Exam 3, Mathematics 11.1, Section TR11 28 copies
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1. *a*) Let f be a real-valued function on the closed interval $I = [a, b]$. (i) Define what a tagged partition is on I. (ii) Define what is meant by the width of a partition. (iii) Define what the Riemann sum is for f associated with a tagged partition of I.

Definition. Let $I = [a, b]$ be a closed interval ($a, b \in \mathbb{R}$ and $a < b$).
(i) A finite subset P of $[a, b]$ such that $a, b \in P$ is called a *partition* of $[a, b]$.[2]
Writing the elements of P in increasing order as

$$P\colon a = x_0 < x_1 < \ldots < x_N = b,$$

where $P = \{x_0, x_1, \ldots, x_N\}$, a *tag* for the interval $I_i \overset{def}{=} [x_{i-1}, x_i]$ is any number $\xi_i \in I_i$ $(1 \le i \le N)$. A *tagged partition* is is a partition that has a tag ξ_i for each interval I_i of the partition $(1 \le i \le N)$.
(ii) The width of the partition P as described above is the number

$$\|P\| \overset{def}{=} \max\{x_i - x_{i-1} : 1 \le i \le N\}.$$

(iii) Let f be a real-valued function on $[a, b]$. The Riemann S sum for f associated with the the tagged partition described above is defined as

$$S = \sum_{i=1}^{N} f(\xi_i)(x_i - x_{i-1}).$$

b) Let f and I be as in Part *a*), and let A be a real number. Define what it means for A to be the Riemann integral of f on I.

Definition. Let f be a real-valued function on the closed interval $[a, b]$ ($a, b \in \mathbb{R}$ and $a < b$). A is said to be the Riemann integral of f on $[a, b]$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that, for every tagged partition of $[a, b]$ that has width $< \delta$, writing S for the Riemann sum for f associated with this tagged partition, we have $|S - A| < \epsilon$.

2. *a*) State the first criterion for a function f on the interval $[a, b]$ to be Riemann integrable (i.e., the criterion that is stated in terms of Riemann sums).

First Criterion for Riemann Integrability. *A real-valued function f on $[a, b]$ is Riemann integrable if and only if for every $\epsilon > 0$ there is a $\delta > 0$ such that for any two Riemann sums S_1 and S_2 for f associated with partitions of $[a, b]$ we have $|S_1 - S_2| < \epsilon$.*

b) Show that the criterion described in Part *a*) is necessary (that is, if f is Riemann integrable, then the condition described in the criterion is satisfied).

Proof of Necessity. Assume f is Riemann integrable on f and write $A = \int_a^b f$. Let $\epsilon > 0$ be arbitrary. According to the definition of the Riemann integral, there is a $\delta > 0$ such that $|A - S| < \epsilon/2$ for any Riemann sum S for f associated with a partition of width $< \delta$ of $[a, b]$. Thus S belongs to the interval $(A - \epsilon/2, A + \epsilon/2)$. Hence, if S and S' are Riemann sums for f associated with partitions of width $< \delta$ of $[a, b]$, then both S and S' belong to this interval; thus $|S' - S| < \epsilon$.

[1]All computer processing for this manuscript was done under Fedora Core Linux. $\mathcal{A}_\mathcal{M}\mathcal{S}$-TeX was used for typesetting.

[2]There are other ways to define a partition. A partition could be defined as a sequence of points of $[a, b]$ or even as a finite set of non-overlapping (i.e., having no interior points in common) closed subintervals of $[a, b]$ whose union is $[a, b]$. Any definition will work that will leave the concept of the Riemann sum unchanged. However, some definitions are easier to work with in certain settings than others.

3. *a*) Let $[a, b]$ be a closed interval and let let α and β be real numbers with $a \le \alpha < \beta \le b$. Define the function f on $[a, b]$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in (\alpha, \beta), \\ 0 & \text{otherwise,} \end{cases}$$

Let $P : a = x_0 < x_1 < \ldots < x_N = b$ be a partition, and define p and q such that $x_{p-1} \le \alpha < x_p$ and $x_{q-1} < \beta \le x_q$. Show that for any Riemann sum S associated with the partition P we have $S \le x_q - x_{p-1}$.

Proof. We have $f(\xi_i) = 0$ if $i < p$ or $i > q$, and $f(\xi_i) = 1$ if $p < i < q$.[3] Thus, for the corresponding Riemann sum

$$S = \sum_{i=1}^{N} f(\xi_i)(x_i - x_{i-1})$$

we have

$$S \le \sum_{i:\ p \le i \le q} (x_i - x_{i-1}).$$

Since we have $p \le q$ by the definitions of p and q, the sum on the right-hand side telescopes:[4]

$$\sum_{i:\ p \le i \le q} (x_i - x_{i-1}) = \sum_{i=p}^{q} (x_i - x_{i-1}) = (x_p - x_{p-1}) + (x_{p+1} - x_p) \\ + (x_{p+2} - x_{p+1}) + \cdots + (x_q - x_{q-1}) = x_q - x_{p-1}.$$

This shows that, indeed $S \le x_q - x_{p-1}$.

b) State the Fundamental Theorem of Calculus. Make sure that the conditions are precisely stated. Do not give a proof.

