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Hopfield Networks

Hopfield network (Discrete) - a recurrent autoassociative network

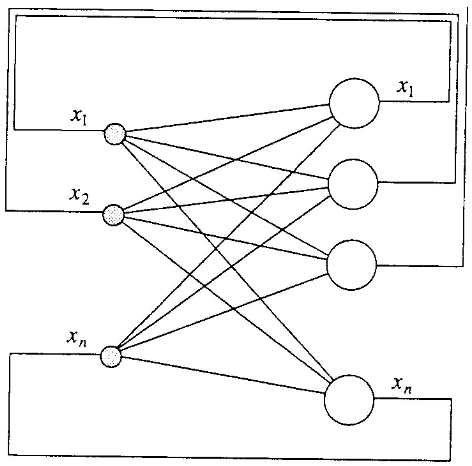


Fig. 12.3. Autoassociative network with feedback

Recurrent autoassociative network

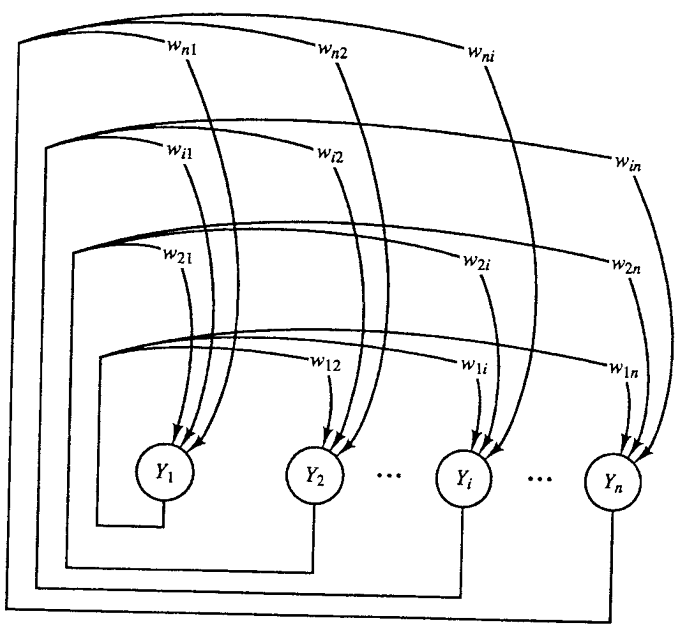


Figure 3.7 Discrete Hopfield net.

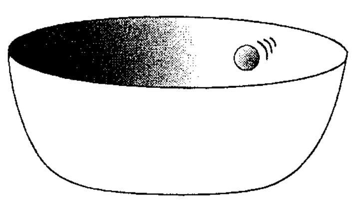
Contrast with recurrent autoassociative network shown above

Note: there are no self-loops in a Hopfield net

- Hopfield nets - example of a dynamical physical system that may be thought of as instantiating "memories" as stable states associated with minima of a suitably defined system energy

• ~ ______ ______ ____ ______ . . .

- Consider a bowl in which a ball bearing is allowed to roll freely . . .



- Suppose we let the ball go from a point somewhere up the side of the bowl.

- The ball will roll back and forth and around the bowl until it comes to rest at the bottom.

- The physical description of what has happened may be couched in terms of the energy of the system.

- The energy of the system is just the potential energy of the ball and is directly related to the height of the ball above the bowl's center; the higher the ball the greater its energy.

- Eventually the ball comes to rest at the bottom of the bowl - potential energy is gone . . .

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- The ball-bowl system settles in an energy minimum at equilibrium when it is allowed to operate under its own dynamics

- The <u>resting state</u> is said to be <u>stable</u> because the <u>system remains there</u> after it has been reached.

- We suppose that the ball comes to rest in the same place each time because it "remembers" where the bottom of the bowl is.

<u>position vector</u> $\bar{x}(t) = (x(t), y(t), z(t))$

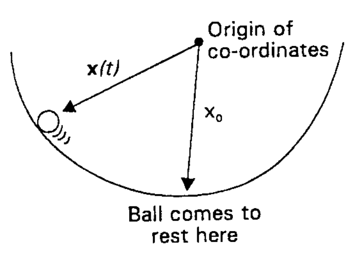

Figure 7.3 Bowl and ball bearing with state description.

