Exam 1, Mathematics 11.1, Section TF10 43 copies
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1. *a*) Define what it means for a number c to be an upper bound of the subset S of $\mathbb{R}$ ($\mathbb{R}$ denotes the set of real numbers).

Solution.

Definition. A number c is called an upper bound of the subset S of $\mathbb{R}$ if $c \ge x$ for every $x \in S$. Formally

$$c \text{ is an upper bound of } S \leftrightarrow \forall x \in S\ [x \le c].$$

b) Define what it means for a number c to be the least upper bound of the subset S of $\mathbb{R}$.

Solution.

Definition. A number c is called the least upper bound of a set S of $\mathbb{R}$ if c is an upper bound of S and for any upper bound y of S we have $c \le y$. Formally,

$$c \text{ is the least upper bound of } S \leftrightarrow \forall x \in S\ [x \le c]\ \ \&\ \ \forall y\ \big(\forall x \in S\ [x \le y] \to [c \le y]\big).$$

c) Define what it means for a number c to be the maximum of the subset S of $\mathbb{R}$.

Solution.

Definition. A number c is called the maximum of the set S if $c \in S$ and c is greater than or equal to every element of S. Formally,

$$c \text{ is the maximum of } S \leftrightarrow \big(c \in S\ \ \&\ \ \forall x \in S\ [x \le c]\big).$$

In other words, the maximum of S is an upper bound of S that belongs to S. In yet other words, if S has a least upper bound and this least upper bound belongs to S, then the least upper bound is the maximum of S. It is easy to see that if S has no least upper bound (because it is either empty or not bounded from above), or if S has a least upper bound but the least upper bound does not belong to S, then S has no maximum.

d) Assume S is a nonempty closed and bounded set of reals. Prove that $a \overset{def}{=} \sup S$ is an element of S.

Solution. Assume that $a \notin S$, i.e., that $a \in \mathbb{R} \setminus S$. As S is closed, the complement of S, i.e, the set $\mathbb{R} \setminus S$, is open. Since a belongs to this set, there is an $\epsilon > 0$ such that the open ball with center a and radius ϵ, i.e., the interval $(a-\epsilon, a+\epsilon)$, is included in $\mathbb{R} \setminus S$. That is, the set $S \cap (a-\epsilon, a+\epsilon)$ is empty. Hence, there is no element $x \in S$ for which $a - \epsilon < x \le a$. Since a is an upper bound of S, we have $x \le a$ for every $x \in S$; therefore, we also have $x \le a - \epsilon$. That is, $a - \epsilon$ is an upper bound of S. This is a contradiction, since $a - \epsilon < a$, and a is the least upper bound of S. This contradiction proves that the assumption $a \notin S$ was incorrect. That is, we indeed have $a \in S$, as we wanted to prove.

[1]All computer processing for this manuscript was done under Fedora Core Linux. $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX was used for typesetting.

This means that for a closed set S of reals, the supremum is in fact the largest element (i.e., the maximum) of S. The maximum a of S can be defined formally as

$$\text{maximum}(S, a) \leftrightarrow \big(a \in S \;\; \& \;\; \forall x \in S\ [x \leq a]\big).$$

The advantage of the supremum over the maximum is that every nonempty bounded set of reals has a supremum, but such a set need not have a maximum. For example, the supremum of the set $(0,1) = \{x \in \mathbb{R} : 0 < x < 1\}$ is 1, while this set has no maximum.

2. *a*) Define what it means for a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers to converge to the real number L.

Definition. We say that the sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers converges to the real number L if for every *epsilon* > 0 there is an N such that $|L - a_n| < \epsilon$ whenever $n > N$. Formally

$$a_n \text{ tends to } L \leftrightarrow \forall\epsilon > 0\ \exists N\ \forall n > N\ [|L - a_n| < \epsilon].$$

b) Define what it means for the sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers to be increasing.

Definition. The sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers is said to be increasing if $a_n \leq a_{n+1}$ for every integer $n \geq 1$.

c) Given an increasing sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers that is bounded from above, prove that $\lim_{n\to\infty} a_n = \sup\{a_n : n \geq 1\}$.

Solution. Write $L = \sup\{a_n : n \geq 1\}$; the supremum exists by the Axiom of Completeness since the set $\sup\{a_n : n \geq 1\}$ is nonempty and is bounded from above. Let $\epsilon > 0$ be arbitrary. Then $L - \epsilon$ is not an upper bound of the set $\sup\{a_n : n \geq 1\}$ (since L is the least upper bound of this set). So there is an element of this set, say a_N, such that $a_N > L - \epsilon$ (here N is an integer ≥ 1). As the sequence $L = \sup\{a_n : n \geq 1\}$ is increasing, we have $a_N < a_n$ whenever $n > N$; thus $L - \epsilon < a_n$ whenever $N < n$.[2] We also have $a_n \leq L$ for such N (in fact, for every $N \geq 1$, since L is an upper bound of the set $\sup\{a_n : n \geq 1\}$). Hence, $a_n < L + \epsilon$ *a fortiori*[3]. Thus $L - \epsilon < a_n < L + \epsilon$ whenever $n > N$; i.e., $|L - a_n| < \epsilon$ whenever $n > N$. Since $\epsilon > 0$ was arbitrary, this shows that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to L.

