Graph Algorithms

Chromatic Polynomials
Chromatic Polynomials – Definition

★ $G$ – a simple labelled graph with $n$ vertices and $m$ edges.

★ $k$ – a positive integer.

★ $P_G(k)$ – number of different ways of coloring the vertices of $G$ with $k$ colors.

★ $P_G(k)$ is an integer function (polynomial) of $k$:
  - If $\chi(G) > k$ then $P_G(k) = 0$.
  - If $\chi(G) \leq k$ then $P_G(k) > 0$.

$\Rightarrow \chi(G)$ is the smallest $k$ such that $P_G(k) > 0$. 

Graph Algorithms
3 Vertices and 0 Edges

\[ P_G(k) = k^3 \]

\( k \) ways to color independently each of the vertices \( u, v, w \).
3 Vertices, 0 Edges, and 2 colors

\[ P_G(2) = 2^3 = 8 \]
3 Vertices and 1 Edge

- $k$ ways to color $v$; $k$ ways to color $u$; $k - 1$ ways to color $w$ that cannot get the color of $u$.

$$P_G(k) = k^2(k - 1)$$
$$= k^3 - k^2$$
3 Vertices, 1 Edge, and 2 colors

\[ P_G(2) = 2^3 - 2^2 = 4 \]
3 Vertices and 2 Edges

- $k$ ways to color $u$; $k - 1$ ways to color $v$ that cannot get the color of $u$; $k - 1$ ways to color $w$ that cannot get the color of $u$.

\[
P_G(k) = k(k - 1)^2
= k^3 - 2k^2 + k
\]
3 Vertices, 2 Edges, and 2 colors

\[ P_G(2) = 2^3 - 2 \cdot 2^2 + 2 = 2 \]
$3$ Vertices and $3$ Edges

$\star \ k$ ways to color $u$; $k - 1$ ways to color $v$ that cannot get the color of $u$; $k - 2$ ways to color $w$ that cannot get the colors of $u$ and $v$.

\[
P_G(k) = k(k - 1)(k - 2)
= k^3 - 3k^2 + 2k
= (k - 1)^3 - (k - 1)
\]
3 Vertices, 3 Edges, and 3 colors

\[ P_G(3) = 3 \cdot 2 \cdot 1 = 6 \]
$k$ ways to color $v$; $k - 1$ ways to color $x$ that cannot get the color of $v$; $k - 2$ ways to color $u$ that cannot get the colors of $v$ and $x$; $k - 2$ ways to color $w$ that cannot get the colors of $v$ and $x$. 

Graph Algorithms
4 Vertices and 5 Edges

\[
P_G(k) = k(k - 1)(k - 2)^2
= k^4 - 5k^3 + 8k^2 - 4k
\]
4 Vertices and 5 Edges

- $k(k - 1)$ ways to color $u$ and $w$ with different colors; $k - 2$ ways to color $v$ that cannot get the colors of $u$ and $w$; $k - 3$ ways to color $x$ that cannot get the colors of $u$, $v$, and $w$.
- $k$ ways to color $u$ and $w$ with the same color; $k - 1$ ways to color $v$ that cannot get the color of $u$ and $w$; $k - 2$ ways to color $x$ that cannot get the colors of $u$, $v$, and $w$. 
\[ P_G(k) = k(k - 1)(k - 2)(k - 3) + k(k - 1)(k - 2) \]
\[ = k(k - 1)(k - 2)^2 \]
\[ = k^4 - 5k^3 + 8k^2 - 4k \]
\( P_G(k) \) – Properties

\( P_G(k) \) is a polynomial in \( k \):

\[
P_G(k) = a_n k^n + a_{n-1} k^{n-1} + \cdots + a_1 k + a_0
\]

\( \star \) The degree of the polynomial is \( n \): the number of vertices in the graph.

\( \star \) All the coefficients are integers (could be 0).

\( \star \) The coefficient of \( k^n \) is 1: \( a_n = 1 \).