<u>The location of the bottom of the bowl</u>, $\bar{X}_p$, represents the <u>pattern that is stored</u>

where $\bar{X} = \bar{X}_p + \Delta \bar{X}$

$\bar{X}$: ball's initial position

- We may think of the <u>ball's initial position</u> as representing the <u>partial Knowledge</u> or cue for recall, since it <u>approximates</u> the <u>memory</u> $\bar{X}_p$

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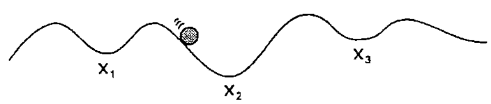
- If a corrugated surface is employed instead of a single depression (as in the bowl), we may store many "memories".



- If the ball is started somewhere on this surface, it will eventually come to rest at the local depression that is closest to its initial starting point.

i.e. it evokes the stored pattern which is closest to its initial partial pattern or cue. This corresponds to an energy minimum of the system.

- The memories shown correspond to states $\bar{X}_1, \bar{X}_2, \bar{X}_3$

- Notice that the energy of each of the equilibria E_i may differ, but each one is the lowest available locally within its basin of attraction.

- John Hopfield (1982) - American physicist proposed an asynchronous neural network model

Hopfield model:

- Individual units preserve their own states until they are selected for an update
- The selection (for updating) is made randomly
- n totally coupled units - each unit is connected to all other units except itself
- The network is symmetric $w_{ij} = w_{ji} \ \forall i,j$

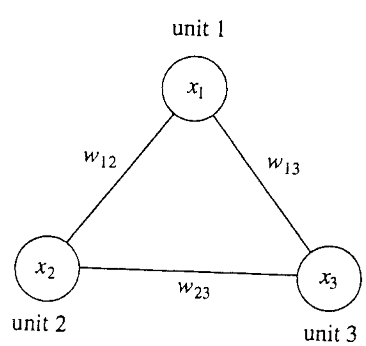


Fig. 13.2. A Hopfield network of three units

- Each unit can assume the state 1 or −1.
- If the weight matrix does not contain a zero diagonal, the network dynamics do not necessarily lead to stable states.

For example:

$$\mathbf{W} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

transforms the state vector $(1,1,1)$ into the state vector $(-1,-1,-1)$ and conversely.

- A connection matrix with a zero diagonal can also lead to oscillations in the case where the weight matrix is not symmetric

- The weight matrix $\mathbf{W} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

  describes this network:

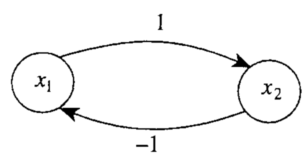


Fig. 13.3. Network with asymmetric connections

- The state vector $(1,-1)$ is transformed into $(1,1)$
  $(-1,1) \rightarrow (-1,-1) \rightarrow (1,-1)$

- The symmetry of the weight matrix and a zero diagonal are thus necessary conditions for the convergence of an asynchronous totally connected network to a stable state (... sufficient as well).

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The Energy function -

Definition 13.2.1. *Let* $\mathbf{W}$ *denote the weight matrix of a Hopfield network of* n *units and let* θ *be the* n*-dimensional row vector of units' thresholds. The energy* $E(\mathbf{x})$ *of a state* $\mathbf{x}$ *of the network is given by*

$$E(\mathbf{x}) = -\frac{1}{2}\mathbf{x}\mathbf{W}\mathbf{x}^{\mathrm{T}} + \theta\mathbf{x}^{\mathrm{T}}.$$

The energy function can also be written in the form

$$E(\mathbf{x}) = -\frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} w_{ij}x_i x_j + \sum_{i=1}^{n}\theta_i x_i.$$

The factor 1/2 is used because the identical terms $w_{ij}x_ix_j$ and $w_{ji}x_jx_i$ are present in the double sum.

The energy function of a Hopfield network is a quadratic form. A Hopfield network always finds a local minimum of the energy function.

An example

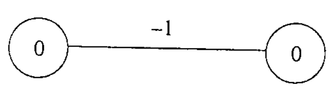

Fig. 13.4. A flip-flop

two units with threshold 0.

