Exam 2, Mathematics 11.1, Section TR11 28 copies
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1. *a*) Let (E, d) be a metric space. Define what it means for (E, d) to be connected.

Definition. The metric space (E, d) is said to be *connected* if the only sets in this space that are both open and closed are the empty set $\emptyset$ and the whole space E.

This definition can be restated to say that (E, d) is connected if there are no nonempty disjoint open sets U and V such that $U \cup V = E$. (Instead of "open," one can say "closed" here.)

b) Prove that for any $a, b \in \mathbb{R}$ with $a < b$ the interval $[a, b]$ is connected.

Solution. Assume, on the contrary that $[a, b] = A \cup B$ where A and B are disjoint nonempty sets that are open in $[a, b]$. Note that this does not mean that A and B are open sets in $\mathbb{R}$; it actually means that there are set U and V that are open in $\mathbb{R}$ such that $A = U \cap [a, b]$ and $B = V \cap [a, b]$.[2] Then the set A is closed in $\mathbb{R}$; indeed, this set is the intersection of two closed sets in $\mathbb{R}$, since we have

$$A = [a, b] \setminus B = [a, b] \cap (\mathbb{R} \setminus V).$$

Similarly, the set B is closed. The point b belongs either to the set A or to the set B; there is no harm in assuming that $b \in B$.

Let c be the maximum of A; since A is a nonempty bounded closed set, A does have a maximum. We have $c \in [a, b]$, since $A \subset [a, b]$. Further, we have $c < b$, since $b \in B$, so $b \notin A$. As $c \in A \subset U$, and the set U is open in $\mathbb{R}$, there is an $\epsilon > 0$ such that the open ball $\{x : c - \epsilon < x < \epsilon\}$ is a subset of U; we may assume that $\epsilon < b - c$ here. Then $c + \epsilon/2$ belongs to this open ball; hence it belongs to U. As $a \le c + \epsilon/2 < b$, $c + \epsilon/2$ belongs to $A = U \cap [a, b]$. Since $c < c + \epsilon/2$, this contradicts the definition of c as the maximum of A. This contradiction shows that the sets A and B as described cannot exists. That is, the interval $[a, b]$ is indeed connected, as we wanted to show.

2. *a*) Let f be a continuous function from the metric space (E, d) to the metric space (E', d'). Show that for every open set $V \subset E'$ the set $f^{-1}(V) \overset{def}{=} \{p \in E : f(p) \in V\}$ is open.

Proof. Consider an arbitrary $p \in f^{-1}[V]$. We need to show that there is an open ball with center p that is included in $f^{-1}[V]$. In other words, we need to show that there is a $\delta > 0$ such that $\{q \in E : d(p, q) < \delta\} \subset f^{-1}[V]$. As V is open and $f(p) \in V$ (since $p \in f^{-1}[V]$), there is an $\epsilon > 0$ such that the open ball $\{u \in E' : d'(f(p), u) < \epsilon\}$ is a subset of V. By the continuity of f, there is a $\delta > 0$ such that $d'(f(p), f(q)) < \epsilon$ holds whenever $d(p, q) < \delta$; in other words,

$$\{f(q) : q \in E \text{ and } d(p, q) < \delta\} \subset \{u \in E' : d'(f(p), u) < \epsilon\}.$$

Since the latter set is a subset of V, this implies that $\{q \in E : d(p, q) < \delta\} \subset f^{-1}[V]$, showing that $f^{-1}[V]$ is indeed open. The proof is complete.

b) Let f be a continuous function from the metric space (E, d) to the metric space (E', d'). Assume E is compact. Show that $f[E] \overset{def}{=} \{f(p) : p \in E\}$ is compact.

Proof. Let $\mathcal{U}$ be a collection of open subsets of E' such that

$$f[E] \subset \bigcup \mathcal{U}, \tag{1}$$

i.e., such that $\mathcal{U}$ cover $f[E]$. We need to show that there is a finite subcollection of $\mathcal{U}$ that covers $f[E]$. (Recall that the word *collection* is just a synonym for set; so we could have described $\mathcal{U}$ as a set of open subsets of E'.)

