Exam 1, Mathematics 11.1, Section TR11 28 copies
11:00–12:15 am, February 26, 2009, 330NE
Instructor: Attila Máté[1]

1. *a*) Define what a cut is.

Definition. With $\mathbb{R}$ denoting the set of all real numbers, a cut is a pair (A, B) where A and B are nonempty sets of reals such that $A \cup B = \mathbb{R}$, and for every $x \in A$ and $y \in B$ we have $x < y$.

b) Define what it means for a cut (A, B) to determine the real number t.

Definition. The cut (A, B) is said to determine the real number t if $x \le t$ for every $x \in A$ and $t \le y$ for every $y \in B$.

c) State the Axiom of Completeness (in terms of cuts).

Solution. The Axiom of Completeness says that every cut determines a real number.

d) Let S be a nonempty set of reals that is bounded from above. In proving that S has a supremum, one defines a cut (A, B) such that the number determined by (A, B) turns out to be the supremum of S. *Without proving anything*, give the definition of such a cut.

Solution. Let A be the set of reals that are not upper bounds of S, and let B be the set of reals that are upper bounds of S. Formally,

$$A = \{a : \exists x \in S\ [a < x]\}$$

and

$$B = \{b : \forall x \in S\ [x \le b]\}.$$

Then (A, B) is a cut, and the number determined by the cut (A, B) is the supremum of S.

2. *a*) Let S be a subset of $\mathbb{R}$ and assume that $c = \sup S$. Let $\epsilon > 0$. Show that there is an $x \in S$ such that $x > c - \epsilon$.

Solution. As c is the least upper bound of S, no smaller number, namely $c - \epsilon$, is an upper bound of S. Saying that this number is not an upper bound means that there is an $x \in S$ bigger than this number, i.e., for which $x > c - \epsilon$.

b) Let S be a subset of $\mathbb{R}$ and assume that $c = \sup S$, and let $d = \max S$ (the maximum of S). Prove that $c = d$.

Solution. By the definition of maximum, we have $d \in S$ and, further, for every $x \in S$ we have $x \le d$. Hence d is an upper bound of S. There can be no upper bound y of S that is smaller than d; in fact, as $d \in S$, we must have $d \le y$ for every upper bound y of S. That is, d is the least upper bound of S; i.e., $d = c$.

3. *a*) Give the definition that the sequence $\{p_n\}_{n=1}^{\infty}$ converges to p in the metric space (E, d) (you need to give the definition *in words*; in addition, you may also give the formal definition).

Definition. The sequence $\{p_n\}_{n=1}^{\infty}$ in the metric space (E, d) is said to converge to the point $p \in E$ if for every $\epsilon > 0$ there is an N (which can be taken to be a positive integer, if so desired) such that, for every $n > N$ we have $d(p, p_n) < \epsilon$. Formally,

$$\text{converges}(\{p_n\}_{n=1}^{\infty}, p) \leftrightarrow \forall \epsilon > 0\ \exists N\ \forall n > N\ [d(p, p_n) < \epsilon].$$

b) Define what it means for the real number a to be the supremum of the set S of reals.

---

[1] All computer processing for this manuscript was done under Fedora Core Linux. $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX was used for typesetting.

Definition. The real number a is the supremum of the set S of reals if a is the least upper bound of S; that is, if a is an upper bound of S and it is smaller than any other upper bound of S. Formally,

$$\text{supremum}(S, a) \leftrightarrow \big(\forall x \in S\ [x \le a]\ \ \&\ \ \forall y\ [\forall x \in S\ (x \le y) \to a \le y]\big).$$

c) Assume S is a nonempty closed and bounded set of reals. Prove that $a \overset{def}{=} \sup S$ is an element of S.

Solution. Assume that $a \notin S$, i.e., that $a \in \mathbb{R} \setminus S$. As S is closed, the complement of S, i.e, the set $\mathbb{R} \setminus S$, is open. Since a belongs to this set, there is an $\epsilon > 0$ such that the open ball with center a and radius ϵ, i.e., the interval $(a - \epsilon, a + \epsilon)$, is included in $\mathbb{R} \setminus S$. That is, the set $S \cap (a - \epsilon, a + \epsilon)$ is empty. Hence, there is no element $x \in S$ for which $a - \epsilon < x \le a$. Since a is an upper bound of S, we have $x \le a$ for every $x \in S$; therefore, we also have $x \le a - \epsilon$. That is, $a - \epsilon$ is an upper bound of S. This is a contradiction, since $a - \epsilon < a$, and a is the least upper bound of S. This contradiction proves that the assumption $a \notin S$ was incorrect. That is, we indeed have $a \in S$, as we wanted to prove.

This means that for a closed set S of reals, the supremum is in fact the largest element (i.e., the maximum) of S. The maximum a of S can be defined formally as

$$\text{maximum}(S, a) \leftrightarrow \big(a \in S\ \ \&\ \ \forall x \in S\ [x \le a]\big).$$

The advantage of the supremum over the maximum is that every nonempty bounded set of reals has a supremum, but such a set need not have a maximum. For example, the supremum of the set $(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$ is 1, while this set has no maximum.

