Exam 2, Mathematics 4201, Section ETY6 32 copies
6:05–7:40 pm, March 26, 2026, 1105 IH
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1. *a)* State Rolle's Theorem without proof.

Rolle's Theorem. *Let $[a, b]$ be a closed interval for some $a, b \in \mathbb{R}$ with $a < b$, and let f be a real-valued continuous function on $[a, b]$ such that $f(a) = f(b) = 0$. Assume, further, that f is differentiable on (a, b). Then there is a $\xi \in (a, b)$ such that $f'(\xi) = 0$.*

Remark. As differentiability implies continuity, assuming that f is continuous in (a, b) is redundant. However, assuming that f is continuous in $[a, b]$ also requires, in addition to the continuity of f in (a, b), that f is continuous from the right at a and from the left at b, and these continuity requirements at the endpoints do not follow from the differentiability assumption (because differentiability at the endpoints is not assumed).

b) Let $n \ge 1$ be an integer. Let U be an open interval in $\mathbb{R}$ and let $f : U \to \mathbb{R}$ be a function that is $n + 1$ times differentiable. For any $a, b \in U$ put

$$R_n(b, a) = f(b) - \sum_{k=0}^{n} f^{(k)}(a)\frac{(b - a)^k}{k!}.$$

Then, as we know,

$$(*) \qquad \frac{d}{dx}R_n(b, x) = -\frac{f^{(n+1)}(x)(b - x)^n}{n!}$$

holds for every $x \in U$. Using this, show that, given any $a, b \in U$ with $a < b$, there is a $\xi \in (a, b)$ such that

$$R_n(b, a) = \frac{f^{(n+1)}(\xi)}{(n + 1)!}(b - a)^{n+1}.$$

Solution. As we assumed $a < b$, we have $b - a \ne 0$, and so the equation

$$(**) \qquad R_n(b, a) = K \cdot \frac{(b - a)^{n+1}}{(n + 1)!}$$

can be solved for K. Let K be the real number for which this equation is satisfied, and write

$$\phi(x) = R_n(b, x) - K \cdot \frac{(b - x)^{n+1}}{(n + 1)!}$$

Then ϕ is differentiable in U; as differentiability implies continuity, it follows that f is continuous on the interval $[a, b]$ and differentiable in (a, b).[2] As $\phi(a) = 0$ by the choice of K and $\phi(b) = 0$ trivially, we can use Rolle's Theorem to obtain the existence of a $\xi \in (a, b)$ such that $\phi'(\xi) = 0$. Using $(*)$, we can see that

$$0 = \phi'(\xi) = -\frac{f^{(n+1)}(\xi)(b - \xi)^n}{n!} - K \cdot \frac{-(n + 1)(b - \xi)^n}{(n + 1)!}.$$

Noting that $\frac{(n+1)}{(n+1)!} = \frac{1}{n!}$ and keeping in mind that $\xi \ne b$, we obtain $K = f^{n+1}(\xi)$ from here. Thus the result follows from $(**)$.

A more general version of the remainder term for the Taylor series can be found in my notes for the course, on pp. 38–39.

2. *a)* Let f be a real-valued function of the closed interval $I = [a, b]$. (i) Define what a tagged partition is of I. (ii) Define what is meant by the width of a partition. (iii) Define what the Riemann sum is for f associated with a tagged partition of I.

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[1]All computer processing for this manuscript was done under Fedora Linux. $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX was used for typesetting. PiCTeX was used for the diagrams, together with the programming language Perl used for generating the data points.

[2]Naturally, ϕ is differentiable also at a and b, but this is not needed for the rest of the argument.

Definition. Let $I = [a, b]$ be a closed interval ($a, b \in \mathbb{R}$ and $a < b$).

(i) A finite subset P of $[a, b]$ such that $a, b \in P$ is called a *partition* of $[a, b]$.[3]

Writing the elements of P in increasing order as

$$P\colon a = x_0 < x_1 < \ldots < x_N = b,$$

where $P = \{x_0, x_1, \ldots, x_N\}$, a *tag* for the interval $I_i \overset{def}{=} [x_{i-1}, x_i]$ is any number $\xi_i \in I_i$ ($1 \le i \le N$). A *tagged partition* is is a partition that has a tag ξ_i for each interval I_i of the partition ($1 \le i \le N$).

(ii) The width of the partition P as described above is the number

$$\|P\| \overset{def}{=} \max\{x_i - x_{i-1} : 1 \le i \le N\}.$$

(iii) Let f be a real-valued function on $[a, b]$. The Riemann S sum for f associated with the the tagged partition described above is defined as

$$S = \sum_{i=1}^{N} f(\xi_i)(x_i - x_{i-1}).$$

b) Let f and I be as in Part *a*), and let A be a real number. Define what it means for A to be the Riemann integral of f on I.

Definition. Let f be a real-valued function on the closed interval $[a, b]$ ($a, b \in \mathbb{R}$ and $a < b$). A is said to be the Riemann integral of f on $[a, b]$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that, for every tagged partition of $[a, b]$ that has width $< \delta$, writing S for the Riemann sum for f associated with this tagged partition, we have $|S - A| < \epsilon$.

