y)$ be the harmonic mean of $x$ and $y$

Then

$$\sqrt{\frac{x^2 + y^2}{2}} \geq \frac{x + y}{2} \geq \sqrt{xy} \geq \frac{2}{1/x + 1/y}$$

Remark

The inequalities are strict ("\(>\)" and not "\(\geq\)"") when $x \neq y$
When $x = y$, all of the means are equal to $x = y$
Which Mean Do You Mean?

The case \( x = y \)

\[
x = \sqrt{\frac{x^2 + x^2}{2}} = \frac{x + x}{2} = \sqrt{x \cdot x} = \frac{2}{1/x + 1/x}
\]

Example: \( x = 8 \) and \( y = 6 \)

\[
\sqrt{\frac{8^2 + 6^2}{2}} = \sqrt{50} \approx 7.07
\]

\[
\frac{8 + 6}{2} = 7 = 7.0
\]

\[
\sqrt{8 \cdot 6} = \sqrt{48} \approx 6.928
\]

\[
\frac{2}{1/8 + 1/6} = \frac{48}{7} \approx 6.857
\]

On line resource

https://www.youtube.com/watch?v=_gTC1IbHreI&t=11s
Three Classic Proofs

From Hippasus about 2500 years ago
- The square root of 2 is an irrational number

From Euclid about 2300 years ago
- There are infinitely many prime numbers

From Pythagoras about 2550 years ago
- The Pythagorean Theorem
The square root of 2 is an irrational number

Hippasus’ proof

- Assume towards contradiction that \( \frac{p}{q} = \sqrt{2} \) for two positive integers \( p \) and \( q \) that have no common divisor.
- The assumption is equivalent to \( p = \sqrt{2}q \).
- Squaring the equation implies \( p^2 = 2q^2 \).
- \( p \) must be even, because a square of an odd integer is odd and \( 2q^2 \) is even.
- Let \( p = 2r \) for a positive integer \( r \).
- It follows that \( 2q^2 = p^2 = (2r)^2 = 4r^2 \).
- This is equivalent to \( q^2 = 2r^2 \).
- \( q \) must be even, because a square of an odd integer is odd and \( 2r^2 \) is even.
- A contradiction since 2 is a common divisor of both \( p \) and \( q \).
The square root of 2 is an irrational number

Another proof

- Assume towards contradiction that \( \frac{p}{q} = \sqrt{2} \) for two positive integers \( p \) and \( q \) with the smallest possible positive integer \( q \).

\[
\sqrt{2} > 1 \implies \frac{p}{q} > 1 \implies p > q
\]

\[
\sqrt{2} < 2 \implies \frac{p}{q} < 2 \implies p < 2q
\]

- The above inequality is equivalent to \( p - q < q \).

- Hence, the fraction \( r = \frac{2q - p}{p - q} \) is a fraction of two positive integers whose denominator is smaller than the fraction \( \frac{p}{q} \).

- The contradiction is established by proving that \( r = \sqrt{2} \).

\[
r = \frac{2q - p}{p - q} = \frac{2}{\frac{p}{q}} - 1 = \frac{2 - \sqrt{2}}{\sqrt{2} - 1} = \sqrt{2}
\]
Resources

The original proof
- https://www.youtube.com/watch?v=sbGjr_awePE

Proofs with animation
- The second proof: https://www.youtube.com/watch?v=-dbzvN4jnfY
- 5 proofs: https://www.youtube.com/watch?v=zEXcsZo4h0Q

The story of $\sqrt{2}$

Videos with longer explanations
- Original proof: https://www.youtube.com/watch?v=rOGqq10lrzI&feature=youtu.be
- Original proof: https://www.youtube.com/watch?v=TwNfWgMHFmI
- Second proof: https://www.youtube.com/watch?v=VeZleYkcFus
What about $\sqrt{2}^{\sqrt{2}}$?

Theorem
- There exist two irrational numbers $s$ and $t$ such that $s^t$ is rational.

Proof
- If $\sqrt{2}^{\sqrt{2}}$ is rational then set $s = t = \sqrt{2}$.
- Otherwise, set $s = \sqrt{2}^{\sqrt{2}}$ and $t = \sqrt{2}$.

$$s^t = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \left(\sqrt{2}\right)^{\sqrt{2} \cdot \sqrt{2}} = \left(\sqrt{2}\right)^2 = 2$$
There are infinitely many prime numbers

Euclid’s proof

- Let \( p_1 < p_2 < \cdots < p_n \) be \( n \) prime numbers
- Let \( Q = p_1 p_2 \cdots p_n + 1 \)
- If \( Q \) is a prime number, then a new prime prime is found
- Otherwise, \( Q \) is a product of two or more prime numbers
  - The Fundamental Theorem of Arithmetic