\( \star \) The coefficient of \( k^0 \) is 0: \( a_0 = 0 \).

\( \star \) The coefficient of \( k^{n-1} \) is \( -m \): \( a_{n-1} = -m \).
Signs of coefficients alternate between positive and negative.

\[ P_G(k) = k^n - mk^{n-1} + b_{n-2}k^{n-2} \cdots + b_1 k + 0 \]

for non-negative coefficients \( b_1, \ldots, b_{n-2} \).

For a graph with at least one edge, the sum of the coefficients is 0.

\[ a_n + a_{n-1} + \cdots + a_1 = 0 \]

for positive or negative or zero coefficients \( a_1, \ldots, a_{n-2} \).
Null Graphs – $N_n$

The null graph $N_n$ has $n$ vertices and no edges.

Each vertex can be colored independently with $k$ colors.

\[ P_{N_n}(k) = k^n = k^n - 0 \cdot k^{n-1} + 0 \cdot k^{n-2} - \cdots + 0 \]
The Complete graph $K_n$ has $n$ vertices and all possible edges: $m = \frac{n(n-1)}{2}$.

The first vertex can be colored with $k$ colors, the second with $k - 1$ colors . . . and the last with $k - n + 1$ colors.

\[
P_{K_n}(k) = k(k - 1)(k - 2) \cdots (k - n + 1) \\
= k^n - (1 + 2 + \cdots + (n - 1))k^{n-1} + \cdots + 0
\]
The Star graph $S_n$ has $n - 1$ edges. A root vertex is connected to the rest of the $n - 1$ vertices each connected only to the root.

The root can be colored with $k$ colors and each of the other $n - 1$ vertices can be colored with $k - 1$ colors.

$$P_{S_n}(k) = k(k - 1)^{n-1}$$
$$= k^n - (n - 1)k^{n-1} + \cdots + 0$$
The Path graph $P_n$ has $n - 1$ edges. The vertices are connected as a path of length $n - 1$ edges.

The first vertex can be colored with $k$ colors and each one of the other $n - 1$ vertices, in order, can be colored with $k - 1$ colors.

\[
P_{P_n}(k) = k(k - 1)^{n-1}
\]
\[
= k^n - (n - 1)k^{n-1} + \cdots + 0
\]
A tree $T_n$ is an acyclic (connected) graph with $n$ vertices and $n - 1$ edges.

The root can be colored with $k$ colors and each of the other $n - 1$ vertices can be colored with $k - 1$ colors if it is colored after its parent and before all of its children.

$$P_{T_n}(k) = k(k - 1)^{n-1} = k^n - (n - 1)k^{n-1} + \cdots + 0$$
Finding the Chromatic Polynomial

★ Let $G$ be a labelled graph with $n$ vertices.
★ Suppose that there are $f(r)$ different ways to partition $G$ into $r$ independent sets.
★ Each color class in any coloring is an independent set.
⇒ A given partition into $r$ independent sets can be colored in $k(k - 1) \cdots (k - r + 1)$ ways with $k$ colors where each independent set gets a different color.

$$P_G(k) = \sum_{r=1}^{n} f(r) \cdot k(k - 1) \cdots (k - r + 1)$$
The Cycle $C_4$\

\[
P_{C_4}(k) = f(1)k + f(2)k(k - 1) + f(3)k(k - 1)(k - 2) + f(4)k(k - 1)(k - 2)(k - 3)
\]

\[
= k^2 - k + 2k^3 - 6k^2 + 4k + k^4 - 6k^3 + 11k^2 - 6k
\]

\[
= k^4 - 4k^3 + 6k^2 - 3k
\]

\[
= (k - 1)^4 + (k - 1)
\]

\* $f(1) = 0$  $f(2) = 1$  $f(3) = 2$  $f(4) = 1$. 
The Coefficients in $P_G(k)$

$$P_G(k) = \sum_{r=1}^{n} f(r) \cdot k(k-1) \cdots (k-r+1)$$

★ $P_G(k)$ is a polynomial.

