Exam 1, Mathematics 11.1, Section TF10 43 copies
10:50–12:00 am, February 22, 2008, 330NE
Instructor: Attila Máté

Follow these instructions carefully:

Work on the paper provided; do not use your own paper. *Work only on one problem on each sheet (you should not work on two different problems on the two sides of the same sheet).* On the top of each page, *print* your name (*encircle your last name*) and indicate the number of the problem you are working on by writing e.g. *"Problem #4"*. Always *encircle* your final answer. If there are several parts to a problem, always indicate the part that you are answering, e.g. by writing *"Answer to Part b)"* (the number of the problem should be on the top of the page). Do not use a *red* pen or a *red* pencil. Do not write in the corner covered up by the staple (top left corner on the front side, top right corner on the back side). Each problem is worth the *same* amount of credit.

1. *a*) Define what it means for a number c to be an upper bound of the subset S of $\mathbb{R}$ ($\mathbb{R}$ denotes the set of real numbers).

b) Define what it means for a number c to be the least upper bound of the subset S of $\mathbb{R}$.

c) Define what it means for a number c to be the maximum of the subset S of $\mathbb{R}$.

d) Assume S is a nonempty closed and bounded set of reals. Prove that $a \overset{def}{=} \sup S$ is an element of S.

2. *a*) Define what it means for a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers to converge to the real number L.

b) Define what it means for the sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers to be increasing.

c) Given an increasing sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers that is bounded from above, prove that $\lim_{n\to\infty} a_n = \sup\{a_n : n \ge 1\}$.

3. *a*) Let (E, d) be a metric space, and let $S \subset E$ be a set. Define what it means for the set S to be open.

b) Let (E, d) be a metric space, and assume the set $S \subset E$ is not closed. Prove that there is a sequence $\{p_n\}_{n=1}^{\infty}$ of elements of S that is convergent and its limit does not belong to S.

c) Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two convergent sequences of reals, and assume that $a_n \le b_n$ for every integer $n \ge 1$. Prove that $\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n$.

4. *a*) Let (E, d) be a metric space and let $\{p_n\}_{n=1}^{\infty}$ be a sequence of elements of E. Show that there is at most one element p of E such that $\{p_n\}_{n=1}^{\infty}$ converges to p. (That is, show that a sequence cannot converge to two different points.)

b) Let (E, d) be a metric space and let $\{p_n\}_{n=1}^{\infty}$ be a convergent sequence of elements of E. Show the set $\{p_n : n \ge 1\}$ is bounded.

c) Let a be a real number with $0 \le a < 1$. Prove that $\lim_{n\to\infty} a^n = 0$. Describe your argument carefully by referring to the theorems and results that the steps in your proof rely on.

5. *a*) Define what it means for the sequence $\{p_n\}_{n=1}^{\infty}$ in a metric space (E, d) to be a Cauchy sequence.

b) Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence of reals, and let a be the supremum of the set $S = \{x :$ there are infinitely many positive integers n for which $x \le a_n\}$. Let ϵ be a positive real number and let N be a positive integer. Show that there is an integer $n > N$ for which $a - \epsilon < a_n < a + \epsilon$.