The only stable states are $(1,-1)$ and $(-1,1)$

$w_{12} = w_{21} = -1$

$$E(x_1, x_2) = x_1 x_2$$

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Energy function for continuous Hopfield model

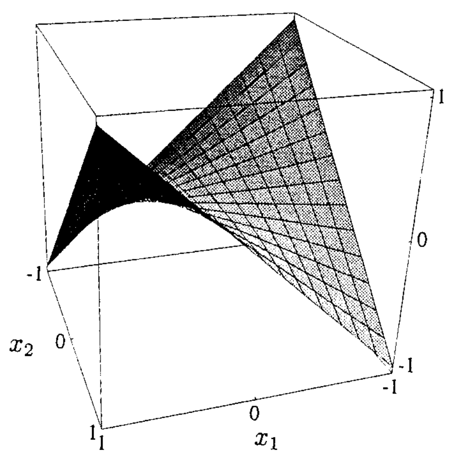

Fig. 13.5. Energy function of a flip-flop

Unit's states can assume all real values between 0 and 1.

The energy function has local minima at (1, −1) and (−1, 1)

Hopfield network for computing logic functions

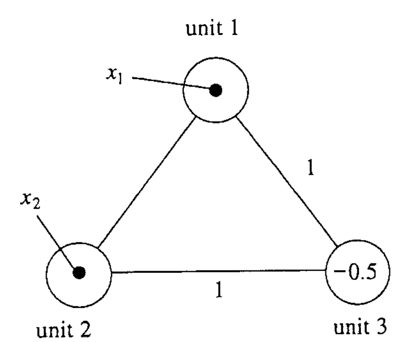

Fig. 13.6. Network for the computation of the OR function

note - the individual units are perceptrons ... hence we suspect XOR cannot be implemented with 3 units.

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- Suppose a Hopfield network with three units should store stable states given by:

unit	1	2	3
state 1	−1	−1	−1
state 2	1	−1	1
state 3	−1	1	1
state 4	1	1	−1

- From point of view of third unit, this is XOR function.

- The third unit should be capable of linearly separating the vectors (−1, −1) and (1, 1) from (−1, 1) and (1, −1) but this is impossible.

XOR problem can be solved if four units employed. The unknown weights can be found using a learning algorithm (Hebbian learning or a variation on Perceptron learning)

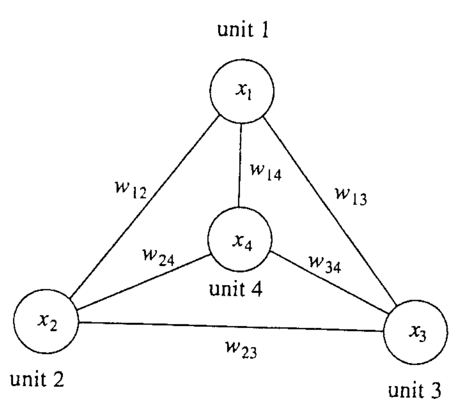

Fig. 13.7. Network for the computation of XOR

unit	1	2	3	4
state 1	−1	−1	−1	1
state 2	1	−1	1	1
state 3	−1	1	1	1
state 4	1	1	−1	−1

Convergence to Stable States

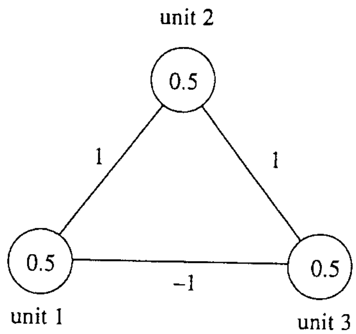

Fig. 13.9. Example of a Hopfield network

- arbitrarily chosen weights and thresholds

- This network can adopt any of eight possible states.