[1]All computer processing for this manuscript was done under Fedora Core Linux. $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX was used for typesetting.

[2]Indeed, for each $x \in A$ there is an open ball $U_x = \{y \in [a, b] : |x - y| < r(x)\}$ of center x and radius $r(x)$ of the space $[a, b]$ such that $U_x \subset A$. Writing $U'_x = \{y \in \mathbb{R} : |x - y| < r(x)\}$, one can find set U as described above in the form $U = \bigcup\{U'_x : x \in A\}$.

First observe that for any collection of $\mathcal{Z}$ of subsets of E',

$$f[E] \subset \bigcup \mathcal{Z} \qquad \text{if and only if} \qquad E \subset \bigcup\{f^{-1}[Z] : Z \in \mathcal{Z}\}. \tag{2}$$

Indeed, the formula on the left says that for every $p \in E$ there is a $Z \in \mathcal{Z}$ such that $f(p) \in Z$, while the formula on the right says that for every $p \in E$ there is a $Z \in \mathcal{Z}$ such that $p \in f^{-1}[Z]$. These two statements are, however, identical, since $f(p) \in Z$ if and only if $p \in f^{-1}[Z]$ (in view of the definition of the latter set).

In virtue of (2), formula (1) means that

$$E \subset \bigcup\{f^{-1}[U] : U \in \mathcal{U}\}.$$

By the continuity of f and the openness of the sets $U \in \mathcal{U}$, the sets $f^{-1}[U]$ are open (see the preceding Lemma), i.e., the set on the right-hand side is an open cover of E. Thus, by the compactness of E, it has a finite subset that is also an open cover; every subset of $\{f^{-1}[U] : U \in \mathcal{U}\}$ has the form $\{f^{-1}[U] : U \in \mathcal{U}'\}$ for some $\mathcal{U}' \subset \mathcal{U}$; a finite subset of the former set has this form with a finite $\mathcal{U}'$. Hence, what we said about the existence of a finite subset that is an open cover can be rephrased as follows: there is a finite $\mathcal{U}' \subset \mathcal{U}$ such that

$$E \subset \bigcup\{f^{-1}[U] : U \in \mathcal{U}'\}.$$

By (2) means that

$$f[E] \subset \bigcup \mathcal{U}';$$

i.e., $\mathcal{U}'$ is a finite open cover of $f[E]$. Since our aim was to show the existence of such a finite open cover, the proof is complete.

3. Let (E, d) and (E', d') be metric spaces. Let $f : E \to E'$ be a function and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions $f_n : E \to E'$.

a) Give a definition of f being continuous on E.

Definition. The function $f : E \to E'$ is said to be *continuous* on E if for every $p \in E$ and for every $\epsilon > 0$ there is a $\delta > 0$ such that for every $q \in E$, if $d(p, q) < \delta$ then $d'(f(p), f(q)) < \epsilon$.

Note. The above definition says that f is continuous at p for every $p \in E$. Using logic notation, the function $f : E \to E'$ is said to be continuous on E if

$$(\forall p \in E)(\forall \epsilon > 0)(\exists \delta > 0)(\forall q \in E)\big(d(p, q) < \delta \to d'(f(p), f(q) < \epsilon\big).$$

The first two quantifiers are interchangeable here, since two quantifiers of the same type (i.e., two universal quantifiers, or two existential quantifiers) are interchangeable, so we can write this also as

$$(\forall \epsilon > 0)(\forall p \in E)(\exists \delta > 0)(\forall q \in E)\big(d(p, q) < \delta \to d'(f(p), f(q) < \epsilon\big). \tag{1}$$

b) Give a definition of f being uniformly continuous on E.

Definition. The function $f : E \to E$ is said to be *uniformly continuous* on E if for every $\epsilon > 0$ there is a $\delta > 0$ such that for every $p \in E$ and for every $q \in E$, if $d(p, q) < \delta$ then $d'(f(p), f(q)) < \epsilon$.

Note. Using logic notation, the function $f : E \to E$ is said to be uniformly continuous on E if

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall p \in E)(\forall q \in E)\big(d(p, q) < \delta \to d'(f(p), f(q) < \epsilon\big). \tag{2}$$

After giving the definition, explain what the difference is between continuity and uniform continuity. (The emphasis is on the word *after*; the definition and the explanation must not be intermixed.)

