Decomposition of Disjunctive Linear Programs for Fixed-Income Portfolio Selection

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Abstract

Structural requirements for optimal fixed-income portfolios can be modeled with *disjunctive constraints*; *e.g.*, a minimum trade requirement is "either the holding in asset j equals zero or the holding in asset j is at least the minimum." The complexity of quadratic programs makes adding disjunctions to a traditional mean-variance model computationally prohibitive. However, if absolute deviation is used to measure dispersion of returns instead of variance, then structural requirements can be added to a portfolio selection model with linear constraints and a linear objective function. Absolute deviation models produce *step-shaped* programs that can be decomposed into smaller linear programs that share a set of constraints. When logical (disjunctive) constraints are added to an absolute deviation model, the resulting programs are disjunctive linear programs that inherit the step shape of the model. We show that these programs can be solved by combining decomposition and branch-and-bound search; the decomposition technique provides a test for local, and in some cases global, optimality. We consider here a model that selects the portfolio that maximizes the trade-off between expected total (horizon) return and expected absolute deviation from a benchmark on scenarios of interest rate movements.

The choice of an optimal portfolio is often subject to structural constraints. There may be integral trading amount requirements for certain instruments, *e.g.* Treasury bills have to be purchased in increments of \$5,000, after an initial investment of \$10,000. Investors may demand a certain level of diversification or a certain credit grade for their portfolios. Dedicated bond portfolios often have sector specifications that have to be satisfied. All these are logical requirements; they can be modeled with disjunctions that describe allowable choices. However, introducing logical constraints into a model makes an otherwise linear and convex problem into a nonconvex programming problem, a disjunctive linear programming problem.

Absolute deviation

Portfolio selection models that use the absolute deviation of returns on different scenarios or time periods as the measure of risk have been developed by several researchers [Ko] [KoYa] [WoVZZe] [ZeKa]. Development of these models followed work by Sharpe [Sh67] [Sh71] using absolute deviation of returns in finding the characteristic or market line for a particular security or portfolio. The absolute deviation models are of interest because they can be formulated as programs with linear objective functions, as opposed to models that use variance as the measure of risk.

In some cases, using absolute deviation as a measure of dispersion has the same effect as using variance. Let $(R_1, \ldots, R_N)$ be a vector of normally distributed random variables, and let $\mu = (\mu_1, \ldots, \mu_N)$ be the mean of $(R_1, \ldots, R_N)$. Konno [Ko] has shown that minimizing the mean absolute deviation

$$w(x) = E[|\sum_{j=1}^{N} R_j x_j - E[\sum_{j=1}^{N} R_j x_j]\ |]$$

is equivalent to minimizing the standard deviation of $\sum_{j=1}^{N} R_j x_j$.

The absolute deviation trade-off model is an extension of these earlier mean absolute deviation models. We use the L_1 distance from a given benchmark as the measure of risk and maximize the trade-off between the expected present value of reinvested cash flows and the difference between benchmark and portfolio returns. This objective function is similar in spirit to one utilized by Hiller and Eckstein [HiEc]; they used downside deviation from a liability stream as the measure of risk. In this context, let R_{ij} be the sample returns for asset j on scenario i; let $bench_i$ be the known returns for the benchmark on scenario i. Then minimizing the risk for the absolute deviation trade-off model is equivalent to minimizing the function

$$f(x) = \sum_{i=1}^{S} \mid bench_i - \sum_{j=1}^{N} R_{ij} x_j \mid .$$

We can look at minimizing $f(x)$ as fitting a linear regression using least absolute deviation. Minimizing the sum of the L_2 norms, or

$$\sum_{i=1}^{S} || \, bench_i - \sum_{j=1}^{N} R_{ij} x_j \, ||_2$$

is equivalent to Markowitz's model for a tracking portfolio [Ma87], and is analogous to minimizing variance about the mean. Research in regression analysis thus is relevant to the question of choosing the measure of risk.

