Discrete Structures: Sets

Amotz Bar-Noy

Department of Computer and Information Science
Brooklyn College
Russell’s Paradox:

- Let $R$ be the set of all sets that are not members of themselves.
- If $R$ is not a member of itself, then its definition dictates that $R$ must contain itself,
- and if $R$ contains itself, then $R$ contradicts its own definition as the set of all sets that are not members of themselves.
- Symbolically: $(R = \{x \mid x \notin x\}) \implies (R \in R \iff R \notin R)$.

https://www.youtube.com/watch?v=83X1YWDXdDA
Paradoxes

Russell's Paradox:
- Let $R$ be the set of all sets that are not members of themselves.
- If $R$ is not a member of itself, then its definition dictates that $R$ must contain itself,
- and if $R$ contains itself, then $R$ contradicts its own definition as the set of all sets that are not members of themselves.
- Symbolically: \((R = \{x | x \notin x\}) \implies (R \in R \iff R \notin R)\).

https://www.youtube.com/watch?v=83X1YWDXdDA

The barber paradox:
- https://www.youtube.com/watch?v=qGjznKRLClA&feature=youtu.be
- https://www.youtube.com/watch?v=qQs2ZHV_WBk
Sets, Sets, and Sets

A 30-minute introductory video:
https://www.youtube.com/watch?v=HoWC0GQWGaa&list=PLZzHxk_TPosRgPtwRZ6KzmkUQBQ8TSWVX&index=6&t=546s

On-line tutorials:
- Short tutorial:

Videos:
- Set theory basics: https://www.youtube.com/watch?v=IqX0emlxVEU
- Types of sets: https://www.youtube.com/watch?v=VBzlvKP-2yI

Amotz Bar-Noy (Brooklyn College)
The De Morgan’s Laws

Notations for the negation operation

\[ X' \equiv \overline{X} \equiv \neg X \, . \]

The two sets case

\[ \overline{A \cup B} = \overline{A} \cap \overline{B} \]
\[ \overline{A \cap B} = \overline{A} \cup \overline{B} \]

Proofs with Venn Diagrams

- https://www.youtube.com/watch?v=wFBFFIb2wa8
- https://www.youtube.com/watch?v=oOx7FSzSav4

The three sets case

\[ \neg (A \cup B \cup C) = (\neg A) \cap (\neg B) \cap (\neg C) \]
\[ \neg (A \cap B \cap C) = (\neg A) \cup (\neg B) \cup (\neg C) \]
The General De Morgan’s Laws

The Complement of the Union is the Intersection of the Complements

\[
\left( \bigcup_{i=1}^{n} A_i \right)' = (A_1 \cup A_2 \cup \cdots \cup A_n)' = A_1' \cap A_2' \cap \cdots \cap A_n' = \bigcap_{i=1}^{n} A_i'
\]

The Complement of the Intersection is the Union of the Complements

\[
\left( \bigcap_{i=1}^{n} A_i \right)' = (A_1 \cap A_2 \cap \cdots \cap A_n)' = A_1' \cup A_2' \cup \cdots \cup A_n' = \bigcup_{i=1}^{n} A_i'
\]
The Complement of the Union is the Intersection of the Complements

Let \( L = (A_1 \cup A_2 \cup \cdots \cup A_n)' \) and \( R = A'_1 \cap A'_2 \cap \cdots \cap A'_n \).

Prove that \( x \in L \iff x \in R \).

\[
\begin{align*}
x \in L & \iff x \in (A_1 \cup A_2 \cup \cdots \cup A_n)' \\
& \iff x \notin (A_1 \cup A_2 \cup \cdots \cup A_n) \\
& \iff \forall i \in \{1, \ldots, n\} \ x \notin A_i \\
& \iff \forall i \in \{1, \ldots, n\} \ x \in A'_i \\
& \iff x \in (A'_1 \cap A'_2 \cap \cdots \cap A'_n) \\
& \iff x \in R
\end{align*}
\]
The Complement of the Intersection is the Union of the Complements

Let \( L = (A_1 \cap A_2 \cap \cdots \cap A_n)' \) and \( R = A'_1 \cup A'_2 \cup \cdots \cup A'_n \).