4. *a*) Define what it means for the sequence $\{p_n\}_{n=1}^{\infty}$ in a metric space (E, d) to be a Cauchy sequence.

Definition. The sequence $\{p_n\}_{n=1}^{\infty}$ of points in the metric space (E, d) is called a Cauchy sequence if for every $\epsilon > 0$ there is an N (which can be taken to be a positive integer, if desired) such that we have $d(p_n, p_m) < \epsilon$ whenever $m, n > N$. Formally,

$$\text{Cauchy}(\{p_n\}_{n=1}^{\infty}) \leftrightarrow \forall \epsilon > 0\ \exists N\ \forall m > N\ \forall n > N\ [d(p_m, p_n) < \epsilon].$$

b) Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence of reals, and let a be the supremum of the set $S = \{x :$ there are infinitely many positive integers n for which $x \le a_n\}$. Let ϵ be a positive real number and let N be a positive integer. Show that there is an integer $n > N$ for which $a - \epsilon < a_n < a + \epsilon$.

Solution. As a is the least upper bound of S, the number $a + \epsilon$ does not belong to S (since the inequality $a + \epsilon < a$ does not hold). Thus, there are only finitely many positive integers n for which $a_n \ge a + \epsilon$.

Further, as a is the least upper bound of S, the number $a - \epsilon$ is not an upper bound of S. Hence there is an $x \in S$ for which $a - \epsilon < x$. Then, according to the definition of S, there are infinitely many positive integers n for which $x \le a_n$. Hence, *a fortiori*,[2] there are infinitely many positive integers n for which $a - \epsilon < a_n$; i.e., the set

$$\{n \in \mathbb{N} : a - \epsilon < a_n\}$$

is infinite. Omitting finitely many elements from this set, namely those n's for which $n \le N$ or for which $a_n \ge a + \epsilon$, we can pick an n from among the remaining elements for which $n > N$ and $a - \epsilon < a_n < a + \epsilon$, showing that an integer n exists as required.

The above argument, with $\epsilon/2$ instead of ϵ, is the key step in a proof that if $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence of reals, then this sequence converges.

5. *a*) Define what it means for a metric space (E, d) to be compact.

---

[2] Even more so, or for a still stronger reason. Latin, occasionally used in mathematical and other specialized (technical, scientific, or legal) writing.

Definition. Let (E, d) be a metric space and let S be a subset of E. The set S is said to be *compact* in (E, d) if the following holds: For any collection $\mathcal{U}$ of open sets in (E, d), if $S \subset \bigcup \mathcal{U}$, there is a finite subcollection $\mathcal{U}' \subset \mathcal{U}$ such that $S \subset \bigcup \mathcal{U}'$.

Further, the space (E, d) is said to be compact if E is a compact set in (E, d).

To say that $\mathcal{U}$ is a collection of open sets in (E, d) means that $\mathcal{U}$ is a collection every element of which is an open subset of (E, d). The inclusion $S \subset \bigcup \mathcal{U}$ is expressed by saying that $\mathcal{U}$ is on open cover of S. One also says that $\mathcal{U}$ covers S, or that the elements of $\mathcal{U}$ cover S. If $\mathcal{U}' \subset \mathcal{U}$ and $S \subset \bigcup \mathcal{U}'$, then $\mathcal{U}'$ is called a subcover of $\mathcal{U}$ (for S, if one wants to avoid repeating the preposition "of"). Thus, one can say shortly that a set is compact if and only if any of its open covers has a finite subcover.

b) Given a metric space (E, d), a set $S \subset E$, and a point $p \in E$, define what it means for p to be the cluster point of S.

Definition. The point $p \in E$ of the metric space (E, d) is said to be a *cluster point* of the set $S \subset E$ if for every $\epsilon > 0$ the open ball

$$U(p, \epsilon) = \{q \in E : d(p, q) < \epsilon\}$$

contains infinitely many elements of S.

c) Assume (E, d) is a compact metric space and $S \subset E$ is an infinite set. Prove that S has a cluster point.

Solution. Assume, on the contrary, that S has no cluster point. Then, for every $p \in E$ there is and $\epsilon(p) > 0$ such that the open ball $B(p) \overset{def}{=} U(p, \epsilon(p))$ contains only finitely many points of p. Then the collection

$$\mathcal{U} = \{B(p) : p \in E\}$$

of open sets covers E (since $p \in B(p)$). As E is compact, a finite subset $\mathcal{U}'$ of $\mathcal{U}$ also covers E; as $S \subset E$, it then also covers S; that is,

$$S \subset \bigcup \mathcal{U}'.$$

Since, by our assumption, the set $B \cap S$ is finite for each $B \in \mathcal{U}$, it follows that the set

$$S = S \cap \bigcup \mathcal{U}' = \bigcup \{S \cap B : B \in U'\},$$

being a union of finitely many finite sets, is finite; the second equality here is true by a distributive rule for the union of sets.[3] This is a contradiction, since the set S is infinite by our assumptions.

---

[3] The distributive rule in question says that

$$X \cap \bigcup \mathcal{A} = \bigcup \{X \cap A : a \in \mathcal{A}\}$$

is valid for any set X and any collection $\mathcal{A}$ of sets. This equation is easy to verify with the aid of the definition of the union of sets.