Regression analysis using least absolute deviation is more robust, *i.e.*, resistant to small changes in the data, than least squares regression; the least absolute deviation method of regression gives less weight to outliers than least squares does [BirDo]. The least squares estimate has the smallest variance of all unbiased estimates if the errors follow a normal distribution, but this advantage disappears if we remove the hypothesis of normality. If the distribution of errors has tails heavier than the probability density function of the normal distribution, or is asymmetric, or if the sample is very large, then minimizing the absolute deviation is more suitable in regression analysis [BirDo].

Because absolute deviation models have linear formulations, it is possible to include disjunctive constraints in these models. Quadratic $0-1$ and general integer programming methods are still in development and encounter complexity theoretic problems [GJ]. Also, in many cases, optimal portfolios for these LP-based models have many fewer nonzero holdings than do portfolios found optimal for quadratic models [Ko] [Bi]. This is important in portfolio optimization because transaction costs occur for every separate purchase.

Disjunctive linear programs

We give a precise definition of programs with *disjunctive constraints*.

DEFINITION: A polyhedral set is the solution set for a finite system of linear inequalities, and a finite union of polyhedral sets is called a disjunctive set. A disjunctive set is defined by a *disjunctive constraint* and a *disjunctive (linear) program* is the problem of maximizing (or minimizing) a linear functional over the intersection of a family of disjunctive sets.

Let $\cup_k F_{ik}$ be a disjunctive set for each $i, i = 1, \dots, n$. Then the intersection of the disjunctive sets F_{ik} can be described by a logical formula in disjunctive normal form (DNF): $\vee_{h \in Q}(x \in K_h)$, where $|Q| = \prod_{i=1}^{n} M_i$. If $M_i = M_j = M$ for $i, j \leq n$, then $|Q| = M^n$. We shall call $K_h, \forall h \in Q$, a *disjunctive feasible region*, since $K_h \subseteq \cap_i \cup_k F_{ik} \quad \forall h \in Q$. The set of vectors $\cup_{h \in Q} K_h$ is a disjunctive set. See [McTr96] [McTr97] for a discussion of disjunctive linear programming.

Step-shaped programs

The programs that correspond to the absolute deviation models share several family characteristics. First, auxiliary variables are introduced to represent the absolute value of the difference between portfolio and benchmark returns; second, there are constraints on the primary variables in which the auxiliary variables don't appear; third, the constraints in which both the primary and auxiliary variables appear do not further constrain the primary variables; and, finally, the only constraints on the auxiliary variables alone are bounding constraints restricting their values to the nonnegative orthant. We will call the primary variables *portfolio variables* and the auxiliary variables *risk variables.*

These characteristics motivate the following definitions.

DEFINITION: Let $R = \{x \in \Re^n : Ax \leq b\}$ and let $Dx + Ey \leq f$ be a system of linear inequalities. Then $Dx + Ey \leq f$ is *free* for $x \in R$ if for every $x \in R$ $\exists y \geq 0$ such that (x, y) satisfies $Dx + Ey \leq f$. $\Box$

Let $x \in \Re_+^N$ and $y \in \Re_+^S$. Consider the following linear program (P):

$$\begin{array}{lllcl} (P) & \max & \{cx + dy\} & = & z_P \\ & \text{s.t.} & A_1^1 x + A_1^2 y & \leq & r \\ & & A_2^1 x + A_2^2 y & \leq & b \\ & & A_3^1 x + A_3^2 y & \leq & 0 \end{array}$$

DEFINITION: Let $R = \{x \in \Re_+^N : A_1^1 x \leq r\}$. A program (P) is a *step-shaped program* if the coefficient matrix for (P) can be written as above and if, in this formulation for (P), (1) A_1^2 and A_3^1 are zero submatrices, and (2) the constraints $A_2^1 x + A_2^2 y \leq b$ are free for $x \in R$. $\square$

Programs for absolute deviation models typically have two constraints that link the portfolio and risk variables for every scenario. In applications it is necessary to include as many scenarios as possible, so these problems quickly become very large; analysis of decomposition techniques applied to step-shaped programs can lead to explicit solutions of relatively large programs.

