Artificial Neural Networks
Lecture Notes

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Part 9

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- Acknowledgments:
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Contents

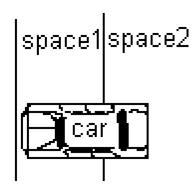
Fuzzy Logic

- Neural networks can learn. However, they are rather opaque.
- "The network acts like a black box by computing a statistically sound approximation to a function known only from a training set.", Rojas.

- Fuzzy logic has an explanatory capability. However, it is unable to learn.

Fuzzy Sets Vs. Crisp Sets

- "Raise your hand if you're male" ... hands down. Now,
- "Raise your hand if you're female" ... hands down.
- These are *crisp sets*.
- "Raise your hand if you're satisfied with your jobs" ... hands down. Now,
- Some hands probably went up both times, and
- Hands may have been only somewhat raised in each case.
- We would say that job satisfaction is a *fuzzy concept*, in that most people are not entirely satisfied or dissatisfied with their jobs.
- Parking spaces in a lot would be another example: Most of the time one has a situation like the one shown on the right.



- *Fuzzy logic* was developed by Lotfi Zadeh during the mid 1960's.
- Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set
- The subset A of X consisting of x_1 alone, can be described by the n-dimensional membership vector

  $$Z(A) = (1, 0, 0, \ldots 0).$$

- The subset B of X with elements x_1 and x_n is described by the vector

  $$Z(A) = (1, 0, 0, \ldots, 1).$$

- Any other crisp subset of X can be represented in the same way by an n-dimensional binary vector.
- Consider the fuzzy set C

  $$Z(C) = (0.5, 0, 0, \ldots, 0).$$

In classical set theory, this would be impossible. For classically, either $x_1 \in C$ or it doesn't. The world is black and white in classical set theory.

- With *Fuzzy* set theory, this type of description is permitted.
- The element x_1 belongs to the set C only to some extent.
- The degree of membership is expressed by a real number in the interval [0,1].
- An example:
  "x_1 is a tall person"
  Is a 6-foot individual tall? ... yes.
- Other diffuse statements:
  x is old
  y is young
  z is mature

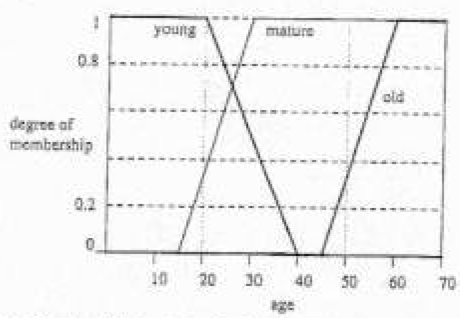


Fig. 11.1. Membership functions for the concepts young, mature and old

Three membership functions in the interval 0:70 years.

- The three functions define the degree of membership of any given age in the sets of young, mature and old ages.
- A 20-year old - degree of membership in the set
  of young people is 1.0
  of mature adults is 0.35 , and
  of old persons is 0.0.
- If someone is 50 years old, the degrees of membership are 0.0, 1.0, 0.3 in the respective sets.
- <u>Definition 11.1</u>:
  Let X be a classical universal set. A real function $\mu_A : X \longrightarrow [0,1]$ is called the membership function of A, and defines the fuzzy set A of X.
  This is the set of all pairs $(x, \mu_A(x))$ with $x \in X$.
- A fuzzy set is completely determined by its membership function.

- The *set of support* of a fuzzy set A, is the set of all elements x of X for which $(x, \mu_A(x)) \in A$, and $\mu_A(x) > 0$ holds.

- A fuzzy set A with finite set of support $\{a_1, a_2, \ldots, a_m\}$ can be described as:

$$A = \mu_1 / a_1 + \mu_2 / a_2 + \ldots + \mu_m / a_m$$

  where $\mu_i = \mu_A(a_i)$ for $i = 1, \ldots, m$.

- Let $X = \{x_1, x_2, x_3\}$.

  The classical (crisp) subsets $A = \{x_1, x_2\}$ and $B = x_2, x_3\}$ can be represented as

$$A = 1/x_1 + 1/x_2 + 0/x_3$$

$$B = 0/x_1 + 1/x_2 + 1/x_3$$

- The union of A and B is computed by taking for each element x_i the maximum of its membership in both sets. I.e.,

$$A \cup B = 1/x_1 + 1/x_2 + 1/x_3 .$$

- The fuzzy union of two fuzzy sets can be computed in the same way.
  Let

$$C = 0.5/x_1 + 0.6/x_2 + 0.3/x_3$$

$$D = 0.7/x_1 + 0.2/x_2 + 0.8/x_3$$

  Then

$$C \cup D = 0.7/x_1 + 0.6/x_2 + 0.8/x_3 .$$

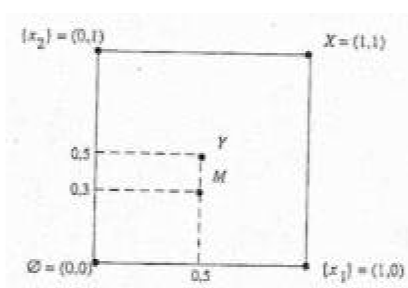
- The fuzzy intersection of two sets can be defined in a similar manner.

