Exam 2, Mathematics 11.1, Section TF10 43 copies
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1. *a*) Define what it means for a metric space (E, d) to be compact.

Definition. Let (E, d) be a metric space and let S be a subset of E. The set S is said to be *compact* in (E, d) if the following holds: For any collection $\mathcal{U}$ of open sets in (E, d), if $S \subset \bigcup \mathcal{U}$, there is a finite subcollection $\mathcal{U}' \subset \mathcal{U}$ such that $S \subset \bigcup \mathcal{U}'$.

Further, the space (E, d) is said to be compact if E is a compact set in (E, d).

To say that $\mathcal{U}$ is a collection of open sets in (E, d) means that $\mathcal{U}$ is a collection every element of which is an open subset of (E, d). The inclusion $S \subset \bigcup \mathcal{U}$ is expressed by saying that $\mathcal{U}$ is on open cover of S. One also says that $\mathcal{U}$ covers S, or that the elements of $\mathcal{U}$ cover S. If $\mathcal{U}' \subset \mathcal{U}$ and $S \subset \bigcup \mathcal{U}'$, then $\mathcal{U}'$ is called a subcover of $\mathcal{U}$ (for S, if one wants to avoid repeating the preposition "of"). Thus, one can say shortly that a set is compact if and only if any of its open covers has a finite subcover.

b) Let (E, d) a compact metric space, and let S be a closed subset of E. Prove that S is compact.

Solution. Let $\mathcal{U}$ be a collection of open sets that covers S. I.e., we have $S \subset \bigcup \mathcal{U}$. Then we have

$$E \subset S \cup (E \setminus S) \subset \left(\bigcup \mathcal{U}\right) \cup (E \setminus S) \subset \bigcup (\mathcal{U} \cup \{E \setminus S\}).$$

Note that $\{E \setminus S\}$ is the one-element set containing $E \setminus S$, i.e., the complement of S, as its only element. Since S is a closed set, $E \setminus S$ is an open set. Thus, $\mathcal{U} \cup \{E \setminus S\}$ is a collection of open sets that covers E. As the set E is compact by our assumptions, this collection of open sets includes a finite subcollection $\mathcal{U}'$ that covers E. Then the set $\mathcal{U}'' \overset{def}{=} \mathcal{U}' \setminus \{E \setminus S\}$ is a finite subcollection of $\mathcal{U}$ that covers S.[2] Indeed, the set S and $E \setminus S$ are disjoint, so the set S will be covered even if we remove the element $E \setminus S$ from the collection $\mathcal{U}'$. Thus, starting with an arbitrary collection $\mathcal{U}$ covering S, we found a finite subcollection $\mathcal{U}''$ that also covers S. It follows that the set S is compact, as we wanted to show.

c) Assume (E, d) be a metric space, and let S be a compact subset of E. Prove that S is bounded.

Solution. For each $p \in S$, let $B(p)$ be an open ball with center p. Then the collection $\{B(p) : p \in S\}$ covers S. By the compactness of S, a finite subcollection of this also covers S. That is, S can be covered with finitely many open balls. One can easily find a single open ball that includes all these open balls.

Indeed, let S' be a finite subset of S such that the finite collection $\{B(p) : p \in S'\}$ of open balls covers S. Let r_p be the radius of the open ball $B(p)$. Further, let p_0 be an arbitrary element of E. Then the open ball with center p_0 and with radius

$$\max\{d(p_0, p) + r_p : p \in S'\}$$

includes all the open balls in the set $\{B(p) : p \in S'\}$. Thus the set S is a subset of this single open ball. This shows that S is bounded.

2. Let (E, d) and (E', d') be metric spaces, and let $f : E \to E'$ be a function.

a) Give a definition of f being continuous on E.

Definition. The function $f : E \to E$ is said to be *continuous* on E if for every $p \in E$ and for every $\epsilon > 0$ there is a $\delta > 0$ such that for every $q \in E$, if $d(p, q) < \delta$ then $d'(f(p), f(q)) < \epsilon$.

Note. The above definition says that f is continuous at p for every $p \in E$. Using logic notation, the function $f : E \to E$ is said to be continuous on E if

$$(\forall p \in E)(\forall \epsilon > 0)(\exists \delta > 0)(\forall q \in E)\big(d(p, q) < \delta \to d'(f(p), f(q) < \epsilon\big).$$

[1]All computer processing for this manuscript was done under Fedora Core Linux. $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX was used for typesetting.