Prove that \( x \in L \iff x \in R \).

\[
\begin{align*}
  x \in L & \iff x \in (A_1 \cap A_2 \cap \cdots \cap A_n)' \\
& \iff x \notin (A_1 \cap A_2 \cap \cdots \cap A_n) \\
& \iff \exists i \in \{1, \ldots, n\} x \notin A_i \\
& \iff \exists i \in \{1, \ldots, n\} x \in A'_i \\
& \iff x \in (A'_1 \cup A'_2 \cup \cdots \cup A'_n) \\
& \iff x \in R
\end{align*}
\]
The Principle of Inclusion Exclusion

Setting
- Let \( A_1, A_2, \ldots, A_n \) be \( n \) sets.
- let \( U = \bigcup_{i=1}^n A_i \) be the union of all the \( n \) sets.
- **Goal:** Find the **cardinality** of \( U \) \((u = |U|)\).

Method
- Include (**add**) the cardinalities of the sets.
- Exclude (**subtract**) the cardinalities of the pairwise intersections.
- Include (**add**) the cardinalities of the triple-wise intersections.
- Exclude (**subtract**) the cardinalities of the quadruple-wise intersections.
- Include (**add**) the cardinalities of the quintuple-wise intersections.
- Continue, until the cardinality of the \( n \)-tuple-wise intersection is
  - included (**added**) if \( n \) is odd or
  - excluded (**subtracted**) if \( n \) even.
The Principle of Inclusion Exclusion for 2, 3, and 4 sets

Two Sets

\[ |A \cup B| = |A| + |B| - |A \cap B| \]

Three Sets

\[ |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \]

Four Sets

\[ |A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| \]

\[ - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \]

\[ + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \]

\[ - |A \cap B \cap C \cap D| \]
Venn Diagrams with Three Categories
- https://www.youtube.com/watch?v=LIzIhmK1YPk

The Trev Tutor Course: Inclusion Exclusion Principle
- https://www.youtube.com/watch?v=GS7dIWA6Hpo&feature=youtu.be

Three parts tutorial
- The principle: https://www.youtube.com/watch?v=4qd9JkYVGBU
- The proof: https://www.youtube.com/watch?v=y9123Dx6cJA
- Basic example: https://www.youtube.com/watch?v=Xd2ZGvMqXsc
The Numbers

- **An integer number**: One of the (counting) numbers \( \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \).
  - \( \mathbb{Z} (\mathbb{Z}^+, \mathbb{Z}^-) \) – The set of all (positive, negative) integers.

- **A rational number**: A number that can be expressed as a fraction \( p/q \) where \( p \) (the numerator) and \( q \) (the denominator) are integers and \( q \neq 0 \).
  - \( \mathbb{Q} (\mathbb{Q}^+, \mathbb{Q}^-) \) – The set of all (positive, negative) rationals.

- **A real number**: A number of the form \( n.b_1b_2b_3\ldots \) where \( n \) is an integer and \( b_i \in \{0, 1\} \) for all \( i = 1, 2, \ldots \).
  - \( \mathbb{R} (\mathbb{R}^+, \mathbb{R}^-) \) – The set of all (positive, negative) reals.
Equality Between Sets

- **Definition:** Two finite or infinite sets $S$ and $T$ have the same
cardinality ($S \approx T$) if and only if there is a one-to-one mapping
$f : S \rightarrow T$ from $S$ to $T$.

- **Mapping:** For all $s \in S$ there exists $t \in T$ such that $f(s) = t$.

- **From $S$:** $s_1 \neq s_2 \Rightarrow f(s_1) \neq f(s_2)$.

- **To $T$:** For all $t \in T$ there exists $s \in S$ such that $f(s) = t$.

- **Remark:** For finite sets $S = T$ is equivalent to $S \approx T$. 
Examples

- $\mathbb{Z} \approx \mathbb{Z}^+$ although it “seems” like $\mathbb{Z}$ is twice as large as $\mathbb{Z}^+$.

- $\mathbb{Q} \approx \mathbb{Z}$ although there are infinite number of rational numbers between any two consecutive integers.

- $\mathbb{R} \not\approx \mathbb{Q}$ although there are infinite number of rational numbers between any two real numbers.
  