Geometric Interpretation of Fuzzy Sets

- Let $X = \{x_1, x_2\}$ be a universal set.

- Each point in the interior represents a subset of X

- The crisp subsets of X are located at the vertices of the unit square.



- The point (1,0) represents the set x_1.
  The point (0,1) represents the set x_2
  etc. ...

- The crisp subsets of X are located at the vertices of the unit square.

- The inner region of a unit hypercube in an n-dimensional space - the fuzzy region.

- The point M (in the figure above) corresponds to the fuzzy set

  $$M = 0.5/x_1 + 0.3/x_2.$$

- Center of square - most diffuse of all fuzzy sets.

- The degree of fuzziness of a fuzzy set can be measured by its entropy (here different from the entropy used in information theory or physics.)

- The **entropy** (index of fuzziness) corresponds inversely to the distance between the representation of the set and the center of the unit square.

- Entropy here is the index of fuzziness, crispness, certitude or ambiguity.

- The set Y in the previous figure has the maximum entropy.

- The vertices represent the crisp sets and have the lowest entropy, i.e. 0.

Entropy of a Fuzzy Set M

- Quotient of the distance of the nearest corner from M to the distance d_2 from the corner which is farthest away.

  $$E(M) = d_1/d_2.$$

  $0 \leqslant$ entropy of a set $\leqslant 1$.

- Maximum entropy - at the center of square.

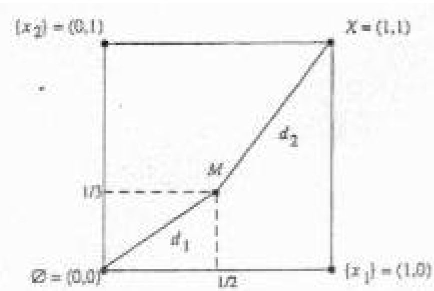


Fig. 11.3. Distance of the set M to the universal and to the void set

Visualization for Union and Intersection of Fuzzy Sets

- The membership function for the *union* of two sets A and B is

$\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x)), \forall x \in X.$

- This corresponds to the maximum of the corresponding coordinates in the geometric visualization.

- The membership function for the *intersection* of two sets A and B is given by

$\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x)), \forall x \in X.$

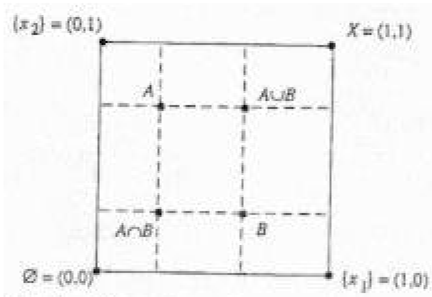


Fig. 11.4. Intersection and union of two fuzzy sets

- The union or intersection of two fuzzy sets is in general a fuzzy, not a crisp set.

- The *complement* A^C of a fuzzy set is given by

$\mu_{A^c}(x) = 1 - \mu_A(x), \forall x \in X.$

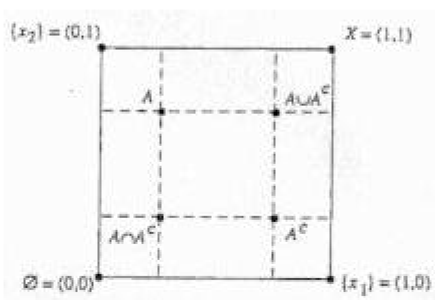


Fig. 11.5. Complement A^c of a fuzzy set

The line joining the representation of A and A^C goes through the center of the square.

- In general, for fuzzy sets we have

$A \cup A^C \neq X$, and

$A \cap A^C \neq X$, and

Fuzzy Sets and Logic:

On the Relationship between Set Theory and the Propositional Logic

- Let A, B be two crisp sets. We have,

  $\mu_A , \mu_B : X \longrightarrow \{0,1\}$.

- The membership function $\mu_{A \cup B}$ for $A \cup B$ is

  $\mu_{A \cup B}(x) = \mu_A(x) \vee \mu_B(x) , \quad \forall x \in X.$

  where $0 \equiv$ False , $1 \equiv$ True.

- Similarly, for the membership function for $A \cap B$, we have

  $\mu_{A \cap B}(x) = \mu_A(x) \wedge \mu_B(x) , \quad \forall x \in X.$

- For the complement A^c,

  $\mu_{A^c}(x) = \neg \mu_A(x)$.