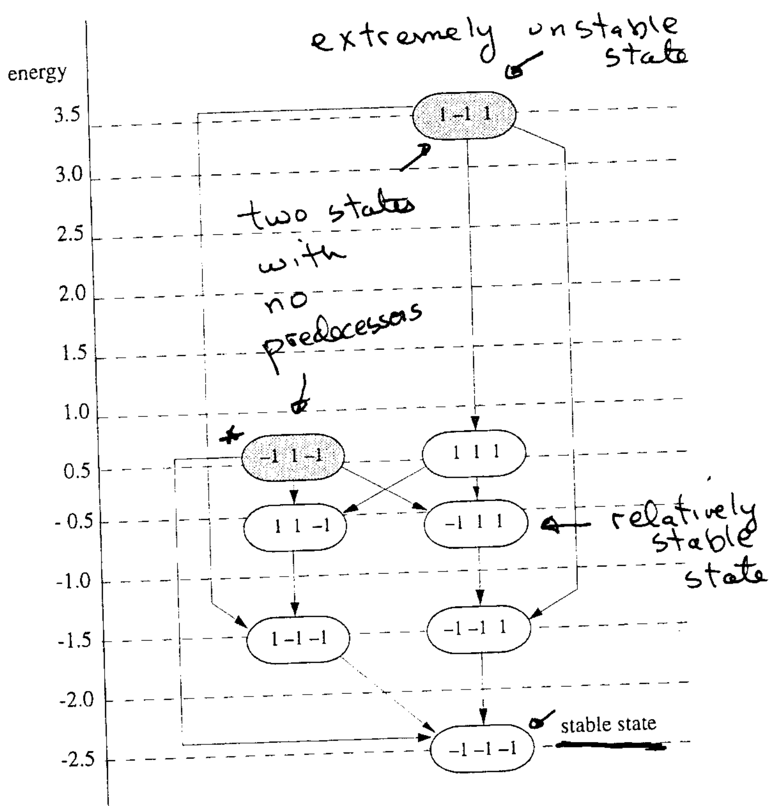

Fig. 13.10. State transitions for the network of Figure 13.9

Each transition has the same probability - the probability of selecting one of the three units for a state transition is uniform and equal to 1/3

energy calculation

$$* \quad E(x) = -\frac{1}{2}\sum_{j=1}^{3}\sum_{i=1}^{3} w_{ij}x_ix_j + \sum_{i=1}^{3}\theta_i x_i$$

for state (111)

$$-\frac{1}{2}\left(\overset{1}{w_{12}}\overset{1}{x_1}\overset{1}{x_2} + \overset{(-1)}{w_{13}}\overset{1}{x_1}\overset{1}{x_3} + \overset{1}{w_{21}}\overset{1}{x_2}\overset{1}{x_1} + \overset{1}{w_{23}}\overset{1}{x_2}\overset{1}{x_3} + \overset{1}{w_{32}}\overset{1}{x_3}\overset{1}{x_2} + \overset{-1}{w_{31}}\overset{1}{x_3}\overset{1}{x_1}\right) + \overset{1/2}{\theta_1}\cdot\overset{1}{x_1} + \overset{1/2}{\theta_2}\cdot\overset{1}{x_2} + \overset{1/2}{\theta_3}\overset{1}{x_3}$$

$$= -\frac{1}{2}\left(1+(-1)+1+1+1+(-1)\right) + \frac{3}{2}$$

$$= -\frac{1}{2}(2) + \frac{3}{2} = -1+\frac{3}{2} = \underline{0.5}$$

- Hopfield networks can be used to find approximate solutions for difficult problems.

- Hopfield networks do not require synchronization; they guarantee that a local minimum of the energy function will be reached.

- If an optimization function can be written in an analytical form isomorphic to the Hopfield energy function – it can be solved by a Hopfield network.

- Every unit in the network is simulated by a small processor.

- The states of the units can be computed asynchronously by transmitting the current unit states from processor to processor.

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The multiflop problem

Binary vector of dimension n whose components are all zero except for a single 1.

Hopfield network to solve this problem:

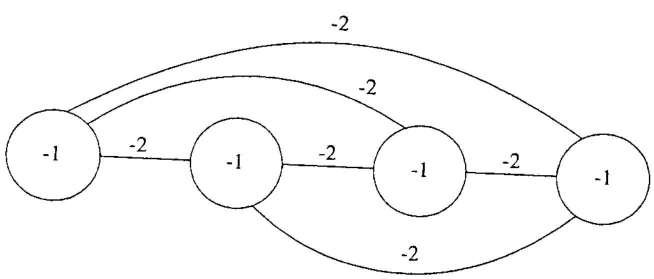

Fig. 13.15. A multiflop network

Whenever a unit is set to 1, it inhibits the other units through the edges with weight −2.