[2]The collection, or set (since the word collection here means the same as set, it is used merely to make the text more lively), $\mathcal{U}''$ is the same as $\mathcal{U}'$ in case the set $E \setminus S$ is not an element of $\mathcal{U}'$; if $E \setminus S$ is an element of $\mathcal{U}'$, then $\mathcal{U}''$ is obtained by removing the element $E \setminus S$ from $\mathcal{U}'$.

The first two quantifiers are interchangeable here, since two quantifiers of the same type (i.e., two universal quantifiers, or two existential quantifiers) are interchangeable, so we can write this also as

$$(\forall\epsilon > 0)(\forall p \in E)(\exists\delta > 0)(\forall q \in E)\big(d(p,q) < \delta \to d'(f(p), f(q) < \epsilon\big). \tag{1}$$

b) Give a definition of f being uniformly continuous on E.

Definition. The function $f : E \to E$ is said to be *uniformly continuous* on E if for every $\epsilon > 0$ there is a $\delta > 0$ such that for every $p \in E$ and for every $q \in E$, if $d(p,q) < \delta$ then $d'(f(p), f(q)) < \epsilon$.

Note. Using logic notation, the function $f : E \to E$ is said to be uniformly continuous on E if

$$(\forall\epsilon > 0)(\exists\delta > 0)(\forall p \in E)(\forall q \in E)\big(d(p,q) < \delta \to d'(f(p), f(q) < \epsilon\big). \tag{2}$$

After giving the definition, explain what the difference is between continuity and uniform continuity. (The emphasis is on the word *after*; the definition and the explanation must not be intermixed.)

The formal difference between continuity and uniform continuity is indicated by the different order of the second and third quantifiers in formulas (1) and (2); aside from this difference, the two formulas are identical. However, these quantifiers are of different type (one is a universal quantifier, the other is an existential quantifier), and so they are not interchangeable. Hence the meanings of these two formulas are different.

To explain the difference in a less formal way, in case of continuity, δ depends on the choice of p as well as on ϵ (and, of course, on the function f itself), whereas in case of uniform continuity, δ depends only on ϵ but not on p (but it does depend on the function f itself).

c) Explain why the function $f(x) = 1/x$ is not uniformly continuous on the interval $(0,1)$. Do not just give a vague argument; show clearly that this function does not satisfy the definition of uniform continuity.

Solution. We have to show that the following is not true: For every $\epsilon > 0$ there is a $\delta > 0$ such that $|1/x - 1/y| < \epsilon$ holds whenever $x, y \in (0,1)$ are such that $|x - y| < \delta$. Formally, we have to show that the following is not true:

$$(\forall\epsilon > 0)(\exists\delta > 0)(\forall x \in (0,1))(\forall y \in (0,1))\left(|x-y| < \delta \to \left|\frac{1}{x} - \frac{1}{y}\right| < \epsilon\right).$$

The negation of this formula is

$$(\exists\epsilon > 0)(\forall\delta > 0)(\exists x \in (0,1))(\exists y \in (0,1))\left(|x-y| < \delta \wedge \left|\frac{1}{x} - \frac{1}{y}\right| \ge \epsilon\right). \tag{3}$$

When forming the negation, put a negation sign $\neg$ on the left of the former formula, and move the negation symbol all the way inside, using the following rules: replace $\neg\forall\mathbf{x}$ with $\exists\mathbf{x}\neg$, $\neg\exists\mathbf{x}$ with $\forall\mathbf{x}\neg$, and $\neg(\mathbf{A} \to \mathbf{B})$ with $\mathbf{A} \wedge \neg\mathbf{B}$.

We need to show that the latter displayed formula is true. That is, we need to show that there is an $\epsilon > 0$ such that for all $\delta > 0$ there are x and y in the interval $(0,1)$ such that $|x - y| < \delta$ and $|1/x - 1/y| \ge 1$. To this end, put $\epsilon = 1$.[3] Let $\delta > 0$ be arbitrary. Put $x = \min\{\delta, 1/2\}$. Then $x \in (0,1)$.[4] In order to find a $y \in (0,1)$ with $|1/x - 1/y| \ge \epsilon = 1$, it is enough to find a y with $0 < y \le x$ such that $1/x - 1/y \le -1$, i.e., such that

$$0 < y \le \frac{1}{\frac{1}{x} + 1}.$$

For example, we tan cake

$$y = \frac{1}{\frac{1}{x} + 1};$$

[3] We could take any fixed $\epsilon > 0$ instead.