- None of these prime numbers can be \( p_1, \ldots, p_n \) because each one of them is a factor of \( Q - 1 \)
- Therefore, the prime factors of \( Q \) are new prime numbers
- The above process can continue forever to find infinitely many prime numbers
There are infinitely many prime numbers

Examples for the Euclid’s proof

- Set of prime numbers \{2, 3\}:
  - \( Q = 2 \cdot 3 + 1 = 7 \)
  - The new prime number is 7

- Set of prime numbers \{3, 5\}:
  - \( Q = 3 \cdot 5 + 1 = 16 = 2 \cdot 2 \cdot 2 \cdot 2 \)
  - The new prime number is 2

- Set of prime numbers \{2, 3, 5\}:
  - \( Q = 2 \cdot 3 \cdot 5 + 1 = 31 \)
  - The new prime number is 31

- Set of prime numbers \{2, 3, 5, 7, 11, 13\}:
  - \( Q = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 1 = 30031 = 59 \cdot 509 \)
  - The new prime numbers are 59 and 509
There are infinitely many prime numbers

A similar proof

- Assume towards \textit{contradiction} that $p$ is the largest prime number
- Let $Q = p! + 1$
- If $Q$ is a prime, then a new prime larger than $p$ exists
- Otherwise, $Q$ is a product of two or more prime numbers
  - The Fundamental Theorem of Arithmetic
- None of the prime numbers $2, 3, \ldots, p$ can be a factor of $Q$ because each one of them is a factor of $Q - 1$
- Therefore, the prime factors of $Q$ are larger than $p$
- A \textit{contradiction} to the assumption about $p$
There are infinitely many prime numbers

Examples for the second proof

- The largest prime number is 3:
  - $Q = 3! + 1 = 7$
  - The new larger prime number is 7

- The largest prime number is 5:
  - $Q = 5! + 1 = 121 = 11 \cdot 11$
  - The new larger prime number is 11

- The largest prime number is 7:
  - $Q = 7! + 1 = 5041 = 71 \cdot 71$
  - The new larger prime number is 71

- The largest prime number is 11:
  - $Q = 11! + 1 = 39916801$
  - The new larger prime number is 39916801
Resources

The original proof by Euclid

- A 2.5 minute video: https://www.youtube.com/watch?v=dQmdHpvfyJs
- A 4 minute video: https://www.youtube.com/watch?v=ZYkZws-23R8
- A 7 minute video: https://www.youtube.com/watch?v=inUkhh8-h-I
- A 7 minute video with more historical background: https://www.youtube.com/watch?v=ctC33JAV4FI

The second proof

- A 1.5 minute video: https://www.youtube.com/watch?v=fOXZgcAsrP8
The Pythagorean Theorem

**Theorem**

- Let $\triangle ABC$ be a right triangle with hypotenuse of length $c$ and two sides of lengths $a$ and $b$
- Then $c^2 = a^2 + b^2$
An algebraic proof

https://www.youtube.com/watch?v=BNCj-K2hd_k

Three proofs in 5 minutes

https://www.youtube.com/watch?v=YompsD1Edtc
“Famous” Proofs

Pythagoras’ proof
https://www.youtube.com/watch?v=4yEyLZUEmQ8&feature=youtu.be

Euclid’s Proof
https://www.youtube.com/watch?v=nxi8gV6_50o

Leonardo da Vinci’s Proof
https://www.youtube.com/watch?v=ZlGaQdNRdqA

James Garfield’s Proof
https://www.youtube.com/watch?v=TsKTteGJpZU
Additional Proofs

Moving objects proofs
- https://www.youtube.com/watch?v=6HbmSRU0uDE
- https://www.youtube.com/watch?v=odfhFSnLaaw
- https://www.youtube.com/watch?v=o1L86rJWkBc

Six proofs without words all of them in 1 minute
- https://www.youtube.com/watch?v=COkhrDbNcuA

Three more 1-minute proof each
- https://www.youtube.com/watch?v=JRjPsuJ7S-g
- https://www.youtube.com/watch?v=q28H5NY5UVU
- https://www.youtube.com/watch?v=pVTLi01nl54
Proofs?