3. *a*) State chain rule of differentiation. Be sure to state all assumptions precisely.

The Chain Rule for Differentiation. *Let U and V be open subsets of $\mathbb{R}$, and let f and g be real-valued functions such that $U \subset \operatorname{dom}(f) \subset \mathbb{R}$ and $V \subset \operatorname{dom}(g) \subset \mathbb{R}$. Let $x_0 \in U$ and assume $f(x_0) \in V$. Assume, further, that f is differentiable at x_0 and g is differentiable at $f(x_0)$. Then*

$$(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0). \tag{1}$$

b) Prove the chain rule of differentiation.

Proof. To begin with, notice that there is an open set $U' \subset \mathbb{R}$ such that $x_0 \in U'$ and $U' \subset \operatorname{dom}(g \circ f)$. Indeed, write $y_0 = f(x_0)$, and let $\epsilon > 0$ be such that the interval $V' = (y_0 - \epsilon, y_0 + \epsilon)$ is a subset of V; there is such an ϵ since the set V is open and $y_0 \in V$. Observing that f is continuous at x_0 since, by our assumptions, it is differentiable there, there is a $\delta > 0$ such that $f(x) \in V'$ for every $x \in (x_0 - \delta, x_0 + \delta)$. Writing $U' = (x_0 - \delta, x_0 + \delta)$, it is clear that U' is an open set such that

$$x_0 \in U' \subset \operatorname{dom}(g \circ f),$$

as claimed.

Define the function h on V by putting

$$h(y) = \begin{cases} \frac{g(y) - g(y_0)}{y - y_0} & \text{for } y \in V \text{ with } y \ne y_0, \\ g'(y_0) & \text{for } y = y_0. \end{cases} \tag{2}$$

[3] There are other ways to define a partition. A partition could be defined as a sequence of points of $[a, b]$ or even as a finite set of non-overlapping (i.e., having no interior points in common) closed subintervals of $[a, b]$ whose union is $[a, b]$. Any definition will work that will leave the concept of the Riemann sum unchanged. However, some definitions are easier to work with in certain settings than others.

Observe that h is continuous at $y_0 = f(x_0)$, since $\lim_{y\to y_0} h(y) = g'(y_0) = h(y_0)$ by the definition of the derivative (together with the definition of h and the assumption that g is differentiable at y_0).

Now,

$$(g\circ f)'(x_0) = \lim_{x\to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} = \lim_{x\to x_0} h(f(x))\frac{f(x) - f(x_0)}{x - x_0}.$$

This equation is correct, since it is easy to see from (2) that the two expressions after the lim signs are equal for every $x \in U' \setminus \{x_0\}$ (if $f(x) = f(x_0)$ then both expressions are 0). Using the product rule for limits, the right-hand side here equals

$$\lim_{x\to x_0} h(f(x)) \cdot \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

The first limit here is equal to $h(f(x_0))$. Indeed, as noted above, f is continuous at x_0, and h is continuous at $y_0 = f(x_0)$; thus, their composition $h\circ f$ is continuous at x_0.[4] The second limit equals $f'(x_0)$ by the definition of the derivative, since f was assumed to be differentiable at x_0. By noting that $h(f(x_0)) = g'(f(x_0))$ by the definition (2) of h, equation (1) follows. The proof is complete.

4. Let (E, d) and (E', d') be metric spaces, and let $f : E \to E'$ be a function.
   a) Give a definition of f being continuous on E.

Definition. The function $f : E \to E$ is said to be *continuous* on E if for every $p \in E$ and for every $\epsilon > 0$ there is a $\delta > 0$ such that for every $q \in E$, if $d(p, q) < \delta$ then $d'(f(p), f(q)) < \epsilon$.

Note. The above definition says that f is continuous at p for every $p \in E$. Using logic notation, the function $f : E \to E$ is said to be continuous on E if

$$(\forall p \in E)(\forall \epsilon > 0)(\exists \delta > 0)(\forall q \in E)\big(d(p, q) < \delta \to d'(f(p), f(q)) < \epsilon\big).$$

The first two quantifiers are interchangeable here, since two quantifiers of the same type (i.e., two universal quantifiers, or two existential quantifiers) are interchangeable, so we can write this also as

$$(\forall \epsilon > 0)(\forall p \in E)(\exists \delta > 0)(\forall q \in E)\big(d(p, q) < \delta \to d'(f(p), f(q)) < \epsilon\big). \tag{1}$$

b) Give a definition of f being uniformly continuous on E.

Definition. The function $f : E \to E$ is said to be *uniformly continuous* on E if for every $\epsilon > 0$ there is a $\delta > 0$ such that for every $p \in E$ and for every $q \in E$, if $d(p, q) < \delta$ then $d'(f(p), f(q)) < \epsilon$.

