Exam 3, Mathematics 11.1, Section TF10 43 copies
10:50–12:00 am, May 2, 2008, 330NE
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1. *a)* State (without proof) the Mean-Value Theorem (for differentiation).

The Mean-Value Theorem. *Let f be a function that is continuous in the interval $[a,b]$ ($a, b \in \mathbb{R}$ and $a < b$), and differentiable in (a,b). Then there is a $\xi \in (a,b)$ such that*

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

b) Let a, b be reals with $a < b$, and let $f : (a,b) \to \mathbb{R}$ be a function. Assume that f is differentiable on (a,b) and $f'(x) > 0$ for every $x \in (a,b)$. Prove that f is strictly increasing on (a,b).

Solution. Let c, d be arbitrary numbers such that $a < c < d < b$. We need to show that $f(c) < (f(d)$. By the Mean-Value Theorem for the interval $[c,d]$ we have

$$f(d) - f(c) = f'(\xi)(d - c)$$

for some $\xi \in (c,d)$. As $f'(\xi) > 0$ by our assumption, the right-hand side here is positive; hence $f(c) < f(d)$ follows.

2. *a)* Let f be a real-valued function on the closed interval $I = [a,b]$. (i) Define what a tagged partition is on I. (ii) Define what is meant by the width of a partition. (iii) Define what the Riemann sum is for f associated with a tagged partition of I.

Definition. Let $I = [a,b]$ be a closed interval ($a, b \in \mathbb{R}$ and $a < b$).

(i) A finite subset P of $[a,b]$ such that $a, b \in P$ is called a *partition* of $[a,b]$.[2]

Writing the elements of P in increasing order as

$$P\colon a = x_0 < x_1 < \ldots < x_N = b,$$

where $P = \{x_0, x_1, \ldots, x_N\}$, a *tag* for the interval $I_i \overset{def}{=} [x_{i-1}, x_i]$ is any number $\xi_i \in I_i$ ($1 \le i \le N$). A *tagged partition* is is a partition that has a tag ξ_i for each interval I_i of the partition ($1 \le i \le N$).

(ii) The width of the partition P as described above is the number

$$\|P\| \overset{def}{=} \max\{x_i - x_{i-1} : 1 \le i \le N\}.$$

(iii) Let f be a real-valued function on $[a,b]$. The Riemann S sum for f associated with the the tagged partition described above is defined as

$$S = \sum_{i=1}^{N} f(\xi_i)(x_i - x_{i-1}).$$

b) Let f and I be as in Part *a)*, and let A be a real number. Define what it means for A to be the Riemann integral of f on I.

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[1] All computer processing for this manuscript was done under Fedora Core Linux. $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX was used for typesetting.

[2] There are other ways to define a partition. A partition could be defined as a sequence of points of $[a,b]$ or even as a finite set of non-overlapping (i.e., having no interior points in common) closed subintervals of $[a,b]$ whose union is $[a,b]$. Any definition will work that will leave the concept of the Riemann sum unchanged. However, some definitions are easier to work with in certain settings than others.

Definition. Let f be a real-valued function on the closed interval $[a, b]$ ($a, b \in \mathbb{R}$ and $a < b$). A is said to be the Riemann integral of f on $[a, b]$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that, for every tagged partition of $[a, b]$ that has width $< \delta$, writing S for the Riemann sum for f associated with this tagged partition, we have $|S - A| < \epsilon$.

3.*a*) Let $[a, b]$ be a closed interval and let $c \in [a, b]$ be real number. Define the function f on $[a, b]$ by

$$f(x) = \begin{cases} 1 & \text{if } x = c, \\ 0 & \text{otherwise,} \end{cases}$$

Using the definition of the Riemann integral, prove that $\int_a^b f = 0$.

Solution. Let $\epsilon > 0$ be arbitrary, and let

$$P\colon a = x_0 < x_1 < \ldots < x_N = b$$

be a partition of $[a, b]$ of width $< \epsilon/2$. For each i with $1 \le i \le N$ let $\xi_i \in [x_{i-1}, c_i]$ be a tag. There are at most two values of i for which $\xi_i = c$. Namely, if $c = x_k$ for some k with $1 \le k \le N - 1$, then it is possible to have $\xi_k = \xi_{k+1} = c$; otherwise, that is, if (i) $c = a = x_0$, or (ii) $c = x_N$, or (iii) $c \in (x_{k-1}, x_k)$ for some k with $1 \le k \le N$, then there is at most one i for which we can have $\xi_i = c$ ($i = 1$ in case (i), $i = N$ is case (ii), and $i = k$ is case (iii)).[3]

Noting that $f(\xi_i) = 1$ if $\xi_i = c$ and $f(\xi_i) = 0$ if $\xi_i \ne c$, and writing S for the corresponding Riemann sum, we have

$$0 \le S = \sum_{i=1}^{N} f(\xi_i)(x_i - x_{i-1}) \le 2\|P\| < 2 \cdot \frac{\epsilon}{2} = \epsilon$$

Since $\epsilon > 0$ was arbitrary, and this inequality holds with any Riemann sum for f associated with a tagged partition of width $< \epsilon/2$, it follows that 0 is the Riemann integral of f on $[a, b]$.

b) State the Fundamental Theorem of Calculus. Make sure that the conditions are precisely stated. Do not give a proof.