Decomposition for step-shaped programs

The absolute deviation trade-off model produces large programs with linear constraints and two kinds of variables, portfolio variables and risk variables. Thus we explore variable decomposition methods for these programs. Benders [Be] used a partition of the variable set to solve mixed integer problems. Van Slyke and Wets [VSWe] reported on a method called L-shaped decomposition, which is akin to Benders decomposition, for large-scale and stochastic linear problems.

The form of the coefficient matrix for step-shaped programs is close to the Van Slyke and Wets description of L-shaped programs. The shape of the nonzero submatrices we are considering differs from the L-shaped form, however, in that we need to explicitly consider the bounding constraints on y since we will be interested in dual multipliers for the constraints. In fact, our definition of *free* makes precise an informal property discussed by [VSWe]. We will first outline variable decomposition of linear step-shaped programs.

Variable decomposition for linear step-shaped programs

Let (P) be a linear step-shaped program:

$$\begin{array}{llrcl} (P) & \max & \{cx + dy\} & = & z_P \\ & \text{s.t.} & A_1^1 x + A_1^2 y & \leq & r \\ & & A_2^1 x + A_2^2 y & \leq & b \\ & & A_3^1 x + A_3^2 y & \leq & 0 \end{array}$$

Let $x \in \Re_+^N$ and $y \in \Re_+^S$. Let m_1 denote the number of rows in $\left(A_1^1 \; A_1^2 \right)$, let m_2 denote the number of rows in $\left(A_2^1 \; A_2^2 \right)$, and let m_3 denote the number of rows in $\left(A_3^1 \; A_3^2 \right)$. Then A_1^2 is an $m_1 \times S$ zero submatrix, and A_3^1 is an $m_3 \times N$ zero submatrix.

Let $M = m_1 + m_2 + m_3$, let $u \in \Re_+^M$ and let π be a projection operator. Then $\pi_1(u) = \vec{u}_1 \in \Re_+^{m_1}$, $\pi_2(u) = \vec{u}_2 \in \Re_+^{m_2}$, and $\pi_3(u) = \vec{u}_3 \in \Re_+^{m_3}$.

To carry out variable decomposition, we rearrange the constraints for (P), and write the program as

$$\max_{\{x \geq 0\}} \{cx + \max_{y \in G} dy\}$$

$$G = \{y \in \Re_+^S : A_1^2 y \leq r - A_1^1 x, A_2^2 y \leq b - A_2^1 x, A_3^2 y \leq 0\}$$

Then by LP-duality, we can write (P) as

$$\max_{\{x \geq 0\}} \{cx + \min_{u \in J} \{\vec{u_1}(r - A_1^1 x) + \vec{u_2}(b - A_2^1 x) + \vec{u_3} 0\}\}$$

$$J = \{u \in \Re_+^M : \vec{u}_1 A_1^2 + \vec{u}_2 A_2^2 + \vec{u}_3 A_3^2 = d\}$$

Notice that since A_1^2 is a zero matrix, $\vec{u}_1 = \pi_1(u)$ is unconstrained in the description of J. Since the constraints $A_2^1 x + A_2^2 y \leq r$ are free for $x \in \Re^N$, then if x is feasible for the constraints $A_1^1 x \leq r$, we can work in J', which is the projection of J, where $J' = \{\vec{u}_2, \vec{u}_3 : \vec{u}_2 A_2^2 + \vec{u}_3 A_3^2 = d\}$.

Therefore, since by weak duality $dy \leq \vec{u}_2(b - A_2^1 x)$ for all feasible x, y, and $\vec{u}_2$, a program equivalent to (P) is the following one, where one continuous variable λ is substituted for dy and is bound by the dual objective function value at extreme points of J'. Let H be an *a priori* bound for y.

$$\begin{array}{llrcl}
(MP) & \max & \{cx + \lambda\} & = & z_{MP} \\
 & s.t. & A_1^1 x & \leq & r \\
 & & |\lambda| & \leq & dH \\
 & & \lambda & \leq & \vec{u}_2^t(b - A_2^1 x) \\
 & & & & \forall u^t = \langle \vec{u}_2^t, \vec{u}_3^t \rangle
\end{array}$$

Here u^t is an extreme point of $\{u \in \Re_+^{m_2 + m_3} : \vec{u}_2 A_2^2 + \vec{u}_3 A_3^2 = d\}$.

