Exam 3, Mathematics 11.1, Section TR11 28 copies
11:00–12:10 am, April 30, 2009, 330NE
Instructor: Attila Máté

Follow these instructions carefully:

Work on the paper provided; do not use your own paper. *Work only on one problem on each sheet (you should not work on two different problems on the two sides of the same sheet).* On the top of each page, *print* your name (*encircle your last name*) and indicate the number of the problem you are working on by writing e.g. *"Problem #4"*. Always *encircle* your final answer. If there are several parts to a problem, always indicate the part that you are answering, e.g. by writing *"Answer to Part b)"* (the number of the problem should be on the top of the page). Do not use a *red* pen or a *red* pencil. Do not write in the corner covered up by the staple (top left corner on the front side, top right corner on the back side). Each problem is worth the *same* amount of credit.

1. *a)* Let f be a real-valued function on the closed interval $I = [a, b]$. (i) Define what a tagged partition is on I. (ii) Define what is meant by the width of a partition. (iii) Define what the Riemann sum is for f associated with a tagged partition of I.

b) Let f and I be as in Part *a)*, and let A be a real number. Define what it means for A to be the Riemann integral of f on I.

2. *a)* State the first criterion for a function f on the interval $[a, b]$ to be Riemann integrable (i.e., the criterion that is stated in terms of Riemann sums).

b) Show that the criterion described in Part *a)* is necessary (that is, if f is Riemann integrable, then the condition described in the criterion is satisfied).

3. *a)* Let $[a, b]$ be a closed interval and let let α and β be real numbers with $a \le \alpha < \beta \le b$. Define the function f on $[a, b]$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in (\alpha, \beta), \\ 0 & \text{otherwise,} \end{cases}$$

Let $P : a = x_0 < x_1 < \ldots < x_N = b$ be a partition, and define p and q such that $x_{p-1} \le \alpha < x_p$ and $x_{q-1} < \beta \le x_q$. Show that for any Riemann sum S associated with the partition P we have $S \le x_q - x_{p-1}$.

b) State the Fundamental Theorem of Calculus. Make sure that the conditions are precisely stated. Do not give a proof.

4. *a)* State the Comparison Test for convergence, involving two infinite series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$.

b) Prove that if $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ also converges.

5. *a)* State, without proof, the result about the differentiation of a sequence of functions.

b) Let $n \ge 1$ be an integer. Let U be an open interval in $\mathbb{R}$ and let $f : U \to \mathbb{R}$ be a function that is $n + 1$ times differentiable. For any $a, b \in U$ put

$$R_n(b, a) = f(b) - \sum_{k=0}^{n} f^{(k)}(a) \frac{(b - a)^k}{k!}.$$

Then, as we know,

$$(*) \qquad \frac{d}{dx} R_n(b, x) = -\frac{f^{(n+1)}(x)(b - x)^n}{n!}$$

holds for every $x \in U$. Using this, show that, given any $a, b \in U$ with $a < b$, there is a $\xi \in (a, b)$ such that

$$R_n(b, a) = \frac{f^{(n+1)}(\xi)}{(n + 1)!}(b - a)^{n+1}.$$