[4] The choice $x = \min\{\delta, 1/2\}$ is somewhat arbitrary; instead of $1/2$, any other number in the interval $(0,1)$ would do.

note that the inequality $0 < y \le x$ is satisfied for this choice, since the right-hand side is easily shown to be less than x. This shows that $1/x$ is not uniformly continuous on $(0, 1)$.

Note. The argument in the last paragraph can be put differently (and perhaps more simply) by noting that if we put $y = x/2$ then the inequality $|x - y| < \delta$ means $|x - y| = x - x/2 = x/2 < \delta$, i.e.,

$$x < 2\delta$$

$(x \in (0, 1))$. On the other hand, the inequality $|1/x - 1/y| \ge \epsilon$ can be written as

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \frac{1}{x} - \frac{1}{x/2} = \frac{1}{x} \ge \epsilon,$$

i.e., that $x \le 1/\epsilon$.

Now, taking $\epsilon = 1$ again, it is clear that for $\delta > 0$ there are x and y such that both inequalities in the matrix[5] of (3) are satisfied; it is enough to take

$$x = \min\left\{\delta,\ \frac{1}{2}\right\}$$

and $y = x/2$, since, as we just saw, these inequalities become $x < 2\delta$ and $x < 1/\epsilon = 1$ in case $y = x/2$, in turn.

3. Let (E, d) and (E', d') be metric spaces and let $f : E \to E'$ be a function. Assume that for every open subset U of E' the set $f^{-1}[U] \overset{def}{=} \{p \in E : f(p) \in U\}$ is open. Prove that f is continuous.

Solution. Let $p \in E$ and $\epsilon > 0$ be arbitrary. According to the definition of continuity, we have to prove that there is a $\delta > 0$ such that for every $q \in E$ with $d(p, q) < \delta$ we have $d'(f(p), f(q)) < \epsilon$. To this end, first note that the set

$$U = \{v \in E' : d'(f(p), v) < \epsilon\}$$

is open. Therefore, according to the assumption, its inverse image

$$f^{-1}[U] = \{q \in E : f(q) \in U\} = \{q \in E : d'(f(p), f(q)) < \epsilon\}$$

is open. As p is clearly an element of this open set, by the definition of being open there is a $\delta > 0$ such that the open ball with center p and radius δ is a subset of this set; that is:

$$\{q \in E : d(p, q) < \delta\} \subset \{q \in E : d'(f(p), f(q)) < \epsilon\}.$$

The meaning is this inclusion is exactly the statement to be proved. That is, $p \in E$ and $\epsilon > 0$ were arbitrary; then we found a $\delta > 0$ such that $d'(f(p), f(q)) < \epsilon$ for every $q \in E$ with $d(p, q) < \delta$. The proof is complete.

4. *a*) Let (E, d) be a metric space. Define what it means for (E, d) to be connected.

Definition. The metric space (E, d) is said to be *connected* if the only sets in this space that are both open and closed are the empty set $\emptyset$ and the whole space E.

This definition can be restated to say that (E, d) is connected if there are no nonempty disjoint open sets U and V such that $U \cup V = E$. (Instead of "open," one can say "closed" here.)

b) Let (E, d) and (E', d') be metric spaces, let $f : E \to E'$ be a continuous function, and assume that E is connected. Prove that the set $f[E] \overset{def}{=} \{f(p) : p \in E\}$ is connected.

[5] The matrix of a logic formula is the part after all the quantifiers. The matrix makes sense only for logic formulas where all the quantifiers are in front.

Solution. Assume that $f[E]$ is not connected. Then there are two nonempty disjoint sets X and Y that are open in[6] $(f[E], d' \restriction E')$ such that $X \cup Y = f[E]$. Then there are set U and V that are open in (E, d') such that $X = U \cap f[E]$ and $Y = V \cap f[E]$.[7] Then the sets $f^{-1}[U]$ and $f^{-1}[V]$ are open subsets of (E, d). They are both nonempty; for example, the set $U \cap f[E] = X$ is not empty, so there must be a $p \in E$ such that $f(p) \in U$; then $p \in f^{-1}[U]$. Similarly, $f^{-1}[V]$ is not empty.