(MP) is the master problem for the variable decomposition reformulation of (P). The variable decomposition method for solving (MP) is (briefly) as follows. Assume $R = \{x \in \Re_+^N : A_1^1 x \leq r\} \neq \emptyset$. At the first iteration, solve the relaxation (MP^0) of (MP) by dropping the constraints $\lambda \leq \vec{u}_2^t(b - A_2^1 x)$ for all extreme points of J' from (MP). Let (x^0, λ^0) be the optimal solution. If we have

$$(1) \quad \min\{\vec{u}_2(b - A_2^1 x) : \vec{u}_2 A_2^2 + \vec{u}_3 A_3^2 = d, \langle \vec{u}_2, \vec{u}_3 \rangle \geq 0\} \geq \lambda^0$$

then $\lambda^0 = dy^0$ is the optimal solution of the program

$$(2) \quad \max\{dy : A_2^2 y \geq b - A_2^1 x^0, A_3^2 y \geq 0\}$$

and (x^0, y^0) is optimal for (P). If

$$\lambda^0 > \min\{\vec{u}_2(b - A_2^1 x^0) : \vec{u}_2 A_2^2 + \vec{u}_3 A_3^2 = d, \langle \vec{u}_2, \vec{u}_3 \rangle \geq 0\}$$

then (x^0, λ^0) is not feasible for (MP) and we have a valid inequality (or cut) that can be added to (MP^0)

$$\lambda \leq \min\{\vec{u}_2(b - A_2^1 x) : \vec{u}_2 A_2^2 + \vec{u}_3 A_3^2 = d\}$$

to produce the relaxed master problem at stage 1, (MP^1).

At each iteration, this method either proves a point optimal or generates a cut to be added to the current relaxed master problem. (1) is called the dual subproblem $(DSP(x))$ and (2) is called the primal subproblem $(SP(x))$.

Using this method, we replace (P) with a series of relaxations of (MP) and easier subproblems. Solving (P) is carried out by first solving a relaxation of the master problem (MP^k) at stage k and then solving a subproblem $(DSP(x^k))$ to obtain the extreme point that will figure in the test for optimality, and if this fails, in an inequality that is added for stage $k+1$. In this way the relaxations of (MP) are progressively tightened. See the Appendix for a full description of the decomposition algorithm.

Decomposition for disjunctive step-shaped programs

Benders decomposition [Be] was developed for mixed integer programs and involves decomposition with respect to a set of "complicating" variables, that is, ones required to take on integer values. Linear subproblems are solved for a fixed integer-valued variable, and the optimal solution over all feasible integer values is found. An account of Benders decomposition can be found in [Sc]. Hooker [Ho] used Benders decomposition after expressing a problem with logical requirements as a $0-1$ program. Here, we will generalize Benders decomposition to disjunctive programs; *i.e.*, we will solve a linear problem (L) with respect to variables required to take on values in certain disjunctive sets. Since each disjunctive set K_h is defined by a "large" linear step-shaped program (L_h), we can then apply variable decomposition to solving each of these linear programs. Note that step-shaped programs are *hereditary* in the sense that adding constraints on the portfolio variable x yields another step-shaped program.

