Graph Algorithms

Special Sets in Graphs
The Input Graph

\[ G = (V, E) \] a simple and undirected graph:

- \( V \): a set of \( n \) vertices.
- \( E \): a set of \( m \) edges.

\[
\begin{array}{ccccccc}
A & B & C & D & E & F \\
\hline
A & 0 & 1 & 1 & 1 & 0 & 0 \\
B & 1 & 0 & 1 & 0 & 1 & 0 \\
C & 1 & 1 & 0 & 0 & 0 & 1 \\
D & 1 & 0 & 0 & 0 & 1 & 1 \\
E & 0 & 1 & 0 & 1 & 0 & 1 \\
F & 0 & 0 & 1 & 1 & 1 & 0 \\
\end{array}
\]
An Independent Set

★ A set of vertices with no edges among them (independent or stable).

★ Denoted $I$.

★ $\forall u \neq v \in I \{(u, v) \notin E\}$. 
Example: An Independent Set
Example: A Maximum Size Independent Set
A Dominating Set

★ A set of vertices that are neighbors (dominate) to all other vertices in the graph.

★ Denoted $D$.

★ $\forall u \in V - D \exists v \in D \{(u, v) \in E\}$. 
Example: A Dominating Set
Example: A Minimum Size Dominating Set
A set of vertices that cover all the edges in the graph.

Denoted $VC$.

$\forall e \in E \exists u \in VC \{ u \in e \}$. 
Example: A Vertex Cover Set
Example: A Minimum Size Vertex Cover Set
A Clique

★ A set of vertices for which all possible edges exist among them (clique).

★ Denoted $K$.

★ $\forall u \neq v \in K \{(u, v) \in E\}$.
Example: A Clique
Example: A Maximum Size Clique
Hard Problems

• Finding an independent set with **maximum** size.
  
  \( I(G) \) – the size of such a set.

• Finding a dominating set with **minimum** size.
  
  \( D(G) \) – the size of such a set.

• Finding a vertex cover with **minimum** size.
  
  \( VC(G) \) – the size of such a set.

• Finding a clique with **maximum** size.
  
  \( K(G) \) – the size of such a set.
The Chess Graph

★ \( V : 64 \) vertices one per each square.

★ \( E : 2 \) vertices are connected by an edge if they either belong to the same row, or belong to the same column, or belong to the same diagonal.
The 8 queens problem:

★ Put 8 queens on a chess board such that no queen is capable of capturing another queen.
★ Find an independent set in the chess graph of size 8.

The 5 queens Problem:

★ Put 5 queens on a chess board such that each of the 64 squares is threatened by at least one queen.
★ Find a dominating set in the chess graph of size 5.
One of the Solutions to the 8 Queens Problem

*   *   *   *   *   *   *   *
*   *   *   *   *   *   *   *
*   *   *   *   *   *   *   *
*   *   *   *   *   *   *   *
*   *   *   *   *   *   *   *
*   *   *   *   *   *   *   *
*   *   *   *   *   *   *   *
*   *   *   *   *   *   *   *
Proof
One of the Solutions to the 5 Queens Problem
Proof
Observation: $I$ is an independent set in a graph iff $V - I$ is a vertex cover in the graph.

Corollary: $I(G) + VC(G) = n$. 
**Observation:** $K$ is a clique in a graph iff $K$ is an independent set in the complement graph.

**Corollary:** $I(G) = K(\tilde{G})$. 
Observation: $VC$ is a vertex cover in a graph iff $V - VC$ is a clique in the complement graph.

Corollary: $VC(G) + K(\tilde{G}) = n$. 
The following three problems on simple undirected graphs are equivalent – all of them are hard to solve:

• finding a maximum size independent set,
• finding a minimum size vertex cover, and
• finding a maximum size clique

A polynomial time solution to any one of them implies polynomial time solutions to the other two problems.
Maximal Independent Sets

**Definition:** An independent set $I$ is maximal if there is no larger independent set that contains $I$.

**Observation:** An independent set $I$ is also a dominating set iff it is maximal.

**Corollary:** $I(G) \geq D(G)$. 
**Notation:** The $d(v)$ neighbors of $v$ are: $v^1, v^2, \ldots, v^{d(v)}$.

**By definition:** A dominating set contains either $v$ or one of its neighbors.

**Membership:** $v$ is **TRUE** iff $v$ is in the dominating set.

**Idea:** Look for a product of minimum length in the boolean formula: 

$$A = \prod_{i=1}^{n} \left( v_i + v_i^1 + \cdots + v_i^{d(v_i)} \right).$$

**Translation:** Addition is a boolean **OR**. Multiplication is a boolean **AND**.
**Lemma:** Simplifying $A$ to a sum of minimal products yields a list of all the minimal dominating sets.