The formal difference between continuity and uniform continuity is indicated by the different order of the second and third quantifiers in formulas (1) and (2); aside from this difference, the two formulas are identical. However, these quantifiers are of different type (one is a universal quantifier, the other is an existential quantifier), and so they are not interchangeable. Hence the meanings of these two formulas are different.

To explain the difference in a less formal way, in case of continuity, δ depends on the choice of p as well as on ϵ (and, of course, on the function f itself), whereas in case of uniform continuity, δ depends only on ϵ but not on p (but it does depend on the function f itself).

c) Give a definition of the sequence $\{f_n\}_{n=1}^{\infty}$ being convergent to f. (Give the definition in terms of ϵ and N, and not in terms of sequences of points.)

Definition. The sequence $\{f_n\}_{n=1}^{\infty}$ of functions is said to *converge* to f if for every $p \in E$ and for every $\epsilon > 0$ there is an $N \in \mathbb{R}$ such that for every integer $n > N$ we have $d'(f_n(p), f(p)) < \epsilon$.

Note. The convergence described is called *pointwise* convergence. Pointwise convergence means that there is convergence at every point. In other words, for every $p \in E$,

$$\lim_{n\to\infty} f_n(p) = f(p).$$

Using logic notation, the sequence $\{f_n\}_{n=1}^{\infty}$ of functions is said to converge to the function f if

$$(\forall p \in E)(\forall \epsilon > 0)(\exists N \in \mathbb{R})(\forall n > N)\ d'(f_n(p), f(p)) < \epsilon.$$

The first two quantifiers are interchangeable here, so we can write this also as

$$(\forall \epsilon > 0)(\forall p \in E)(\exists N \in \mathbb{R})(\forall n > N)\ d'(f_n(p), f(p)) < \epsilon. \tag{3}$$

d) Give a definition of the sequence $\{f_n\}_{n=1}^{\infty}$ being uniformly convergent to f.

Definition. The sequence $\{f_n\}_{n=1}^{\infty}$ of functions is said to *converge to f uniformly* if for every $\epsilon > 0$ there is an $N \in \mathbb{R}$ such that for every integer $n > N$ for every $p \in E$ we have $d'(f_n(p), f(p)) < \epsilon$.

Note. Formally, the sequence $\{f_n\}_{n=1}^{\infty}$ of functions is said to converge to f uniformly if

$$(\forall \epsilon > 0)(\exists N \in \mathbb{R})(\forall n > N)\ (\forall p \in E) d'(f_n(p), f(p)) < \epsilon.$$

The third and fourth quantifiers here are of the same type, and so they can be interchanged:

$$(\forall \epsilon > 0)(\exists N \in \mathbb{R})(\forall p \in E)(\forall n > N)\ d'(f_n(p), f(p)) < \epsilon. \tag{4}$$

After giving the definition, explain what the difference is between convergence and uniform convergence. (The emphasis is on the word *after*; the definition and the explanation must not be intermixed.)

The difference between convergence and uniform convergence can most easily be appreciated by noting the difference in the orders of the second and third quantifiers in formulas (3) and (4). In other words, in case of pointwise convergence, N depends on p and ϵ (and, of course, on the sequence $\{f_n\}_{n=1}^{\infty}$ of functions), whereas in case of uniform convergence, N depends only on ϵ (and, of course, on the sequence $\{f_n\}_{n=1}^{\infty}$ of functions).

4. *a*) Let $U \subset \mathbb{R}$ be an open set, let $x_0 \in U$, and assume that the function $f : U \to \mathbb{R}$ is differentiable at x_0. Prove that f is continuous at x_0.

Solution. We will give two proofs. The first proof relies on the property that the limit of a product equals the product of the limits.