The feasible region for the dual subproblem $(DSP_h \bar{x})$ is the same for every (L_h) and is in fact identical to the feasible region for the dual subproblem for the linear problem (MP). Let U be the collection of inequalities that are in a full master problem for (ML_h); *i.e.*, U contains all inequalities of the form $\lambda \leq \vec{u}_2(b - A_2^1 x)$ where $(\vec{u}_2, \vec{u}_3)$ is an extreme point of (DSP). So U contains all the inequalities for a full master problem for (P). Note that $\cup_{h \in Q} K_h$ consists of points feasible for the original linear program (P), and thus for (MP). Therefore, we consider two types of master problems: an *outer* master problem that only contains inequalities from U, and an *inner* master problem that contains inequalities from U plus inequalities that define disjunctive constraints. We can test for optimality for any $x \in \cup_{h \in Q} K_h$ both for some L_h and for (P). We also can use the fact that if a point x satisfies $x \in \cup_{h \in Q} K_h$ and tests optimal for an outer master problem and therefore for (P), then this point must be optimal for (L).

Decomposition and depth-first search

The constraint set for each polyhedral set K_h has a linear step-shaped coefficient matrix and if $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, M\}$ there are M^N linear step-shaped programs to solve in any exhaustive search of this problem. Since the number of programs is exponential in the number of variables with disjunctive requirements, it is necessary computationally to shorten this search. We will use a depth-first search among the disjunctive regions and interleave the steps of the search with the cut generation steps of Benders (variable) decomposition. The point of this interleaving is to use the decomposition algorithm to direct the search for the maximal objective function among the disjunctive sets, and to use the maximum objective function value of any disjunctive feasible set as a lower bound to eliminate branches of the search tree.

In combined depth-first-search and variable decomposition, a solution is found at a node and then tested for disjunctive feasibility. If the solution is in one of the disjunctive regions, then it is tested for optimality at a node. If it does not satisfy the disjunctive constraints, then a new disjunctive set is branched on. If a point x satisfies the disjunctive constraints, there is some K_h such that $x \in K_h$; and whether this point is optimal for (ML_h) or not, its objective function value can be used as a lower bound. Further, an inner master problem at node n, (ML_n^k) at stage k is a relaxation of (ML_n^{k+1}) and all nodes descendent from it. So if a disjunctive feasible point is optimal for (ML_n^k) then all children of (ML_n^k) can be pruned; if the initial objective function value for (ML_n^k) is less than the current lower bound, (ML_n^k) and its children can be pruned. Further, if the candidate solution satisfies the disjunctive constraints, we can test if for local and global optimality at the same time, by checking if it is a solution to the global master problem without the disjunctive constraints. Details of the depth-first decomposition algorithm are in the Appendix.

The portfolio problem with disjunctive constraints

Standard structural requirements for a fixed-income portfolio are that asset holdings satisfy minimum trade requirements and that specified levels of diversification be maintained. The depth-first search and decomposition algorithm was carried out with these requirements; the program (L^*) that models the portfolio selection problem with minimum trade requirements and diversification requirements is as follows:

$$\begin{array}{llrcl}
(L^*) & \max & \frac{1}{S}\sum_{i=1}^{S}\sum_{j=1}^{N} ret_{ij}x_j & - & \frac{1}{S}\sum_{i=1}^{S} y_i \\
 & \text{s.t.} & \sum_{j=1}^{N} p_j x_j & \leq & B \\
 & & -\sum_{j=1}^{N} x_j & \leq & -w \\
 & & x & \in & \cap_i \cup_k F_{ik} \\
 & \forall j & 0 \leq d_j & \leq & 1 \\
 & & \sum_{j=1}^{N} d_j & \geq & m \\
 & & d & \in & \cap_r \cup_s D_{rs}
\end{array}$$

where M is an upper bound for x and we have $x \in \cap_i \cup_k F_{ik}$ if for all j, $x_j = 0 \vee x_j \geq v$ and $d \in \cap_r \cup_s D_{rs}$ if for all $j, d_j = 0 \vee d_j = 1$. See the Appendix for a description of the absolute deviation trade-off model.

For each disjunctive region K_h there is a linear program (L_h^*), where $x \in K_h$ and for all $j, d_j = 0 \vee d_j = 1$. Since the optimal vertex solutions to (L^*) are the union of the optimal vertex solutions to (L_h^*), for all $h \in Q$, we can characterize the optimal vertex solutions to (L^*) by considering the polyhedron of vectors feasible for (L_h^*). The vertices of this polyhedron are all portfolios with more than m nonzero holdings.

