Introduction to Analysis*

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1 Connected spaces

Theorem 1.1. *All intervals on the real line $\mathbb{R}$ are connected.*

By interval we mean any kind of interval: closed, open, semiclosed, finite or infinite. This result is given in [2, Theorem, Ch. III. §6, p. 60].

Proof. Call the interval S. Assume, on the contrary, that $S = A \cup B$ and $A \cap B = \emptyset$ for some nonempty sets A and B that are closed in S. Then there are sets $A_1, B_1 \subset \mathbb{R}$ closed in $\mathbb{R}$ such that $A = S \cap A_1$ and $B = S \cap B_1$ (see [1, Second Lemma on p. 14, §6]). Let $a \in A$ and $b \in B$. Without loss of generality, we may assume that $a < b$. As $[a, b] \subset S$, S being an interval, for the sets $A_2 \overset{def}{=} A \cap [a, b]$ and $B_2 \overset{def}{=} B \cap [a, b]$ we have $A_2 = A_1 \cap [a, b]$ and $B_2 = B_1 \cap [a, b]$; thus A_2 and B_2, being intersections of sets that are closed in $\mathbb{R}$, are sets closed in $\mathbb{R}$.

The set A_2 has a greatest element according to [2, Proposition on p. 44], let this be c. Then $c \in [a.b]$ since $A_2 \subset [a, b]$. We have $c < b$ since $b \in B_2$ and $A_2 \cap B_2 = \emptyset$. Since we also have $A_2 \cup B_2 = [a, b]$, it follows that

$$(c, b] \subset B_2, \tag{1.1}$$

since A_2 has no elements in this interval. Now,

$$c = \lim_{n\to\infty} \left(c + \frac{1}{n} \right),$$

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For large enough n we have $c+1/n \in (c, b]$.[1.1] Since B_2 is closed in $\mathbb{R}$, it therefore follows from (1.1) according to [2, Theorem on p. 47] that $c \in B_2$. As we also have $c \in A_2$ and $A_2 \cap B_2 = \emptyset$, this is a contradiction, completing the proof. $\square$

2 The generalized Dirichlet convergence criterion

Theorem 2.1 (Extended Dirichlet's convergence criterion)**.** *Let a_n and b_n for $n \geq 1$ be complex numbers. Writing*

$$A_n = \sum_{k=1}^{n} a_k \qquad (n \geq 1), \tag{2.1}$$

assume that

$$\lim_{n\to\infty} A_n b_n = 0. \tag{2.2}$$

Assume, further, that the sum

$$\sum_{n=1}^{\infty} A_n(b_n - b_{n+1}) \tag{2.3}$$

converges. Then the series

$$\sum_{n=1}^{\infty} a_n b_n \tag{2.4}$$

converges.

Proof. First, note that the convergence of the sum in (2.3) means that $\lim_{n\to\infty} A_n(b_n - b_{n+1}) = 0$, and so we also have

$$\lim_{n\to\infty} A_n b_{n+1} = 0 \tag{2.5}$$

in view of (2.2).

We have $a_n = A_n - A_{n-1}$, so, given integers M and N with $0 \leq M < N$ we have

$$\begin{aligned} \sum_{n=M+1}^{N} a_n b_n &= \sum_{n=M+1}^{N} (A_n - A_{n-1}) b_n \\ &= A_N b_{N+1} - A_M b_{M+1} + \sum_{n=M+1}^{N} A_n(b_n - b_{n+1}) \to 0 \end{aligned} \tag{2.6}$$

as $M, N \to \infty$; the second equation follows by summation by parts, and the fact that the limit is 0 follows from (2.5) and the convergence of the series in (2.3). This shows that the series in (2.4) indeed converges. $\square$

[1.1]Formally:

$$(\exists N \in \mathbb{N})(\forall n > N)\left[c + \frac{1}{n} \in (c, b]\right].$$

Corollary 2.1 (Dirichlet's convergence criterion)**.** *Let a_n and b_n for $n \ge 1$ be complex numbers such that*

$$\lim_{n\to\infty} b_n = 0 \tag{2.7}$$

and

$$\sum_{n=1}^{\infty} |b_n - b_{n+1}| < \infty. \tag{2.8}$$

Assume that there is a real number A such that

$$\left|\sum_{n=1}^{N} a_n\right| \le A \tag{2.9}$$

for all $N \ge 1$. Then the series

$$\sum_{n=1}^{\infty} a_n b_n \tag{2.10}$$

converges.

This result is the Generalized Dirichlet Convergence Test. In the original version of the Dirichlet Test, instead of (2.8) one assumes that b_n is real and $b_n \ge b_{n+1} > 0$ for all $n \ge 1$. The Alternating Series Test is a consequence of the original version of the Dirichlet Test. Indeed, one obtains the Alternating Series Test if one takes $a_n = (-1)^{n+1}$.