  Since $\mathbb{Q} \subset \mathbb{R}$ (every rational number is also a real number), it follows that $\mathbb{Q} \leq \mathbb{R}$ from a cardinality point of view.
The mapping $f : \mathbb{Z} \rightarrow \mathbb{Z}^+$:
- $f(n) = 2n$ for a positive integer $n$.
- $f(-n) = 2n + 1$ for a positive integer $n$.
- $f(0) = 1$. 
\[ \mathbb{Z}^+ \approx \mathbb{Q}^+ \]

**The mapping:** From \( \mathbb{Z}^+ \) to \( \mathbb{Q}^+ \) by following the arrows and skipping the already assigned fractions.
Assume that a mapping \( f : \mathbb{Z}^+ \rightarrow \mathbb{R}^+ \) exists.

\[
\begin{align*}
    f(1) &= n_1.b_1^1 b_2^1 b_3^1 \ldots b_i^1 \ldots \\
    f(2) &= n_2.b_1^2 b_2^2 b_3^2 \ldots b_2^2 \ldots \\
    f(3) &= n_3.b_1^3 b_2^3 b_3^3 \ldots b_3^3 \ldots \\
    &\vdots \\
    f(i) &= n_i.b_1^i b_2^i b_3^i \ldots b_i^i \ldots
\end{align*}
\]

Let \( x = n_1.\bar{b}_1^1 \bar{b}_2^1 \bar{b}_3^1 \ldots \bar{b}_i^1 \ldots \) where \( \bar{b}_j^i = 1 - b_j^i \).

\( x \neq f(i) \) for any \( i = 1, 2, \ldots \).

Therefore \( f \) is not a valid mapping: a contradiction.
The Zeno Paradoxes

- The dichotomy paradox:

- Achilles and the Tortoise I:
  - https://www.youtube.com/watch?v=skM37PcZmWE

- Achilles and the Tortoise II:
  - https://www.youtube.com/watch?v=3vNlf2zGLaE

- Achilles and the Tortoise and beyond:
  - https://www.youtube.com/watch?v=NCtw5f6XPf4
Infinity and Beyond

How Big is Infinity
- https://www.youtube.com/watch?v=UPA3bwVVzGI&feature=youtu.be

Infinity is bigger than you think!
- https://www.youtube.com/watch?v=elvOZm0d4H0

The Infinite Hotel
- https://www.youtube.com/watch?v=Uj3_KqkI9Zo
Empty Hotel with Infinite Number of Guests

- Infinite number of buses each with infinite number of passengers arrive at 11:00 then 11:30 then 11:45 and so on.
- With each arrival the occupant of room $i$ moves to room $2i$ and the new guests get the odd number rooms.
- At 12:00 the owner of the hotel finds all the rooms to be empty!

Infinite IDs

- If each person gets an infinite binary ID.
- Then there exists a person that is not in the hotel.
- Its ID is different in its $i$’s coordinate from the $i$’s coordinate of the guest of room $i$. 
$1 - 1 + 1 - 1 + 1 - \cdots = \ ???$
\[ 1 - 1 + 1 - 1 + 1 - \cdots = ??? \]

0 = 1?

https://www.youtube.com/watch?v=aOxMTgCORhA

Manipulating the sequence \( S = 1 - 1 + 1 - 1 + 1 + \cdots \)

- \( S = (1 - 1) + (1 - 1) + \cdots = 0 \)
- \( S = 1 + (-1 + 1) + (-1 + 1) + \cdots = 1 \)
- \( S = n + (1 - 1) + (1 - 1) + \cdots = n \)

\[ S = 1_1 - 1_2 + 1_3 - 1_4 + 1_5 - 1_6 + \cdots + 1_{2n-1} - 1_{2n} + \cdots \]
\[ = (1_1 + 1_3 + \cdots + 1_{2n-1}) + (1_{2n+1} - 1_2) + (1_{2n+3} - 1_4) + \cdots \]
\[ = n + 0 + 0 + \cdots \]
\[ = n \]
Functions

Relations: https://www.youtube.com/watch?v=j5cj_4Mk9_M&list=PLZzHxk_TPOStgPtqRZ6KzmkUQBQ8TSWXV&index=9&t=1346s

Functions: https://www.youtube.com/watch?v=Tk5_B7w5fiY&list=PLZzHxk_TPOStgPtqRZ6KzmkUQBQ8TSWXV&index=10&t=891s