The Fundamental Theorem of Calculus. *Let I be an open interval of $\mathbb{R}$, and let f be a continuous function on I. Further, let $a \in I$. Then, for every $x \in I$, we have*

$$\frac{d}{dx} \int_a^x f = f(x).$$

The existence of the integral in the theorem follows from the fact that continuous functions are integrable.

4.*a*) Describe what a step function on an interval $[a, b]$ is.

Definition. A real-valued function on the interval $[a, b]$ is a step function if there is a partition

$$P : a = x_0 < x_1 < \ldots < x_N = b$$

of the interval $[a, b]$ and numbers c_i for each integer i with $1 \le i \le N$ such that, for each i with $1 \le i \le N$, we have $f(x) = c_i$ whenever $x \in (x_{i-1}, x_i)$.

b) State the second criterion (the one involving step functions) of Riemann integrability.

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[3] It is of course possible that $\xi_i \ne c$ for any value of i.

The Second Criterion of Riemann Integrability. *The real-valued function f on the interval $[a,b]$ is Riemann-integrable if and only if for every $\epsilon > 0$ there are step functions f_1 and f_2 on the interval $[a,b]$ such that $f_1(x) \le f(x) \le f_2(x)$ whenever $x \in [a,b]$ and*

$$\int_a^b (f_2 - f_1) < \epsilon.$$

c) Let f be a real-valued function that is Riemann integrable on interval $[a,b]$. Prove that f is bounded on $[a,b]$.

Solution. Let $\epsilon > 0$. Let $\delta > 0$ be such that, for any partition of width $< \delta$ we have

$$\left| \int_a^b f - R \right| < \epsilon$$

for any associated Riemann sum.[4]

Now, let

$$P : a = x_0 < x_1 < \ldots < x_N = b$$

be a partition with width $< \delta$. Let the numbers $\xi_i, \eta_i \in [x_{i-1}, x_i]$, to be specified later, for i with $1 \le i \le N$ form two sets of tags for the partition P, and let S_1 and S_2 be the corresponding Riemann sums. Then we have $\left| \int_a^b f - S_1 \right| < \epsilon$ and $\left| \int_a^b f - S_2 \right| < \epsilon$ and so

$$|S_1 - S_2| < 2\epsilon.$$

Let k be a fixed integer with $1 \le k \le N$ and pick ξ_i and η_i such that $\xi_i = \eta_i$ for all i with $i \ne k$ and $1 \le i \le N$. Then all but one term in the difference $S_1 - S_2$ cancel, so the above inequality becomes

$$|S_1 - S_2| = |(f(\xi_k) - f(\eta_k))(x_k - x_{k-1})| < 2\epsilon.$$

That is,

$$|f(\xi_k)| < |f(\eta_k)| + \frac{2\epsilon}{x_k - x_{k-1}}.$$

Picking a fixed $\eta_k \in [x_{k-1}, x_k]$ and letting $\xi_k \in [x_{k-1}, x_k]$ be arbitrary, this inequality shows that f is bounded on the interval $[x_{k-1}, x_k]$. Since this is true for each k with $1 \le k \le N$, and the union of these intervals is $[a,b]$, it follows that f is bounded on $[a,b]$.

5. *a*) State the necessary and sufficient condition for the series $\sum_{n=1}^{\infty} a_n$ of real numbers to be convergent (the condition in question is usually called the *Cauchy criterion* for convergence of series).

Cauchy's Convergence Criterion for Series. *The series $\sum_{n=1}^{\infty} a_n$ of real numbers converges if and only if for every $\epsilon > 0$ there is an N such that*

$$\left| \sum_{k=m+1}^{n} a_k \right| < \epsilon$$

whenever $n > m \ge N$.

Formally

$$\sum_{n=1}^{\infty} a_n \text{ converges } \leftrightarrow (\forall \epsilon > 0)\,(\exists N)\,(\forall m)\,(\forall n)\,\Big(n > m \ge N \to \Big| \sum_{k=m+1}^{n} a_k \Big| < \epsilon\Big)$$

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[4] The Riemann sum is of course not determined by the partition alone; we also need tags to determine the Riemann sum uniquely. This is why we said *any* Riemann sum rather than *the* Riemann sum.

b) Prove that the harmonic series $\sum_{n=1}^{\infty} 1/n$ is divergent. Use material covered in the course (that is, *do not* use the Integral Test for convergence).

Proof. We will show that the harmonic series does not satisfy the condition in Cauchy's Convergence Criterion. Let N be arbitrary, and let j be a nonnegative integer such that $2^j > N$. We have

$$\sum_{2^j+1}^{2^{j+1}} \frac{1}{n} \geq \sum_{2^j+1}^{2^{j+1}} \frac{1}{2^{j+1}} = 2^j \cdot \frac{1}{2^{j+1}} = \frac{1}{2}.$$

Thus, Cauchy's convergence Criterion fails with $\epsilon = 1/2$ for the harmonic series, showing that the harmonic series diverges.