Exam 1, Mathematics 11.1, Section TR11 28 copies
11:00–12:15 am, February 26, 2009, 330NE
Instructor: Attila Máté

Follow these instructions carefully:

Work on the paper provided; do not use your own paper. *Work only on one problem on each sheet (you should not work on two different problems on the two sides of the same sheet).* On the top of each page, *print* your name (*encircle your last name*) and indicate the number of the problem you are working on by writing e.g. *"Problem #4"*. Always *encircle* your final answer. If there are several parts to a problem, always indicate the part that you are answering, e.g. by writing *"Answer to Part b)"* (the number of the problem should be on the top of the page). Do not use a *red* pen or a *red* pencil. Do not write in the corner covered up by the staple (top left corner on the front side, top right corner on the back side). Each problem is worth the *same* amount of credit.

1. *a*) Define what a cut is.
b) Define what it means for a cut (A, B) to determine the real number t.
c) State the Axiom of Completeness (in terms of cuts).
d) Let S be a nonempty set of reals that is bounded from above. In proving that S has a supremum, one defines a cut (A, B) such that the number determined by (A, B) turns out to be the supremum of S. *Without proving anything*, give the definition of such a cut.

2. *a*) Let S be a subset of $\mathbb{R}$ and assume that $c = \sup S$. Let $\epsilon > 0$. Show that there is an $x \in S$ such that $x > c - \epsilon$.
b) Let S be a subset of $\mathbb{R}$ and assume that $c = \sup S$, and let $d = \max S$ (the maximum of S). Prove that $c = d$.

3. *a*) Give the definition that the sequence $\{p_n\}_{n=1}^{\infty}$ converges to p in the metric space (E, d) (you need to give the definition *in words*; in addition, you may also give the formal definition).
b) Define what it means for the real number a to be the supremum of the set S of reals.
c) Assume S is a nonempty closed and bounded set of reals. Prove that $a \stackrel{def}{=} \sup S$ is an element of S.

4. *a*) Define what it means for the sequence $\{p_n\}_{n=1}^{\infty}$ in a metric space (E, d) to be a Cauchy sequence.
b) Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence of reals, and let a be the supremum of the set $S = \{x :$ there are infinitely many positive integers n for which $x \leq a_n\}$. Let ϵ be a positive real number and let N be a positive integer. Show that there is an integer $n > N$ for which $a - \epsilon < a_n < a + \epsilon$.

5. *a*) Define what it means for a metric space (E, d) to be compact.
b) Given a metric space (E, d), a set $S \subset E$, and a point $p \in E$, define what it means for p to be the cluster point of S.
c) Assume (E, d) is a compact metric space and $S \subset E$ is an infinite set. Prove that S has a cluster point.