First proof. Write $f'(x_0)$ for the derivative of f at x_0. We have

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Therefore,

$$\begin{aligned}\lim_{x\to x_0}(f(x)-f(x_0)) &= \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\cdot(x-x_0)\\ &= \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\cdot\lim_{x\to x_0}(x-x_0) = f'(x_0)\cdot 0.\end{aligned}$$

Hence

$$\begin{aligned}\lim_{x\to x_0} f(x) &= \lim_{x\to x_0}\big((f(x)-f(x_0)+f(x_0)\big)\\ &= \lim_{x\to x_0}(f(x)-fx_0)+\lim_{x\to x_0} f(x_0) = 0+f(x_0)=f(x_0).\end{aligned}$$

This shows that f is indeed continuous at x_0, completing the proof.

The second proof proceeds directly from the definition of continuity.

Proof. Write $f'(x_0)$ for the derivative of f at x_0. As before, we have

$$f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0};$$

that is, for every $\epsilon_0>0$ there is a $\delta_0>0$ such that

$$\left|\frac{f(x)-f(x_0)}{x-x_0}-f'(x_0)\right|<\epsilon_0$$

for every $x\in(x_0-\delta_0,x_0+\delta_0)$ with $x\neq x_0$. Pick an $\epsilon_0>0$ (any such ϵ_0 will do; e.g., we can simply pick $\epsilon_0=1$), and let $\delta_0>0$ be a number such that the above inequality holds for every $x\in(x_0-\delta_0,x_0+\delta_0)$ with $x\neq x_0$ (by choosing δ_0 small enough, this interval will be a subset of U, as U is open). Multiplying both sides by $|x-x_0|$, we obtain

$$|f(x)-f(x_0)-f'(x_0)(x-x_0)|\le\epsilon_0|x-x_0|$$

for every $x\in(x_0-\delta_0,x_0+\delta_0)$. We relaxed the inequality from $\le$ to $<$; as a result, we can now allow $x=x_0$, since we obviously have equality here in that case.

Let $\epsilon>0$ be arbitrary. Given $x\in(x_0-\delta_0,x_0+\delta_0)$, by the above inequality we obtain

$$\begin{aligned}|f(x)-f(x_0)| &\le |f(x)-f(x_0)-f'(x_0)(x-x_0)|+|f'(x_0)(x-x_0)|\\ &\le \epsilon_0|x-x_0|+|f'(x_0)|\cdot|x-x_0| = (\epsilon_0+|f'(x_0)|)|\cdot|x-x_0|.\end{aligned}$$

If we choose $\delta=\min\left\{\delta_0,\frac{\epsilon}{\epsilon_0+|f'(x_0)|}\right\}$, then for any x with $|x-x_0|<\delta$ the right-hand side here is less than ϵ, ensuring that the left-hand side, i.e., $|f(x)-f(x_0)|$, is also less than ϵ.

To summarize, given an arbitrary $\epsilon>0$, we found a $\delta>0$ such that for every x with $|x-x_0|<\delta$ we have $|f(x)-f(x_0)|<\epsilon$. This means that f is indeed continuous at x_0. The proof is complete

b) Let $U\subset\mathbb{R}$ be an open set, let $x_0\in U$, and assume that the functions $f:U\to\mathbb{R}$ and $g:U\to\mathbb{R}$ are differentiable at x_0. Using the identity

$$\frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0}=\frac{f(x)-f(x_0)}{x-x_0}g(x)+f(x_0)\frac{g(x)-g(x_0)}{x-x_0},$$

prove that $f(x)g(x)$ is differentiable at x_0 and $(f(x_0)g(x_0))'=f'(x_0)g(x_0)+f(x_0)g'(x_0)$. *Clearly explain* how the result of Part *a*) is used in the proof.

Solution. The symbol $(f(x_0)g(x_0))'$ is somewhat sloppy; the proper notation would be $(fg)'(x_0)$ to indicate that the derivative of the product function fg (defined as $(fg)(x) = f(x)g(x)$ for every x where the right-hand side in meaningful) is taken. We have

$$\begin{aligned}(fg)'(x_0) &= \lim_{x\to x_0} \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} = \lim_{x\to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} g(x) + f(x_0)\frac{g(x) - g(x_0)}{x - x_0} \right) \\ &= \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} g(x) + \lim_{x\to x_0} f(x_0)\frac{g(x) - g(x_0)}{x - x_0} \\ &= \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x\to x_0} g(x) + \lim_{x\to x_0} f(x_0) \lim_{x\to x_0} \cdot \frac{g(x) - g(x_0)}{x - x_0} = f'(x_0)g(x_0) + f(x_0)g'(x_0).\end{aligned}$$