★ All the coefficients in $P_G(k)$ are integers.

★ The degree of $P_G(k)$ is $n$ because $f(r) = 0$ for $r > n$.

★ The coefficient of $k^n$ is 1 because $f(n) = 1$.

★ The coefficient of $k^0$ is 0. because $f(0) = 0$. 

Graph Algorithms
The Sum of All the Coefficients

**Lemma:** Let $G$ be a graph with $n$ vertices and at least 1 edge. Then the sum of all the coefficients in $P_G(k)$ is 0.

**Proof:**

* It is impossible to color $G$ with 1 color $\Rightarrow P_G(1) = 0$.
* By definition, $P_G(1) = a_n 1^n + a_{n-1} 1^{n-1} + \cdots + a_1 1^1$.
* Therefore, $\sum_{i=1}^{n} a_i = 0$. 
The Coefficient of $k^{n-1}$

Lemma: Let $G$ be a graph with $n$ vertices and $m$ edges. Then the coefficient of $k^{n-1}$ in $P_G(k)$ is $-m$.

Proof:
- The coefficient of $k^{n-1}$ in $k(k-1) \cdots (k-n+1)$ is $-\frac{1}{2}n(n-1)$ and $f(n) = 1$.
- The coefficient of $k^{n-1}$ in $k(k-1) \cdots (k-n+2)$ is $1$ and $f(n-1)$ is equal to the number of non-adjacent pairs of vertices: $f(n-1) = \frac{1}{2}n(n-1) - m$.
- The coefficient of $k^{n-1}$ in $k(k-1) \cdots (k-r+1)$ for $r < n-1$ is $0$.
- The coefficient of $k^{n-1}$ in $P_G(k)$ is
  $$1 \cdot (-\frac{1}{2}n(n-1)) + (\frac{1}{2}n(n-1) - m) \cdot 1 = -m.$$
Disconnected Graphs

**Lemma:** Let $G_1, G_2, \ldots, G_h$ be the $h$ connected components of $G$. Then $P_G(k) = P_{G_1}(k) \cdot P_{G_2}(k) \cdots P_{G_h}(k)$.

**Proof:** The colorings of the $h$ connected components are independent.

**Example:**

![Disconnected Graph Example](image)

$$P_G(k) = P_{K_2}(k) \cdot P_{K_1}(k)$$

$$= k(k - 1)k$$

$$= k^3 - k^2$$
**Disconnected Graphs**

**Corollary:** If $G$ is composed of $h$ connected components, then the coefficient of $k^\ell$ for $\ell < h$ is 0.

**Proof:** The coefficient of $k^0$ is 0 in $P_{G_i}(k)$ for all $1 \leq i \leq h$ ⇒ in the product of the $h$ polynomials the smallest degree with a positive coefficient is $k^h$.

**Example:** The null graph $N_n$ has $n$ connected components ⇒ all the coefficients are 0 except the coefficient of $k^n$ which is 1 ⇒ $P_{N_n}(k) = k^n$. 
Lemma: Assume that the chromatic polynomial of a graph $G$ is $P_G(k) = k(k - 1)^{n-1}$. Then $G$ is a tree with $n$ vertices.

Proof:

★ The degree of $P_G(k)$ is $n \Rightarrow G$ has $n$ vertices.
★ The coefficient of $k^{n-1}$ is $-(n - 1) \Rightarrow G$ has $n - 1$ edges.
★ The coefficient of $k$ in a disconnected graph is 0 and the coefficient of $k$ in $k(k - 1)^{n-1}$ is greater than 0 $\Rightarrow G$ is connected.
★ A connected graph with $n$ vertices and $n - 1$ edges is a tree.
Three Transformations

Delete an edge: \( G - (u, v) \) is \( G \) without the old edge \((u, v)\).