The Fundamental Theorem of Calculus. *Let U be an open interval of $\mathbb{R}$, and let $f : U \to \mathbb{R}$ be a continuous function. Let $a \in U$, and define the function $F : U \to \mathbb{R}$ by putting $F(x) = \int_a^x f$ for every $x \in U$. Then $F'(x) = f(x)$ for every $x \in U$.*

4. *a*) State the Comparison Test for convergence, involving two infinite series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$.

Comparison Test. *Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two infinite series such that $|a_n| \le b_n$ for every integer $n \ge 1$. If the latter series is convergent, then the former series is also convergent.*

b) Prove that if $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ also converges.

Solution. The assertion follows from the Comparison Test for the series $\sum_{n=1}^{\infty} a_n$ and the series $\sum_{n=1}^{\infty} b_n$ with $b_n = |a_n|$.

A proof avoiding a reference to the Comparison Test would essentially just repeat the argument in the proof of the Comparison Test: Assume that $\sum_{n=1}^{\infty} |a_n|$ is convergent. By Cauchy's Criterion, then, given an arbitrary $\epsilon > 0$, there is an N such that

$$\Big| \sum_{k=m+1}^{n} |a_k| \Big| < \epsilon$$

holds whenever $n > m \ge N$; the absolute value signs on the outside on the left-hand side have of course no effect, and they can be omitted. Thus

$$\Big| \sum_{k=m+1}^{n} a_k \Big| \le \sum_{k=m+1}^{n} |a_k| < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, the inequality here between the extreme left and the extreme right implies that $\sum_{n=1}^{\infty} a_n$ is convergent in view of Cauchy's Convergence Criterion.

5. *a*) State, without proof, the result about the differentiation of a sequence of functions.

[3] To see this, it is important to remember that $f(\alpha) = f(\beta) = 0$ by the definition of f. For $i = p$ or $i = q$ we may have $f(\xi_i) = 0$ or $f(\xi_i) = 1$.

[4] I.e., all its terms except for the extreme ones will cancel. The significance of the inequality $p \le q$ is that it guarantees that the sum in question is not the empty sum, so we indeed do have cancelations.

Theorem. *Let $U \in \mathbb{R}$ be an open interval, and for $n \geq 1$ let $f_n : U \to \mathbb{R}$ be functions continuously differentiable on U such that the sequence $\{f'_n\}_{n=1}^{\infty}$ of derivatives converges uniformly on U. Assume, further, that for some $a \in U$ the sequence $\{f_n(a)\}_{n=1}^{\infty}$ is convergent. Then the sequence of functions $\{f_n\}_{n=1}^{\infty}$ converges (pointwise) to a limit f on U that is differentiable on U, and $f'(x) = \lim_{n\to\infty} f'_n(x)$ for every $x \in U$.*

Note. One can show that under the same assumptions that, in case U is a finite interval, then the sequence of functions $\{f_n\}_{n=1}^{\infty}$ converges uniformly to a limit f. If U is an infinite interval, then the convergence need not be uniform.

b) Let $n \geq 1$ be an integer. Let U be an open interval in $\mathbb{R}$ and let $f : U \to \mathbb{R}$ be a function that is $n+1$ times differentiable. For any $a, b \in U$ put

$$R_n(b,a) = f(b) - \sum_{k=0}^{n} f^{(k)}(a)\frac{(b-a)^k}{k!}.$$

Then, as we know,

$$(*) \qquad \frac{d}{dx}R_n(b,x) = -\frac{f^{(n+1)}(x)(b-x)^n}{n!}$$

holds for every $x \in U$. Using this, show that, given any $a, b \in U$ with $a < b$, there is a $\xi \in (a,b)$ such that

$$R_n(b,a) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(b-a)^{n+1}.$$

Solution. As we assumed $a < b$, we have $b - a \neq 0$, and so the equation

$$(**) \qquad R_n(b,a) = K \cdot \frac{(b-a)^{n+1}}{(n+1)!}$$

can be solved for K. Let K be the real number for which this equation is satisfied, and write

$$\phi(x) = R_n(b,x) - K \cdot \frac{(b-x)^{n+1}}{(n+1)!}$$

Then ϕ is differentiable in U; as differentiability implies continuity, it follows that f is continuous on the interval $[a,b]$ and differentiable in (a,b).[5] As $\phi(a) = 0$ by the choice of K and $\phi(b) = 0$ trivially, we can use Rolle's Theorem to obtain the existence of a $\xi \in (a,b)$ such that $\phi'(\xi) = 0$. Using $(*)$, we can see that

$$0 = \phi'(\xi) = -\frac{f^{(n+1)}(\xi)(b-\xi)^n}{n!} - K \cdot \frac{-(n+1)(b-\xi)^n}{(n+1)!}.$$

Noting that $\frac{(n+1)}{(n+1)!} = \frac{1}{n!}$ and keeping in mind that $\xi \neq b$, we obtain $K = f^{n+1}(\xi)$ from here. Thus the result follows from $(**)$.

A more general version of the remainder term for the Taylor series can be found in my notes for the course.

[5] Naturally, ϕ is differentiable also at a and b, but this is not needed for the rest of the argument.