Proof. It is easy to show that the numbers a_n and b_n safisfying the assumptions of this corollary also satisfy the assumptions of Theorem 2.1. Indeed, (2.9) says that the numbers A_n defined in (2.1) are bounded, hence (2.2) follows from (2.7), and (2.3) follows from (2.8). □

3 Differentiability at a missing point

Theorem 3.1. *Let (a, b) be an interval on the real line, and let $c \in (a, b)$. Assume the real-valued function f is continuous on (a, b), and differentiable at every point x in (a, b) that is different from c. Assume, further, that the limit $d = \lim_{x\to c} f'(x)$ exists (in particular, it is finite). Then f is also differentiable at c, and $f'(c) = d$.*

Proof. Let $x \in (a, b) \setminus \{c\}$. By the Mean Value Theorem of differentiation, there is a ξ_x strictly between x and c (that is, if $x < c$ then $x < \xi_x < c$, and if $x > c$ then $x > \xi_x > c$) such that

$$\frac{f(x) - f(c)}{x - c} = f'(\xi_x).$$

Making $x \to c$, the number ξ_x will approach c, so $f'(\xi_x)$ will approach $d = \lim_{x\to c} f'(x)$. Hence, it follows that

$$f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c} = \lim_{x\to c} f'(\xi_x) = \lim_{x\to c} f'(x) = d.$$

□

Remark 3.1. The way this proof is formulated, one defines a function that maps x to ξ_x. Since the choice of ξ_x is not unique, the definition of such a function involves the Axiom of Choice. Since the the Axiom of Choice is commonly used in arguments of introductory analysis,[3.1] one might not be worried about its use here. It is however easy to rewrite this proof in a way that requires no reference to the Axiom of Choice if one uses the Cauchy definition, i.e., the ϵ–δ definition of limit.[3.2]

Here is a proof that clearly avoids the Axiom of Choice.

Second Proof. Let $\epsilon > 0$ be arbitrary, and let $\delta > 0$ such that $\delta \le \min(c-a, b-c)$ and such that for every ξ with $0 < |\xi - c| < \delta$ we have $|f'(\xi) - d| < \epsilon$. Let x be such that $0 < |x - c| < \delta$. By the Mean Value Theorem of differentiation, there is a ξ strictly between x and c (that is, if $x < c$ then $x < \xi < c$, and if $x > c$ then $x > \xi > c$) such that

$$\frac{f(x) - f(c)}{x - c} = f'(\xi).$$

Then we also have $0 < |\xi - c| < \delta$, and so

$$\left| \frac{f(x) - f(c)}{x - c} - d \right| = |f'(\xi) - d| < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, this shows that

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = d.$$

□

Perhaps the first proof we gave has greater intuitive appeal, but, as we pointed out, it ignores important foundational issues. Close attention to foundational issues is, however, uncommon even in rigorous introductions to analysis.

Remark 3.2. If one wants to differentiate the function

$$f(x) = \frac{(x-1)^7 (x+2)^9}{(x^2+4)^6},$$

one may simplify the calculation by using logarithmic differentiation, i.e., by calculating the derivative of $\log |f(x)|$ and then finding $f'(x)$ with the aid of the formula

$$\log |f(x)| = \frac{f'(x)}{f(x)};$$

here $\log t$ denotes the natural logarithm of t.[3.3] One may, however, wonder whether the result is correct also for $x = 1$ or $x = -2$, since $\log |f(x)|$ is not defined at these points. One may use the result of the present problem to confirm that the result is indeed correct also at these points.

[3.1] Actually, most uses of the Axiom of Choice can be replaced by the use of its significant weakening, called the Axiom of Dependent Choice, in introductory analysis. To show, however, that there is a set of reals that is not Lebesgue measurable the Axiom of Dependent Choice is not sufficient, according to a result of Robert M. Solovay. There are deep foundational issues we are skirting here that would require a much more extended discussion. Suffice it to say here that Solovay assumed the existence of an inaccessible cardinal in his proof, a natural assumption but unprovable in standard axiomatic set theory. Much later, Saharon Shelah showed that Solovay's assumption is indispensable in establishing the result.

[3.2] The Heine definition involving sequences is more problematic. In any case, one needs the Axiom of (Dependent) Choice to establish the equivalence of the Cauchy and Heine definitions.

[3.3] This is the usual notation in mathematical writing. The symbol $\ln t$ is rarely used in mathematics outside of introductory calculus courses.

References

[1] Attila Máté. Supplementary notes on Introduction to Analysis. http://www.sci.brooklyn.cuny.edu/~mate/anl/analysis.pdf, January 2013.

[2] Maxwell Rosenlicht. *Introduction to Analysis*. Dover, New York, 1986.