- *De Morgan's Laws*

  $(A \cup B)^c \equiv A^c \cap B^c$ corresponds to $\neg(A \vee B) \equiv \neg A \wedge \neg B$

  $(A \cap B)^c \equiv A^c \cup B^c$ corresponds to $\neg(A \wedge B) \equiv \neg A \vee \neg B$

- *Fuzzy Counterparts to Logic Operators*
  We may identify:

  the OR operation ($\bar{\vee}$) with *maximum*,
  the AND ($\bar{\wedge}$) with *minimum*, and
  Complementation ($\bar{\neg}$) with $x \mapsto 1 - x$.

- Then we may write:

  $\mu_{A \cup B}(x) = \mu_A(x) \,\bar{\vee}\, \mu_B(x) , \forall x \in X,$
  $\mu_{A \cap B}(x) = \mu_A(x) \,\bar{\wedge}\, \mu_B(x) , \forall x \in X,$
  $\mu_{A^c}(x) = \bar{\neg} \mu_A(x) , \forall x \in X.$

- The properties of the isomorphism present in the classical theories are preserved.

- Many classical logic rules are valid for fuzzy operators.
  E.g., *min* and *max* are commutative and associative.

- But note: the principle of no contradiction does *not* hold.
  E.g., proposition A with truth value 0.4:

  $A \bar{\wedge} \neg A = \min(0.4, (1 - 0.4)) \neq 0.$

- Similarly, the *Law of Excluded Middle* is not valid either.

Axiomatic Definitions of Fuzzy Operators

Fuzzy OR Operator

- Axiom $\mathcal{U}_1$: *Boundary Conditions*
  $0 \bar{\vee} 0 = 0$
  $1 \bar{\vee} 0 = 1$
  $0 \bar{\vee} 1 = 1$
  $1 \bar{\vee} 1 = 1.$
- Axiom $\mathcal{U}_2$: *Commutativity*
  $a \bar{\vee} b = b \bar{\vee} a.$
- Axiom $\mathcal{U}_3$: *Monotonicity*
  If $a \leqslant a'$ and $b \leqslant b'$,
  then $a \bar{\vee} b \leqslant a' \bar{\vee} b'$.
- Axiom $\mathcal{U}_4$: *Associativity*
  $a \bar{\vee} (b \bar{\vee} c) = (a \bar{\vee} b) \bar{\vee} c.$
- Axiom $\mathcal{U}_5$: *Idempotence*
  $a \bar{\vee} a = a.$

  - The *maximum* function satisfies $\mathcal{U}_1 \longrightarrow \mathcal{U}_4$

Fuzzy AND Operator

- The fuzzy AND is monotonic, commutative and associative.

  Boundary Conditions
  $0 \bar{\wedge} 0 = 0$
  $1 \bar{\wedge} 0 = 0$
  $0 \bar{\wedge} 1 = 0$
  $1 \bar{\wedge} 1 = 1.$

Fuzzy Negation

- Axiom N_1: *Boundary Conditions*
  $\bar{\neg} 0 = 1$
  $\bar{\neg} 1 = 0$
- Axiom N_2: *Monotonicity*

If $a \leqslant b$, then $\neg b \leqslant \neg a$.

- Axiom N_3: *Involution*
  $a = \neg\neg a$.

- Graphs of the functions max and min (i.e., a possible fuzzy AND and fuzzy OR combination,)

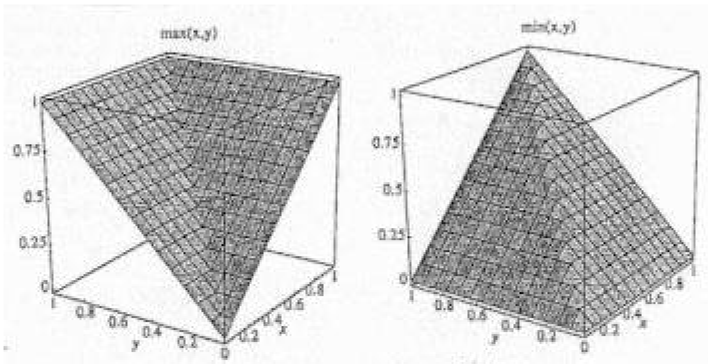


Fig. 11.7. The max and min functions

- They could be used as activation functions in neural networks.

- Using output functions derived from fuzzy logic can have the added benefit of providing a logical interpretation of the neural output.

Fuzzy Inferences

- Fuzzy logic operators can be used as the basis for inference systems.

- Fuzzy inference rules have the same structure as classical ones.
  E.g., rule R1: if $(A \bar{\wedge} B)$ Then C
  $(A \bar{\vee} B)$ Then D.

- Note $\bar{\wedge}$ ~ min and $\bar{\vee}$ ~ max, though other choices are possible.

- Let the truth values of A and B be 0.4 and 0.7. Then

$$A \bar{\wedge} B = \min(0.4, 0.7) = 0.4$$
$$A \bar{\vee} B = \max(0.4, 0.7) = 0.7$$

- Fuzzy inference mechanism - Rules R1 and R2 can only be partially applied (i.e., rule R1 is applied 40% and rule R2 70%.)

- The result of the inference is a combination of the propositions C and D.