Add an edge: \( G + (u, v) \) is \( G \) with the new edge \((u, v)\).

Contract 2 vertices: \( G/(u, v) \) is \( G \)

★ without the old vertices \( u \) and \( v \) and all the edges that are connected to them,

★ with a new vertex \( uv \) that is connected to all the neighbors of \( u \) and \( v \).
First Recursive Formula for $P_G(k)$

**Theorem:** For any non-adjacent vertices $u$ and $v$,

$$P_G(k) = P_{G+(u,v)}(k) + P_{G/(u,v)}(k)$$

**Proof:**

- $P_{G+(u,v)}(k)$ covers all the colorings in which the color of $u$ is different than the color of $v$.
- $P_{G/(u,v)}(k)$ covers all the colorings in which the color of $u$ is the same as the color of $v$. 
The Null Graph $N_2$

\[ P_{N_2}(k) = P_{K_2}(k) + P_{K_1}(k) \]
\[ = k(k - 1) + k \]
\[ = k^2 \]
First Recursive Formula for $P_G(k)$

**Corollary:** The chromatic polynomial of $G$ is a linear combination of chromatic polynomials of complete graphs with at most $n$ vertices,

$$P_G(k) = P_{K_n}(k) + b_{n-1}P_{K_{n-1}}(k) + \cdots + b_1P_{K_1}(k)$$

for some non-negative integers $b_{n-1}, \ldots, b_1$. 

Graph Algorithms
The Cycle $C_4$

\[ P_{C_4}(k) = P_{K_4}(k) + 2P_{K_3}(k) + P_{K_2}(k) \]
\[ = k(k - 1)(k - 2)(k - 3) + 2k(k - 1)(k - 2) + k(k - 1) \]
\[ = k^4 - 4k^3 + 6k^2 - 3k \]
\[ = (k - 1)^4 + (k - 1) \]
Theorem: For any edge \((u, v)\),

\[ P_G(k) = P_{G-(u,v)}(k) - P_{G/(u,v)}(k) \]

Proof:

\(<\>
\begin{itemize}
  \item \( P_{G-(u,v)}(k) \) covers all the colorings in which the color of \( u \) is the same as the color of \( v \) and all the colorings in which the color of \( u \) is different than the color of \( v \).
  \item \( P_{G/(u,v)}(k) \) covers all the colorings in which the color of \( u \) is the same as the color of \( v \).
\end{itemize>
The Complete Graph $K_2$

\[ P_{K_2}(k) = P_{N_2}(k) - P_{N_1}(k) \]
\[ = k^2 - k \]
\[ = k(k - 1) \]
Corollary: The chromatic polynomial of $G$ is a linear combination of chromatic polynomials of null graphs with at most $n$ vertices,

$$P_G(k) = P_{N_n}(k) + c_{n-1}P_{N_{n-1}}(k) + \cdots + c_1P_{N_1}(k)$$

for integers (positive, negative, or 0) $c_{n-1}, \ldots, c_1$. 
The Cycle $C_4$

$$P_{C_4}(k) = P_{N_4}(k) - 4P_{N_3}(k) + 6P_{N_2}(k) - 3P_{N_1}(k)$$

$$= k^4 - 4k^3 + 6k^2 - 3k$$

$$= (k - 1)^4 + (k - 1)$$
The Chromatic Polynomial of the Cycle $C_n$

\[ P_{C_n}(k) = P_{P_n}(k) - P_{C_{n-1}}(k) \]
\[ = P_{P_n}(k) - P_{P_{n-1}}(k) + P_{C_{n-2}}(k) \]
\[ \vdots \]
\[ = P_{P_n}(k) - P_{P_{n-1}}(k) + \cdots + P_{P_2}(k) \]
\[ = k(k - 1)^{n-1} - k(k - 1)^{n-2} + \cdots + k(k - 1) \]
Proposition: For \( n \geq 3 \), \( P_{C_n}(k) = (k - 1)^n + (-1)^n(k - 1) \).