3. *a*) Let (E, d) be a metric space, and let $S \subset E$ be a set. Define what it means for the set S to be open.

Definition. Let (E, d) be a metric space, and let $S \subset E$ be a set. The set S is said to be open if for every $p \in S$ there is a real number $\epsilon > 0$ such that the open ball

$$U(p, \epsilon) \overset{def}{=} \{q \in E : d(p, q) < \epsilon\}$$

is a subset of S.

[2] We conveniently ignore the fact that this inequality holds even if $n = N$; we do this to arrive at the formal definition of limit. Of course, this is a point of only minor importance.

[3] Latin for even more so, or for a still stronger reason.

b) Let (E, d) be a metric space, and assume the set $S \subset E$ is not closed. Prove that there is a sequence $\{p_n\}_{n=1}^\infty$ of elements of S that is convergent and its limit does not belong to S.

Solution. As the set S is not closed, its complement $E \setminus S$ (also denoted as $\complement S$) is not open. Therefore, there is a $p \in E \setminus S$ such that for every $\epsilon > 0$ the open ball

$$U(p, \epsilon) = \{q \in E : d(p, q) < \epsilon\}$$

is not a subset of $E \setminus S$; that is, for every $\epsilon > 0$, the set $U(p, \epsilon) \cap S$ is not empty. For each $n \geq 1$, pick an element p_n of the set $\epsilon > 0$, the set $U(p, 1/n) \cap S$. Then, clearly, all elements of the sequence $\{p_n\}_{n=1}^\infty$ belong to S. Furthermore, this sequence converges to p. Indeed, given an arbitrary $\epsilon > 0$, pick N to be $1/\epsilon$.[4] Then we have

$$d(p, p_n) < \frac{1}{n} < \frac{1}{N} = \frac{1}{1/\epsilon} = \epsilon$$

whenever $n > N$. The proof is complete.

c) Let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be two convergent sequences of reals, and assume that $a_n \leq b_n$ for every integer $n \geq 1$. Prove that $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.

Solution. We have $b_n - a_n \geq 0$, i.e., $b_n - a_n \in [0, +\infty)$. Since the interval $[0, +\infty)$ is a closed set, it follows, that the limit of this sequence, that is, $\lim_{n\to\infty}(b_n - a_n) = \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n$, also belongs to the interval $[0, +\infty)$. Therefore $\lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n \geq 0$, i.e., $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$, as we wanted to show.

4. *a*) Let (E, d) be a metric space and let $\{p_n\}_{n=1}^\infty$ be a sequence of elements of E. Show that there is at most one element p of E such that $\{p_n\}_{n=1}^\infty$ converges to p. (That is, show that a sequence cannot converge to two different points.)

Solution. Assume that the sequence $\{p_n\}_{n=1}^\infty$ converges both to p and q, and $p \neq q$. Let $\epsilon = d(p, q)$; then $\epsilon > 0$ (since, according to the definition of a metric space, the distance between two different points is always positive). Let N_1 be such that $d(p, p_n) < \epsilon/2$ whenever $n > N_1$; such an N_1 exists, since $\{p_n\}_{n=1}^\infty$ converges to p. Let N_2 be such that $d(q, p_n) < \epsilon/2$ whenever $n > N_2$; such an N_2 exists, since $\{p_n\}_{n=1}^\infty$ converges to q. Let n be an integer such that $n > \max\{N_1, N_2\}$. Then $d(p, p_n) < \epsilon/2$ and $d(q, p_n) < \epsilon/2$. Hence, using the triangle inequality, we have

$$d(p, q) \leq d(p, p_n) + d(p_n, q) = d(p, p_n) + d(q, p_n) < \epsilon/2 + \epsilon/2 = \epsilon;$$

the first equality here holds by the symmetry property of the distance function. This inequality means that $d(p, q) < \epsilon$; this, however, contradicts the choice of ϵ, according to which $\epsilon = d(p, q)$. This contradiction shows that the sequence $\{p_n\}_{n=1}^\infty$ cannot converge to two different points.

b) Let (E, d) be a metric space and let $\{p_n\}_{n=1}^\infty$ be a convergent sequence of elements of E. Show the set $\{p_n : n \geq 1\}$ is bounded.

Solution. Choose a number $\epsilon > 0$.[5] Let $p = \lim_{n\to\infty} p_n$, and let N be such that $d(p, p_n) < \epsilon$ whenever $n > N$. Put

$$M_0 = \max\{d(p, p_n) : 1 \leq n \leq N\}.$$

[4] Here it is advantageous not to require that N be an integer. Nevertheless, requiring that N be an integer does not cause a serious problem; in that case, one needs to pick N as the least integer greater than or equal to $1/\epsilon$.

[5] For the argument that follows, one need not consider an arbitrary $\epsilon > 0$. The argument will work with any fixed choice of ϵ; for example, one can pick $\epsilon = 1$.