The first equality above is just the definition of the derivative applied to the function fg at the point x_0; the second equality uses the identity mentioned in the question; the third equality is based on the addition rule for limits, and the fourth equality is based on the product rule of limits. The fifth, i.e., the last, equality uses the definition of the derivative to obtain $f'(x_0)$ and $g'(x_0)$ on the right-hand side. Further, to show the fifth equality, we used the equality $\lim_{x\to x_0} f(x_0) = f(x_0)$; this equality holds because on the left-hand side the limit of a constant is taken (because $f(x_0)$ does not depend on x, i.e., it is constant as far as x is concerned; that is, $f(x_0)$ represents the constant function whose value is $f(x_0)$ at every x). Finally, and *most importantly*, we used that $\lim_{x\to x_0} g(x) = g(x_0)$. This holds because g is continuous at x_0 according the the result proved in Part *a*), since we assumed that g is differentiable at x_0.

5. *a*) State and prove Rolle's Theorem.

Rolle's Theorem. *Let $[a, b]$ be a closed interval for some $a, b \in \mathbb{R}$ with $a < b$, and let f be a real-valued continuous function on $[a, b]$ such that $f(a) = f(b) = 0$. Assume, further, that f is differentiable on (a, b). Then there is a $\xi \in (a, b)$ such that $f'(\xi) = 0$.*

Remark. As differentiability implies continuity, assuming that f is continuous in (a, b) is redundant. However, assuming that f is continuous in $[a, b]$ also requires, in addition to the continuity of f in (a, b), that f is continuous from the right at a and from the left at b, and these continuity requirements at the endpoints do not follow from the differentiability assumption (because differentiability at the endpoints is not assumed).

Proof. Let ξ_0 be a place of maximum for f in $[a, b]$ (by the Maximum-Value Theoem, f assumes its maximum on $[a, b]$). If $\xi_0 \in (a, b)$ then $f'(\xi_0) = 0$, so the assertion is satisfied with $\xi = \xi_0$. If $\xi_0 \notin (a, b)$, then we must have $\xi_0 = a$ or $\xi_0 = b$; in this case, we have $f(x) \le 0$ for $x \in [a, b]$.

Let ξ_1 be a place of minimum of f in $[a, b]$; according to the Maximum-Value Theorem,[3] f has a minimum on $[a, b]$. If $\xi_1 \in (a, b)$ then $f'(\xi_1) = 0$,[4] so the assertion is satisfied with $\xi = \xi_1$. If $\xi_1 \notin (a, b)$, then must have $\xi_1 = a$ or $\xi_1 = b$; in this case, we have $f(x) \ge 0$ for all $x \in [a, b]$.

That is, if neither $\xi = \xi_0$ nor $\xi = \xi_1$ works (because we have both $\xi_0 \notin (a, b)$ and $\xi_1 \notin (a, b)$), then $f(x) = 0$ holds for all $x \in [a, b]$. Therefore, we have $f'(\xi) = 0$ for any choice of $\xi \in (a, b)$. The proof is complete.

b) State (without proof) the Mean-Value Theorem (for differentiation).

The Mean-Value Theorem. *Let f be a function that is continuous in the interval $[a, b]$ ($a, b \in \mathbb{R}$ and $a < b$), and differentiable in (a, b). Then there is a $\xi \in (a, b)$ such that*

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

[3] One is tempted to say Minimum-Value Theorem here, but one usually does not formulate the Minimum-Value Theorem. Instead, to find a place of minimum of f on $[a, b]$, one applies the Maximum-Value Theorem to find a place of maximum of $-f$ on $[a, b]$.

[4] The result we showed says that $f'(\xi) = 0$ at a place of *maximum* inside an open interval. As ξ_1 is a place of minimum of f, it is a place of maximum of $-f$; that is, $(-f)'(\xi_1) = 0$ according the the result we showed. Hence $f'(\xi_1) = 0$.