Proof:

\[ P_{C_3} = k(k-1)(k-2) = k^3 - 3k^2 + 2k = (k-1)^3 - (k-1) \]
\[ P_{C_4} = k^4 - 4k^3 + 6k^2 - 3k = (k-1)^4 + (k-1) \]

\[
P_{C_n}(k) = P_{P_n}(k) - P_{C_{n-1}}(k) = k(k - 1)^{n-1} - (k - 1)^{n-1} - (-1)^{n-1}(k - 1) = (k - 1)^n + (-1)^n(k - 1)
\]
The Chromatic Polynomial of the Broken Wheel $B_n$

\[ P_{B_2} = k(k - 1). \]

\[ P_{B_3} = k(k - 1)(k - 2). \]

\[ P_{B_4} = k(k - 1)(k - 2)^2. \]
The Chromatic Polynomial of the Broken Wheel $B_n$

\[
P_{B_n}(k) = P_{B'_{n-1}}(k) - P_{B_{n-1}}(k)
= (k - 1)P_{B_{n-1}}(k) - P_{B_{n-1}}(k)
= (k - 2)P_{B_{n-1}}(k)
\vdots
= (k - 2)^{n-2}P_{B_2}(k)
= k(k - 1)(k - 2)^{n-2}
\]
The Chromatic Polynomial of the Wheel $W_n$

$\star \ P_{W_4} = k(k - 1)(k - 2)(k - 3)$.

$\star \ P_{W_5} = k(k - 1)(k - 2)(k^2 - 5k + 7)$.

$\star \ P_{W_6} = k(k - 1)(k - 2)(k - 3)(k^2 - 4k + 5)$. 
The Chromatic Polynomial of the Wheel $W_n$

\[
P_{W_n}(k) = P_{B_n}(k) - P_{W_{n-1}}(k) \\
= k(k - 1)(k - 2)^{n-2} - P_{W_{n-1}}(k) \\
= k(k - 1) \left[ (k - 2)^{n-2} - (k - 2)^{n-3} \right] + P_{W_{n-2}}(k) \\
\vdots \\
= k(k - 1) \left[ (k - 2)^{n-2} - (k - 2)^{n-3} \cdots + (k - 2) \right] \\
= k(k - 2)^{n-1} + (-1)^{n-1} k(k - 2) \\
= k(k - 2) \left[ (k - 2)^{n-2} + (-1)^{n-1} \right]
\]
The Signs of the Coefficients of $P_G(k)$

**Lemma:** Let $G$ be a graph with $n$ vertices and $m$ edges. Then the coefficients of $P_G(k)$ alternate between positive and negative.

**Proof:**

⭐ By induction on $m$.

⭐ If $m = 0$ then $P_G(k) = k^n$ and $0$ can be $+0$ or $-0$.

⭐ Assume correctness for graphs with $m - 1$ edges or less.

⭐ Let $(u, v)$ be an edge in $G$. 
Both $G - (u, v)$ and $G/(u, v)$ have at most $m - 1$ edges. $G - (u, v)$ has $n$ vertices and $G/(u, v)$ has $n - 1$ vertices.

By induction, $P_{G - (u,v)} = k^n - b_{n-1}k^{n-1} + b_{n-2}k^{n-2} - \ldots$ for non-negative integers $b_1, \ldots, b_{n-1}$.

By induction, $P_{G/(u,v)} = k^{n-1} - c_{n-2}k^{n-2} + c_{n-3}k^{n-3} - \ldots$ for non-negative integers $c_1, \ldots, c_{n-2}$.

Recall that $P_G(k) = P_{G - (u,v)}(k) - P_{G/(u,v)}(k)$.

$P_G(k) = k^n - (b_{n-1} + 1)k^{n-1} + (b_{n-2} + c_{n-2})k^{n-2} - \ldots$

The signs alternate since all $b_i$ and $c_i$ are not negative.