If network is started with all units set to zero, then the excitation of every unit is 0, which is greater than the threshold and ∴ the first unit to be asynchronously selected will flip its state to 1. No other unit can change its state after this first unit has been set to 1.

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The eight rooks problem

- n rooks must be positioned on an $n \times n$ chessboard so no one figure can take another. Each rook must be positioned on a different row and column to others.

- A solution to the 4-rooks problem.

Hopfield network

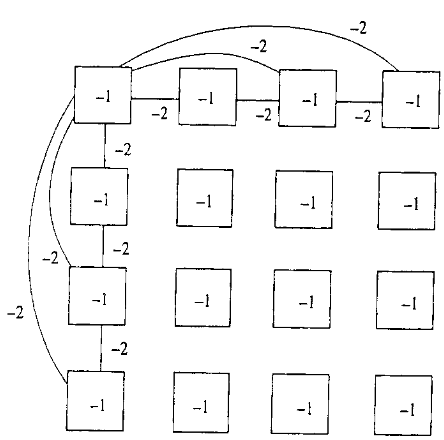

Fig. 13.16. Network for the solution of a four rooks problem

Each field is represented by a unit.

The connections of each unit to all elements in the same row or column have the weight -2, all others have a weight 0.

All units have a threshold of -1.

Any unit set to 1 inhibits any other units in the same row or column.

If a row or column is all set to zero, when one of its elements is selected, it will immediately switch to state 1.

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- Eight Queens Problem
- 8 Queens on an 8×8 chessboard so that no two Queens are attacking.
- Problem is solved by overlapping multiflop problems at each square.

Q
Q
Q
Q

Solution to 4-queens problem

- Hopfield network ... must inhibit rows, columns and diagonals ...

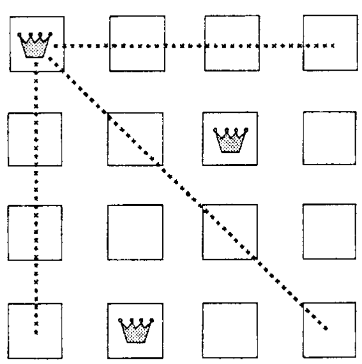

Fig. 13.17. The eight queens problem

- three multiflop chains for each field
- $W_{ij} = -2$ when unit i is different from unit j and belongs to the same row, column, or diagonal as unit j.
- Otherwise $w_{ij} \leftarrow 0$.

Thresholds of all units $\leftarrow -1$

- We do not always get a correct solution. Energy function has local minima with less than n-queens.

The Traveling Salesperson Problem (TSP)

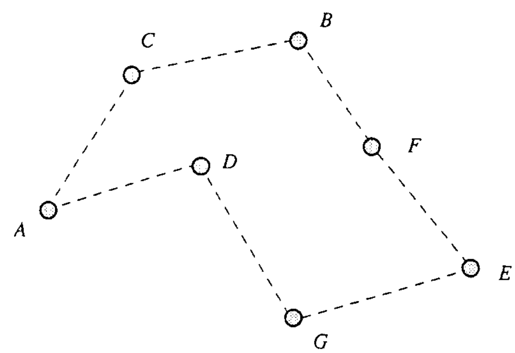

Fig. 13.18. A TSP and its solution

A round trip can be represented using a matrix.

- Each of the n rows of the matrix is associated with a city.
- The columns are labeled 1 through n and they correspond to the n necessary visits.

	1	2	3	4
S_1	1	0	0	0
S_2	0	1	0	0
S_3	0	0	1	0
S_4	0	0	0	1

- This matrix shows a path going through the cities S_1, S_2, S_3, and S_4 in that order.

A single 1 is allowed in each row and each column.
(same conditions as rooks problem)

We need to minimize $L = \frac{1}{2} \sum_{i,j,k}^{n} d_{ij} X_{ik} X_{j,k+1}$

We let last column be equal to the first to ensure a round trip.

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Hopfield and Tank – were the first to use a Hopfield network to solve a difficult problem.

– Could NP-Complete problems be solved in polynomial time? (or at least approximate solutions obtained).

– However, to obtain an optimal solution to an NP-Complete problem – the size of the network explodes exponentially.

– Large Hopfield networks may however be implemented if electrical circuits or optical circuits are employed.