Exam 2, Mathematics 11.1, Section TR11 28 copies
11:00–12:10 am, March 26, 2009, 330NE
Instructor: Attila Máté

Follow these instructions carefully:

Work on the paper provided; do not use your own paper. *Work only on one problem on each sheet (you should not work on two different problems on the two sides of the same sheet).* On the top of each page, *print* your name (*encircle your last name*) and indicate the number of the problem you are working on by writing e.g. *"Problem #4"*. Always *encircle* your final answer. If there are several parts to a problem, always indicate the part that you are answering, e.g. by writing *"Answer to Part b)"* (the number of the problem should be on the top of the page). Do not use a *red* pen or a *red* pencil. Do not write in the corner covered up by the staple (top left corner on the front side, top right corner on the back side). Each problem is worth the *same* amount of credit.

1. *a*) Let (E, d) be a metric space. Define what it means for (E, d) to be connected.
   b) Prove that for any $a, b \in \mathbb{R}$ with $a < b$ the interval $[a, b]$ is connected.

2. *a*) Let f be a continuous function from the metric space (E, d) to the metric space (E', d'). Show that for every open set $V \subset E'$ the set $f^{-1}(V) \overset{def}{=} \{p \in E : f(p) \in V\}$ is open.

3. Let (E, d) and (E', d') be metric spaces. Let $f : E \to E'$ be a function and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions $f_n : E \to E'$.

a) Give a definition of f being continuous on E.

b) Give a definition of f being uniformly continuous on E. *After giving the definition*, explain what the difference is between continuity and uniform continuity. (The emphasis is on the word *after*; the definition and the explanation must not be intermixed.)

c) Give a definition of the sequence $\{f_n\}_{n=1}^{\infty}$ being convergent to f. (Give the definition in terms of ϵ and N, and not in terms of sequences of points.)

d) Give a definition of the sequence $\{f_n\}_{n=1}^{\infty}$ being uniformly convergent to f. *After giving the definition*, explain what the difference is between convergence and uniform convergence. (The emphasis is on the word *after*; the definition and the explanation must not be intermixed.)

4. *a*) Let $U \subset \mathbb{R}$ be an open set, let $x_0 \in U$, and assume that the function $f : U \to \mathbb{R}$ is differentiable at x_0. Prove that f is continuous at x_0.

b) Let $U \subset \mathbb{R}$ be an open set, let $x_0 \in U$, and assume that the functions $f : U \to \mathbb{R}$ and $g : U \to \mathbb{R}$ are differentiable at x_0. Using the identity

$$\frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} = \frac{f(x) - f(x_0)}{x - x_0}g(x) + f(x_0)\frac{g(x) - g(x_0)}{x - x_0},$$

prove that $f(x)g(x)$ is differentiable at x_0 and $(f(x_0)g(x_0))' = f'(x_0)g(x_0) + f(x_0)g'(x_0)$. *Clearly explain* how the result of Part *a*) is used in the proof.

5. *a*) State and prove Rolle's Theorem.
   b) State (without proof) the Mean-Value Theorem (for differentiation).