CLAS 28 copies
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BROOKLYN COLLEGE

FINAL EXAMINATION – SPRING 2009
MATHEMATICS 11.1 SECTION TR11

SHOW ALL WORK.

1. *a*) Define what a cut is.
b) Define what it means for the cut (A, B) to determine the real number t.
c) Define what it means for a number c to be the least upper bound of the subset S of $\mathbb{R}$.
d) Given a nonempty set S that is bounded from above, consider the cut (A, B) where B is the set of real numbers that are upper bounds of S and A is the set of real numbers that are not upper bounds of S. Prove that the real number t determined by the cut (A, B) is an upper bound of S. (In fact, t is the least upper bound of S, but you are not required to prove this now.)

2. *a*) Let (E, d) be a metric space, and let $S \subset E$ be a set. Define what it means for the set S to be open.
b) Let (E, d) be a metric space, and assume the set $S \subset E$ is not closed. Prove that there is a sequence $\{p_n\}_{n=1}^{\infty}$ of elements of S that is convergent and its limit does not belong to S.
c) Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two convergent sequences of reals, and assume that $a_n \le b_n$ for every integer $n \ge 1$. Prove that $\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n$.

3. *a*) Let f be a continuous function from the metric space (E, d) to the metric space (E', d'). Show that for every open set $V \subset E'$ the set $f^{-1}[V] \overset{def}{=} \{p \in E : f(p) \in V\}$ is open.
b) Let f be a continuous function from the metric space (E, d) to the metric space (E', d'). Assume E is compact. Show that $f[E] \overset{def}{=} \{f(p) : p \in E\}$ is compact.

4. *a*) Let $U \subset \mathbb{R}$ be an open set, let $x_0 \in U$, and assume that the function $f : U \to \mathbb{R}$ is differentiable at x_0. Prove that f is continuous at x_0.
b) Let $U \subset \mathbb{R}$ be an open set, let $x_0 \in U$, and assume that the functions $f : U \to \mathbb{R}$ and $g : U \to \mathbb{R}$ are differentiable at x_0. Using the identity

$$\frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} = \frac{f(x) - f(x_0)}{x - x_0} g(x) + f(x_0) \frac{g(x) - g(x_0)}{x - x_0},$$

prove that $f(x)g(x)$ is differentiable at x_0 and $(f(x_0)g(x_0))' = f'(x_0)g(x_0) + f(x_0)g'(x_0)$. *Clearly explain* how the result of Part *a*) is used in the proof.

5. *a*) Describe what a step function on an interval $[a, b]$ is.
b) State the second criterion (the one involving step functions) of Riemann integrability.
c) Let f be a real-valued function that is Riemann integrable on interval $[a, b]$. Prove that f is bounded on $[a, b]$.

6. *a*) State Cauchy's criterion for the series $\sum_{n=1}^{\infty} a_n$ of real numbers to be convergent.
b) Prove that the harmonic series $\sum_{n=1}^{\infty} 1/n$ is divergent. Use material covered in the course (that is, *do not* use the Integral Test for convergence).
c) Assume that $\sum_{n=1}^{\infty} a_n$ of real numbers is absolutely convergent. Prove that it is also convergent.

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7. *a*) State the result on the termwise integrability of a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$.

b) According to the result (it is hoped) you correctly stated in Part *a*), the geometric series

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

can be integrated termwise to obtain

$$\arctan x = \int_0^x \frac{1}{1+t^2}\,dt = \sum_{n=0}^{\infty}\int_0^x (-1)^n t^{2n}\,dt = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$$

for $-1 < x < 1$. Termwise integration is not justified in case $x = 1$. Show that, nevertheless, the result is true even in case $x = 1$. That is, noting that $\arctan 1 = \pi/4$, we have

$$\frac{\pi}{4} = \sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots.$$

End of Examination