Let T_h equal the set of vectors feasible for (L_h^*). If we set the sum of nonzero holdings equal to m, then the vertices of T_h are m-asset portfolios. Included in the vertices of T_h are the portfolios (x, y) where $x =$

$(v, \ldots, v, \ldots, \frac{B-(m-1)v}{p_j}, \ldots, 0)$, or the portfolios which have minimum holdings in $m-1$ assets and the balance of the budget invested in the jth asset, and where $y_i = f_i(x)$ for all i. Other vertices of T_h are points (x, y) such that $(x, y) \in T_h$ and $y_i = 0$ for at least one i.

It is interesting to note that the occurrence of multiple optimal solutions in this problem provides information about the range of optimal investments: a kind of "efficient frontier" of investments with different levels of risk and return is delineated by these multiple solutions. If the solution to (L^*) is unique, then introducing a parameter τ to vary allowable levels of risk can provide a range of investments from more to less risky. In this approach, τ is increased until the portfolio with the least risk is found optimal *c.f.*[HiEc]. Conversely, introduction of the parameter τ will lead to discovery of multiple optimal solutions with different levels of risk if they exist.

Disjunctive linear programs for the absolute deviation trade-off model were coded in the constraint programming language 2LP [McTr96] linked to CPLEX.

Appendix

Absolute deviation trade-off model

We assume that there is a cash-flow model available to calculate the present value of each security's cash payments and embedded options; the nonlinearity of the portfolio selection problem can, to a great extent, be captured in these models. Further, we have fixed a benchmark with known returns on the set of scenarios. We assume that the only constraints are that all asset holdings are nonnegative, *i.e.*, there is no short selling, and there is a set budget for the investment. We also assume that we have a model of the term structure of interest rates of the Heath-Jarrow-Morton type and that from this model we have computed short rates at every node of a binomial tree. Also, we have calculated the value of reinvesting the cash flows from each of the available assets at the current spot rate at every node. If there are N available assets and the holding period for the portfolio comprises M dates on which cash flows are collected, then there are 2^M scenarios. If the number of scenarios becomes too large, sampling techniques or Monte Carlo simulation methods are used in conjunction with the basic model. In this research, we have used explicit computation on a discrete model of the future evolution of spot rates [Ja].

For every asset, the present value of investing the cash flows at the current spot rate over the holding period of the portfolio is calculated. This figure is then "normalized" by multiplying by the price of the asset (per \$100 face) divided by the budget amount. A feasible portfolio is a linear combination of the assets for which the price does not exceed the budget. A variable y_i is introduced for every scenario i to model the absolute value of the difference between a portfolio and the benchmark yielded by this scenario. Two constraints are added to the constraint set for every scenario:

$$\forall i \quad (\sum_{j=1}^{N} ret_{ij} x_j) - y_i \leq benchmark_i$$

$$\forall i \quad (-\sum_{j=1}^{N} ret_{ij} x_j) - y_i \leq -benchmark_i$$

The objective function then is the difference between expected return and expected risk:

$$\max \frac{1}{S} \sum_{i=1}^{S} \sum_{j=1}^{N} ret_{ij} x_j - \frac{1}{S} \sum_{i=1}^{S} y_i$$

At optimality, since the y_i are nonnegative, subtracting the average of the y_i is equivalent to minimizing the y_i, or forcing y_i to the absolute value of the difference between benchmark and portfolio on each scenario. The definitions and constraints for the model are as follows:

Definitions:

S number of scenarios: 2^T where T is investment horizon

N number of securities available

B budget limit

ret_{ij} present value of investing cash flow from security j on scenario i

$benchmark_i$ return from benchmark portfolio on scenario i

p_j current price of \$100 face amount of security j

x_j amount held of security j

y_i absolute value of the deviation of the portfolio return from the benchmark return on scenario i

v minimum trade amount

w minimum investment amount

The linear program that models the problem of selecting the best tracking portfolio is:

$$\text{(P)} \quad \max\{\frac{1}{S}\sum_{i=1}^{S}\sum_{j=1}^{N} ret_{ij}x_j - \frac{1}{S}\sum_{i=1}^{S} y_i\} = z_P$$

$$\begin{array}{lrcl}
\text{s.t.} & p_1x_1 + \ldots + p_Nx_N & \leq & B \\
 & -x_1 - \ldots - x_N & \leq & -d \\
\forall j & -x_j & \leq & 0 \\
\forall i & ret_{i1}x_1 + \ldots + ret_{iN}x_N - y_i & \leq & benchmark_i \\
\forall i & -ret_{i1}x_1 - \ldots - ret_{iN}x_N - y_i & \leq & -benchmark_i \\
\forall i & -y_i & \leq & 0
\end{array}$$

(P) can be written formally as:

$$\begin{array}{llrcl}
(P) & \max & \{cx + dy\} & = & z_P \\
 & \text{s.t.} & A_1^1x + A_1^2y & \leq & r \\
 & & A_2^1x + A_2^2y & \leq & b \\
 & & A_3^1x + A_3^2y & \leq & 0
\end{array}$$

We have the following results for this model:

Lemma 1 *The optimal objective function value for (P) is less than or equal to the average over all scenarios of the return of the benchmark portfolio.*

Lemma 2 *For $x \in \Re^N$, let $f_i(x) : \Re^N \to \Re$ for $i = 1, \ldots, S$ be the sequence of functions $f_i(x) = |benchmark_i - \sum_{j=1}^{N} ret_{ij}x_j|$. Then for every $x \in \Re^N$, the point $(x_1, \ldots, x_N, f_1(x), \ldots, f_S(x))$ satisfies S inequalities from the constraint set $A_2^1x + A_2^2y \leq b$ of (P) with equality.*

For a full discussion of the model and proofs of the lemmas, see [Wy].

Decomposition algorithm for linear programs

The decomposition algorithm consists of the following steps:

Initialization: Set $k := 0$. Let H be a bound for y. The initial master problem is:

$$\begin{array}{llrcl} (MP^0) & \max & z_{MP^0} & = & cx + \lambda \\ & \text{subject to} & A_1^1 x & \leq & r \\ & & |\lambda| & \leq & dH \end{array}$$

1. Solve (MP^k) for (x^k, λ^k). If $\{x \in \Re_+^N : A_1^1 x \leq r\} = \emptyset$, end. (P) is infeasible.

2. Solve (DSP(x^k)) and obtain $\langle \vec{u}_2^k, \vec{u}_3^k \rangle$.

3. Use value of $\vec{u}_2^k$ found to test for optimality of (x^k, λ^k). A point is optimal if it is dual feasible, *i.e*, (x^k, λ^k) is optimal if

$$\lambda^k \leq \vec{u}_2^k (b - A_2^1 x^k).$$

4. If (x^k, λ^k) is optimal, solve $(SP(x^k))$ to obtain y^k, and end.

5. If (x^k, λ^k) fails test for optimality, then add cut with respect to $\vec{u}_2^k$ to (MP^k) to obtain (MP^{k+1}):

$$\lambda \leq \vec{u}_2^k (b - A_2^1 x)$$

   This is a valid cut since $\vec{u}_2^k$ is a projection of an extreme point of the feasible region of (DSP) for all x.

6. Set $k := k + 1$. Return to (1).

Depth-first search and decomposition

The "inner" master problems for decomposition at a node (ML_n) and at a leaf (ML_n) are below. All the disjunctive requirements are satisfied at a leaf; only some of them may satisfied at a node. The "outer" master problem at a node contains only inequalities of the form $\lambda \leq u_2^t(b - A_2^1 x)$.