Since the set on the right-hand side is finite, the maximum indicated on the right-hand side exists (every finite set of real numbers has a maximum). Next, put

$$M = \max\{M_0, \epsilon\}.$$

Then $d(p, p_n) \le M$ for every integer $n \ge 1$. Indeed, for n with $1 \le n \le N$ this holds because $d(p, p_n) \le M_0 \le M$ then, and for $n > N$ this holds because $d(p, p_n) \le \epsilon \le M$ then. Thus, the set $\{p_n : n \ge 1\}$ is a subset of the open ball $U(p, M+1) = \{q : d(p, q) < M+1\}$.[6]

c) Let a be a real number with $0 \le a < 1$. Prove that $\lim_{n\to\infty} a^n = 0$. Describe your argument carefully by referring to the theorems and results that the steps in your proof rely on.

Solution. The sequence $\{a^n\}_{n=1}^\infty$ is decreasing; indeed, we have $a^{n+1} - a^n = a^n(a-1) < 0$, and so $a^{n+1} < a^n$. Since this sequence is bounded from below ($a^n \ge 0$ for every $n \ge 1$), it is known that this sequence is convergent. Let x be the limit of this sequence, that is, write $x = \lim_{n\to\infty} a^n$. Then we have

$$ax = a \lim_{n\to\infty} a^n = \lim_{n\to\infty} (a \cdot a^n) = \lim_{n\to\infty} a^{n+1} = \lim_{n\to\infty} a^n = x.$$

Here the fourth equality holds since $\{a^{n+1}\}_{n=1}^\infty$ is a subsequence of $\{a^n\}_{n=1}^\infty$,[7] hence its limit is the same as that of the latter sequence. Thus $ax = x$; i.e., $(a-1)x = 0$. As $a - 1 \ne 0$ (since $0 \le a < 1$), we must have $x = 0$, as we wanted to show.

5. *a*) Define what it means for the sequence $\{p_n\}_{n=1}^\infty$ in a metric space (E, d) to be a Cauchy sequence.

Definition. The sequence $\{p_n\}_{n=1}^\infty$ of points in the metric space (E, d) is called a Cauchy sequence if for every $\epsilon > 0$ there is an N (which can be taken to be a positive integer, if desired) such that we have $d(p_n, p_m) < \epsilon$ whenever $m, n > N$. Formally,

$$\text{Cauchy}(\{p_n\}_{n=1}^\infty) \leftrightarrow \forall \epsilon > 0\ \exists N\ \forall m > N\ \forall n > N\ [d(p_m, p_n) < \epsilon].$$

b) Let $\{a_n\}_{n=1}^\infty$ be a bounded sequence of reals, and let a be the supremum of the set $S = \{x :$ there are infinitely many positive integers n for which $x \le a_n\}$. Let ϵ be a positive real number and let N be a positive integer. Show that there is an integer $n > N$ for which $a - \epsilon < a_n < a + \epsilon$.

Solution. As a is the least upper bound of S, the number $a + \epsilon$ does not belong to S (since the inequality $a + \epsilon < a$ does not hold). Thus, there are only finitely many positive integers n for which $a_n \ge a + \epsilon$.

Further, as a is the least upper bound of S, the number $a - \epsilon$ is not an upper bound of S. Hence there is an $x \in S$ for which $a - \epsilon < x$. Then, according to the definition of S, there are infinitely many positive integers n for which $x \le a_n$. Hence, *a fortiori*,[8] there are infinitely many positive integers n for which $a - \epsilon < a_n$; i.e., the set

$$\{n \in \mathbb{N} : a - \epsilon < a_n\}$$

[6] Instead of this one can say that the set $\{p_n : n \ge 1\}$ is a subset of the closed ball $\overline{U}(p, M) = \{q : d(p, q) \le M\}$ if one defines a bounded set as any set included in a ball. However, instead of this, we said that a set is bounded if it is included in an open ball. It is easy to see that these two definitions are equivalent.

[7] $\{c_n\}_{n=1}^\infty$ is a subsequence of the sequence $\{d_n\}_{n=1}^\infty$ if there is an increasing function f from the set $\{n : n \ge 1\}$ into itself (i.e., f is such that $f(n) \le f(n+1)$ for every integer $n \ge 1$) such that $d_n = c_{f(n)}$. In the present case, the sequence $\{a^n\}_{n=1}^\infty$ is the sequence $\{c_n\}_{n=1}^\infty$ with $c_n = a^n$, and the sequence $\{a^{n+1}\}_{n=1}^\infty$ is the sequence $\{d_n\}_{n=1}^\infty$ with $d_n = c_{f(n)}$, where $f(n) = n + 1$.

[8] Even more so, or for a still stronger reason. Latin, occasionally used in mathematical and other specialized (technical, scientific, or legal) writing.

is infinite. Omitting finitely many elements from this set, namely those n's for which $n \le N$ or for which $a_n \ge a + \epsilon$, we can pick an n from among the remaining elements for which $n > N$ and $a - \epsilon < a_n < a + \epsilon$, showing that an integer n exists as required.

The above argument, with $\epsilon/2$ instead of ϵ, is the key step in a proof that if $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence of reals, then this sequence converges.