Note. Using logic notation, the function $f : E \to E$ is said to be uniformly continuous on E if

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall p \in E)(\forall q \in E)\big(d(p, q) < \delta \to d'(f(p), f(q) < \epsilon\big). \tag{2}$$

After giving the definition, explain what the difference is between continuity and uniform continuity. (The emphasis is on the word *after*; the definition and the explanation must not be intermixed.)

The formal difference between continuity and uniform continuity is indicated by the different order of the second and third quantifiers in formulas (1) and (2); aside from this difference, the two formulas are identical. However, these quantifiers are of different type (one is a universal quantifier, the other is an existential quantifier), and so they are not interchangeable. Hence the meanings of these two formulas are different.

To explain the difference in a less formal way, in case of continuity, δ depends on the choice of p as well as on ϵ (and, of course, on the function f itself), whereas in case of uniform continuity, δ depends only on ϵ but not on p (but it does depend on the function f itself).

c) Explain why the function $f(x) = 1/x$ is not uniformly continuous on the interval $(0, 1)$. Do not just give a vague argument; show clearly that this function does not satisfy the definition of uniform continuity.

[4] see Rosenlicht, the Proposition on p. 71.

Solution. We have to show that the following is not true: For every $\epsilon > 0$ there is a $\delta > 0$ such that $|1/x - 1/y| < \epsilon$ holds whenever $x, y \in (0,1)$ are such that $|x - y| < \delta$. Formally, we have to show that the following is not true:

$$(\forall\epsilon > 0)(\exists\delta > 0)(\forall x \in (0,1))(\forall y \in (0,1)) \left(|x-y| < \delta \to \left|\frac{1}{x} - \frac{1}{y}\right| < \epsilon \right).$$

The negation of this formula is

$$(\exists\epsilon > 0)(\forall\delta > 0)(\exists x \in (0,1))(\exists y \in (0,1)) \left(|x-y| < \delta \wedge \left|\frac{1}{x} - \frac{1}{y}\right| \geq \epsilon \right). \tag{3}$$

When forming the negation, put a negation sign $\neg$ on the left of the former formula, and move the negation symbol all the way inside, using the following rules: replace $\neg\forall\mathbf{x}$ with $\exists\mathbf{x}\neg$, $\neg\exists\mathbf{x}$ with $\forall\mathbf{x}\neg$, and $\neg(\mathbf{A} \to \mathbf{B})$ with $\mathbf{A} \wedge \neg\mathbf{B}$.

We need to show that the latter displayed formula is true. That is, we need to show that there is an $\epsilon > 0$ such that for all $\delta > 0$ there are x and y in the interval $(0,1)$ such that $|x - y| < \delta$ and $|1/x - 1/y| \geq 1$. To this end, put $\epsilon = 1$.[5] Let $\delta > 0$ be arbitrary. Put $x = \min\{\delta, 1/2\}$. Then $x \in (0,1)$.[6] In order to find a $y \in (0,1)$ with $|1/x - 1/y| \geq \epsilon = 1$, it is enough to find a y with $0 < y \leq x$ such that $1/x - 1/y \leq -1$, i.e., such that

$$0 < y \leq \frac{1}{\frac{1}{x} + 1}.$$

For example, we tan cake

$$y = \frac{1}{\frac{1}{x} + 1};$$

note that the inequality $0 < y \leq x$ is satisfied for this choice, since the right-hand side is easily shown to be less than x. This shows that $1/x$ is not uniformly continuous on $(0,1)$.

Note. The argument in the last paragraph can be put differently (and perhaps more simply) by noting that if we put $y = x/2$ then the inequality $|x - y| < \delta$ means $|x - y| = x - x/2 = x/2 < \delta$, i.e.,

$$x < 2\delta$$

($x \in (0,1)$). On the other hand, the inequality $|1/x - 1/y| \geq \epsilon$ can be written as

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \frac{1}{x} - \frac{1}{x/2} = \frac{1}{x} \geq \epsilon,$$

i.e., that $x \leq 1/\epsilon$.

Now, taking $\epsilon = 1$ again, it is clear that for $\delta > 0$ there are x and y such that both inequalities in the matrix[7] of (3) are satisfied; it is enough to take

$$x = \min\left\{\delta,\ \frac{1}{2}\right\}$$

and $y = x/2$, since, as we just saw, these inequalities become $x < 2\delta$ and $x < 1/\epsilon = 1$ in case $y = x/2$, in turn.

5. Let a, b, α, and β be real numbers with $a \leq \alpha < \beta \leq b$, and define the function f on $[a,b]$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in (\alpha,\beta), \\ 0 & \text{otherwise.} \end{cases}$$

Show that

$$\int_a^b f = \beta - \alpha.$$

Solution. See my online notes `analysis.pdf` p. 58.

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[5] We could take any fixed $\epsilon > 0$ instead.

[6] The choice $x = \min\{\delta, 1/2\}$ is somewhat arbitrary; instead of $1/2$, any other number in the interval $(0,1)$ would do.

[7] The matrix of a logic formula is the part after all the quantifiers. The matrix makes sense only for logic formulas where all the quantifiers are in front.