$$(L_n) \quad \begin{array}{ll} \max & cx + dy \\ \text{s.t.} & A_1^1 x \leq r \\ & x \in \cup_j F_{i,j(i)},\ i \text{ selected} \\ & A_2^1 x + A_2^2 y \leq b \\ & A_3^2 y \leq 0 \end{array} \qquad (ML_n) \quad \begin{array}{ll} \max & cx + dy \\ \text{s.t.} & A_1^1 x \leq r \\ & x \in \cup_j F_{i,j(i)},\ i \text{ selected} \\ & \lambda \leq \vec{u}_2^t(b - A_2^1 x) \\ & \text{all } u^t \text{ extreme points of } F(DSP) \end{array}$$

$$(L_h) \quad \begin{array}{ll} \max & cx + dy \\ \text{s.t.} & A_1^1 x \leq r \\ & x \in K_h \\ & A_2^1 x + A_2^2 y \leq b \\ & A_3^2 y \leq 0 \end{array} \qquad (ML_h) \quad \begin{array}{ll} \max & cx + \lambda \\ \text{s.t.} & A_1^1 x \leq r \\ & x \in K_h \\ & \lambda \leq \vec{u}_2^t(b - A_2^1 x) \\ & \text{all } u^t \text{ extreme points of } F(DSP) \end{array}$$

The algorithm is as follows:

Initialization: Assume we have u^0 an extreme point for the feasible region for (DSP). Let $n = 1, \ldots, T$ index the nodes of the search tree. Set $k := 0$. Select the root node as the first node to examine. Set lower bound equal to $-\infty$. Let $(ML^0) = (L_0^1)$ be the current outer master problem.

$$(L_0^1) \quad \begin{array}{ll} \max & z \\ \text{s.t.} & A_1^1 x \leq r \\ & z \leq cx + \vec{u}_2(b - A_2^1 x) \end{array}$$

Stage $k (k > 0)$:

(1) Solve the relaxed inner master problem (ML_n) at node n to get $(\bar{x}_n, \bar{\lambda}_n)$ and the relaxed objective function value $\bar{z}_n$. If $\bar{z}_n \geq$ lower bound, go to (2). If $\bar{z}_n <$ lower bound, go to (3). If (ML_n) is infeasible, go to (3).

(2) Check if $\bar{x} \in \cup_{h \in Q} K_h$.

(a) If $\bar{x}$ is disjunctive feasible, then solve $(DSP\bar{x})$ to get $\bar{u}$. Test $(\bar{x}_n, \bar{\lambda}_n)$ for optimality for (ML_n).

(i) If $(\bar{x}_n, \bar{\lambda}_n)$ is optimal for (ML_n) then update lower bound if possible and go to (3). If all nodes have been explored, stop. If the lower bound is greater than or equal to 0, then lower bound is optimal.

(ii) If $(\bar{x}_n, \bar{\lambda}_n)$ is not optimal for (ML_n) then add cut with respect to $\bar{u}$ to (ML_n).

(b) If $\bar{x}_n \notin \cup_{h \in Q} K_h$, then branch on a new disjunctive set; *i.e.*, select a node descendent from n.

Go to (1).

(3) Set $k{:=}k+1$. Backtrack to last unexplored node m. Go to (1). If all nodes have been explored, stop. The lower bound is optimal.

Theorem 1 *Assume that the linear relaxation (P) of the disjunctive program (L) is feasible. If (L) is feasible, then the algorithm combining variable decomposition and depth-first search will terminate with the optimal solution. If (L) is infeasible, then the algorithm will end with no solution.*

Proof. If no nodes are pruned in the search for the optimal solution to (L), then all M^N leaves in the search tree will be explored and the optimal objective function value will be found for each disjunctive feasible set. In this case, the algorithm clearly terminates and the optimal solution is the best objective function value found at a leaf. Since any nodes pruned by this algorithm achieved relaxed objective function values strictly less than the current lower bound, and since any parent node is a relaxation of its descendents, leaves which contain optimal solutions are not removed. Therefore, the algorithm will terminate with the optimal value.

However, if there is no feasible solution to (L), then every inner master problem at a leaf is infeasible. This infeasibility may be detected at a node higher up in the search tree, in which case the descendents of this node will not be explored. But since every disjunctive region is considered, the algorithm will terminate with no solution. □

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