**Corollary:** A product with the minimum size represents a minimum size dominating set. Its size is $D(G)$.

**Complexity:** Not a polynomial time algorithm. There could be exponential number of products in $A$. 
\[ A = (a + b + d + e)(b + a + c + d)(c + b + d)(d + a + b + c + e)
\] 
\[ (e + a + d + f)(f + e) \]
\[ A = (a + b + d + e)(b + c + d)(e + f) \]  
\[ (* (x + y)y = y *) \]
\[ A = abe + abf + ace + acf + ade + adf + be + bf + bce 
\] 
\[ +bcf + bde + bdf + dbf + dbe + dbf + dce + dcf + de + df 
\] 
\[ +eb + ef + ec + ecf + ed + edf \]  
\[ (* (yy = y *) \]
\[ A = be + de + ce + bf + df + acf \]  
\[ (* xy + y = y *) \]
\[ D(G) = 2: \text{ For example the set } \{b, e\}. \]
Let $G$ be a perfect matching graph: There are $n/2$ edges each connects 2 different vertices.

Any minimal dominating set contains exactly one vertex from each edge.

There are $2^{n/2}$ different minimal dominating sets.
By definition: A vertex cover contains at least one vertex from each edge.

Membership: \( v \) is TRUE iff \( v \) is in the vertex cover set.

Idea: Look for a product of minimum length in the boolean formula: 
\[
B = \Pi_{(u,v)\in E} (u + v).
\]

Translation: Addition is a boolean OR. Multiplication is a boolean AND.
Lemma: Simplifying $B$ to a sum of minimal products yields a list of all the minimal vertex cover sets.

Corollary: A product with the minimum size represents a minimum size vertex cover set. Its size is $VC(G)$.

Corollary: $I(G) = n - VC(G)$.

Corollary: $K(G) = n - VC(\tilde{G})$.

Complexity: Not a polynomial time algorithm. There could be exponential number of products in $B$. 
Example

\[ B = (a + b)(a + d)(a + e)(b + c)(b + d)(c + d)(d + e)(e + f). \]

\[ B = abce + abdf + acde + acdf + bde. \]

\[ VC(G) = 3: \text{ the set } \{b, d, e\}. \]

\[ I(G) = 3: \text{ the set } \{a, c, f\}. \]
Let $G$ be a perfect matching graph: There are $n/2$ edges each connects 2 different vertices.

Any minimal vertex cover set contains exactly one vertex from each edge.

There are $2^{n/2}$ different minimal vertex cover sets.
Clique Cover

Definition I:
★ A collection of cliques covering all the vertices.
★ A partition $V = K_1 \cup K_2 \cup \cdots \cup K_k$ such that $K_j$ is a clique for all $1 \leq j \leq k$.

Definition II:
★ An assignment of IDs to the vertices such that if two vertices are assigned the same ID then they are connected by an edge.
★ A function $ID : V \to \{1, \ldots, k\}$ such that if $ID(u) = ID(v)$ then $(u, v) \in E$.

Observation: Both definition are equivalent.
Example: A Clique Cover
Example: A Clique Cover with Minimum Number of Cliques
The Clique Cover Problem

The Optimization Problem: Find a clique cover with minimum number of cliques.

Notation: $C'(G)$ – the minimum number of cliques required to cover all the vertices of $G$.

Hardness: A very Hard problem (an NP-Complete problem).
Properties

**Observation:** $I(G) \leq C(G)$.

- Because in any clique cover, each member of an independent set must be covered by a different clique.

**Observation:** $C(G) \geq \left\lceil \frac{n}{K(G)} \right\rceil$.

- A *pigeon hole* argument: each clique in a clique cover is of size at most $K(G)$. 
Example: $I(G') < C(G')$

$I(G') = 5 \quad C(G') = 6$