

Follow these instructions carefully:

Work on the paper provided; do not use your own paper. *Work only on one problem on each sheet (you should not work on two different problems on the two sides of the same sheet).* On the top of each page, *print* your name (*encircle your last name*) and indicate the number of the problem you are working on by writing e.g. “*Problem #4*”. Always *encircle* your final answer. If there are several parts to a problem, always indicate the part that you are answering, e.g. by writing “*Answer to Part b*” (the number of the problem should be on the top of the page). Do not use a *red* pen or a *red* pencil. Do not write in the corner covered up by the staple (top left corner on the front side, top right corner on the back side). Each problem is worth the *same* amount of credit.

1. *a)* Define what it means for a metric space (E, d) to be compact.
b) Let (E, d) a compact metric space, and let S be a closed subset of E . Prove that S is compact.
c) Assume (E, d) be a metric space, and let S be a compact subset of E . Prove that S is bounded.
2. Let (E, d) and (E', d') be metric spaces, and let $f : E \rightarrow E'$ be a function.
a) Give a definition of f being continuous on E .
b) Give a definition of f being uniformly continuous on E .
c) Explain why the function $f(x) = 1/x$ is not uniformly continuous on the interval $(0, 1)$. Do not just give a vague argument; show clearly that this function does not satisfy the definition of uniform continuity.
3. Let (E, d) and (E', d') be metric spaces and let $f : E \rightarrow E'$ be a function. Assume that for every open subset U of E' the set $f^{-1}[U] \stackrel{\text{def}}{=} \{p \in E : f(p) \in U\}$ is open. Prove that f is continuous.
4. *a)* Let (E, d) be a metric space. Define what it means for (E, d) to be connected.
b) Let (E, d) and (E', d') be metric spaces, let $f : E \rightarrow E'$ be a continuous function, and assume that E is connected. Prove that the set $f[E] \stackrel{\text{def}}{=} \{f(p) : p \in E\}$ is connected.
5. *a)* State Rolle’s Theorem without proof.
b) Let $U \subset \mathbb{R}$ be an open set, and let $a \in U$. Let f be a function differentiable in U such that $f(a) \geq f(x)$ for all $x \in U$. Prove that $f'(a) = 0$.