Furthermore, $f^{-1}[U]$ and $f^{-1}[V]$ are disjoint. Indeed, assume $p \in E$ is such that both $p \in f^{-1}[U]$ and $p \in f^{-1}[V]$. Then $f(p) \in U$ and $f(p) \in V$. As $f(p) \in f[E]$, this means that $f(p) \in X = U \cap f[E]$ and $f(p) \in Y = V \cap f[E]$; this, however, is not possible, since the sets X and Y are disjoint by assumption.

Finally, $f^{-1}[U] \cup f^{-1}[V] = E$. Indeed, let $p \in E$ be arbitrary. Then $f(p) \in f[E] = X \cup Y \subset U \cap V$; that is, we have $f(p) \in U$ or $f(p) \in V$. That is, $p \in f^{-1}[U]$ $p \in f^{-1}[V]$. Thus $f^{-1}[U]$ and $f^{-1}[V] = E$ are two nonempty disjoint open sets in (E, d) whose union is E, contradicting the assumption that (E, d) is connected. The proof is complete.

5. *a*) State Rolle's Theorem without proof.

Rolle's Theorem. *Let a, b be real numbers with $a < b$, and assume that f is continuous on $[a, b]$ and differentiable on (a, b), and, further, that $f(a) = f(b) = 0$. Then there is a $\xi \in (a, b)$ such that $f'(\xi) = 0$.*

b) Let $U \subset \mathbb{R}$ be an open set, and let $a \in U$. Let f be a function differentiable in U such that $f(a) \geq f(x)$ for all $x \in U$. Prove that $f'(a) = 0$.

Solution. Assume $f'(a) \neq 0$ and write $\epsilon = |f'(a)|/2$. Then, clearly, $\epsilon > 0$; as

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

In view of the definition of the limit, then there is a $\delta > 0$ such that

$$\left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| < \epsilon$$

whenever $0 < |x - a| < \delta$; choose δ small enough so that $\{x : |x - a| < \delta \subset U\}$ (this is possible since $a \in U$ and U is open), i.e., that

$$-\epsilon < \frac{f(x) - f(a)}{x - a} - f'(a) < \epsilon.$$

Noting that $\epsilon = |f'(a)|/2$, this means that

$$f'(a) - \frac{|f'(a)|}{2} < \frac{f(x) - f(a)}{x - a} - f'(a) < f'(a) + \frac{|f'(a)|}{2},$$

whenever $0 < |x - a| < \delta$. Note that the two extreme members of this inequality, $f'(a) - \frac{|f'(a)|}{2}$ and $f'(a) + \frac{|f'(a)|}{2}$, are either both positive (when $f'(a) > 0$) or both negative (when $f'(a) < 0$). This is not possible, however, since the middle member, $\frac{f(x)-f(a)}{x-a}$ is is positive or zero for $x < a$ and negative or zero for $x > a$ ($x \in U$). This is because $f(a) \geq f(x)$ for all $x \in U$; that is, the numerator is never positive; on the other hand, the denominator is negative for $x < a$ and it is positive for $x > a$. That is, the last displayed inequality cannot be true for every x with $0 < |x - a| < \delta$. This is a contradiction, proving that the assumption $f'(a) \neq 0$ cannot be true. That is, we must have $f'(a) = 0$.

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[6] $d' \restriction f[E]$ is the restriction of d' to $f[E]$. That is, $(d' \restriction f[E])(p, q) = d'(p, q)$ if $p \in f[E]$ and if $q \in f[E]$, and $(d' \restriction f[E])(p, q)$ is undefined if either $p \notin f[E]$ or $q \notin f[E]$.

[7] See the section entitled "Subspaces and Compact Sets" in my supplementary notes. The issue is that that X and Y need not be open in the sense of (E', d'), so one cannot immediately conclude that $f^{-1}[X]$ and $f^{-1}[Y]$ are open. However, one can put this argument differently, by considering f to be a function from (E, d) to $(f[E], d' \restriction E')$ (instead of (E', d'). Since f from E to $f[E]$ is also a continuous function, one can conclude that the sets $f^{-1}[X]$ and $f^{-1}[Y]$ are open, without having to introduce the sets U and V.

One usually expresses the change of point of view involved in considering f to be a function from E to E' to a function from E to $f[E]$ by saying that, without loss of generality, we may assume that $E' = f[E]$.