An “experimental” proof

https://www.youtube.com/watch?v=CAkMUdeB06o
https://www.youtube.com/watch?v=_e6w5GtkcGI

An Origami Proof

https://www.youtube.com/watch?v=z61L83wl31E
Beyond the Traditional Theorem

Replacing a right triangle with two lunes

https://www.youtube.com/watch?v=p3_7b0jiWm8

Visualization and generalizations

https://www.youtube.com/watch?v=p-0SOWbzUYI
Texts About Proofs

370 proofs from 900 B.C. to 1940 A.D.
- An 1940 book: “The Pythagorean Proposition” by Elisha S. Loonis

Online list of 122 proofs
- http://www.cut-the-knot.org/pythagoras/

Wikipedia: More about the Theorem and its proofs
- https://en.wikipedia.org/wiki/Pythagorean_theorem
The Pythagorean Trigonometric Identity

**Theorem**
- Let $\triangle ABC$ be a right triangle with hypotenuse $c$ and sides $a$ and $b$
- Let $\theta$ be the angle between the side $b$ and the hypotenuse $c$
- Then $\sin^2(\theta) + \cos^2(\theta) = 1$

**Proof**
- $\sin(\theta) = \frac{a}{c}$, $\cos(\theta) = \frac{b}{c}$
- $\sin^2(\theta) + \cos^2(\theta) = \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = \frac{a^2+b^2}{c^2}$
- $c^2 = a^2 + b^2 \iff \sin^2(\theta) + \cos^2(\beta) = 1$

**Resources**
- [https://www.youtube.com/watch?v=MyO2MFJbfI4](https://www.youtube.com/watch?v=MyO2MFJbfI4)
- [https://www.youtube.com/watch?v=Zb0An8dxDas](https://www.youtube.com/watch?v=Zb0An8dxDas)
The Pythagorean Tree

Constructing the tree

The construction of the Pythagoras tree begins with a square. Upon this square are constructed two squares, each scaled down by a linear factor of $\sqrt{2}/2$, such that the corners of the squares coincide pairwise. The same procedure is then applied recursively to the two smaller squares, ad infinitum.

Resources

- [https://en.wikipedia.org/wiki/Pythagoras_tree_(fractal)#/media/File:Pythagoras_tree_1_1_13.svg](https://en.wikipedia.org/wiki/Pythagoras_tree_(fractal)#/media/File:Pythagoras_tree_1_1_13.svg)
- [https://www.youtube.com/watch?v=0Ih8LuIJ_go](https://www.youtube.com/watch?v=0Ih8LuIJ_go)
1 = 2

\[ a = b \]
\[ a^2 = ab \]
\[ a^2 + a^2 = a^2 + ab \]
\[ 2a^2 = a^2 + ab \]
\[ 2a^2 - 2ab = a^2 + ab - 2ab \]
\[ 2a^2 - 2ab = a^2 - ab \]
\[ 2(a^2 - ab) = 1(a^2 - ab) \]
\[ 2 = 1 \]

What is wrong?
Division by zero is forbidden!
False Proofs

1 = 2

\[
\begin{align*}
a &= b \\
a^2 &= ab \\
a^2 + a^2 &= a^2 + ab \\
2a^2 &= a^2 + ab \\
2a^2 - 2ab &= a^2 + ab - 2ab \\
2a^2 - 2ab &= a^2 - ab \\
2(a^2 - ab) &= 1(a^2 - ab) \\
2 &= 1
\end{align*}
\]

What is wrong?

- Division by zero is **forbidden**!
False Proofs

\[ 2 + 2 = 5 \]

\[
\begin{align*}
-20 & = -20 \\
16 - 36 & = 25 - 45 \\
16 - 36 + \frac{81}{4} & = 25 - 45 + \frac{81}{4} \\
\left(4 - \frac{9}{2}\right)^2 & = \left(5 - \frac{9}{2}\right)^2 \\
4 - \frac{9}{2} & = 5 - \frac{9}{2} \\
4 & = 5 \\
2 + 2 & = 5
\end{align*}
\]

What is wrong?

\( (−a)^2 \) does not imply \( −a = a \)
False Proofs

2 + 2 = 5

-20 = -20
16 - 36 = 25 - 45
16 - 36 + \frac{81}{4} = 25 - 45 + \frac{81}{4}
\left(4 - \frac{9}{2}\right)^2 = \left(5 - \frac{9}{2}\right)^2
4 - \frac{9}{2} = 5 - \frac{9}{2}
4 = 5
2 + 2 = 5

What is wrong?

\bullet \ (-a)^2 = a^2 \text{ does not imply that } -a = a
False Proofs

1 = -1

1 = \sqrt{1}
1 = \sqrt{-1} \cdot -1
1 = \sqrt{-1} \cdot \sqrt{-1}
1 = i \cdot i
1 = i^2
1 = -1
False Proofs

1 = −1

\[
\begin{align*}
1 &= \sqrt{1} \\
1 &= \sqrt{-1} \cdot -1 \\
1 &= \sqrt{-1} \cdot \sqrt{-1} \\
1 &= i \cdot i \\
1 &= i^2 \\
1 &= -1
\end{align*}
\]

What is wrong?

- \( \sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b} \) only if \( a \geq 0 \) and \( b \geq 0 \)

Online resource

- 2 = 0: https://www.youtube.com/watch?v=1irvvZzbJkU
A False Visual Proof

60=59=58?

https://www.youtube.com/watch?v=iMgFDhpa000