Exam 2, Mathematics 4201, Section TR11 28 copies
11:00–12:10 pm, April 11, 2013, 4411N
Instructor: Attila Máté

Follow these instructions carefully:

Work on the paper provided; do not use your own paper. *Work only on one problem on each sheet (you should not work on two different problems on the two sides of the same sheet).* On the top of each page, *print* your name (*encircle your last name*) and indicate the number of the problem you are working on by writing e.g. *"Problem #4"*. Always *encircle* your final answer. If there are several parts to a problem, always indicate the part that you are answering, e.g. by writing *"Answer to Part b)"* (the number of the problem should be on the top of the page). Do not use a *red* pen or a *red* pencil. Do not write in the corner covered up by the staple (top left corner on the front side, top right corner on the back side). Each problem is worth the *same* amount of credit.

1. *a)* Let (E, d) be a metric space. Define what it means for (E, d) to be connected.
   b) Prove that for any $a, b \in \mathbb{R}$ with $a < b$ the interval $[a, b]$ is connected.

2. *a)* Let (E, d) and (E', d') be metric spaces, and let $f : E \to E'$ be a continuous function. Further, let $V \subset E'$ be an open set. Prove that $f^{-1}[V] \overset{def}{=} \{p \in E : f(p) \in V\}$ is open.
   b) Let (E, d) and (E', d') be metric spaces, and assume E is compact. Further, let $f : E \to E'$ be a continuous function. Prove that $f[E] \overset{def}{=} \{f(p) : p \in E\}$ is compact.

3. *a)* Let $U \subset \mathbb{R}$ be an open set, let $x_0 \in U$, and assume that the function $f : U \to \mathbb{R}$ is differentiable at x_0. Prove that f is continuous at x_0.
   b) Let $U \subset \mathbb{R}$ be an open set, let $x_0 \in U$, and assume that the functions $f : U \to \mathbb{R}$ and $g : U \to \mathbb{R}$ are differentiable at x_0. Using the identity

$$\frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} = \frac{f(x) - f(x_0)}{x - x_0} g(x) + f(x_0) \frac{g(x) - g(x_0)}{x - x_0},$$

prove that $f(x)g(x)$ is differentiable at x_0 and $(f(x_0)g(x_0))' = f'(x_0)g(x_0) + f(x_0)g'(x_0)$. *Clearly explain* how the result of Part *a)* is used in the proof.

4. *a)* State the Mean-Value Theorem of differentiation without proof.
   b) Let $U \subset \mathbb{R}$ be an open set, and let $a \in U$. Let f be a function differentiable at a such that $f(a) \ge f(x)$ for all $x \in U$. Prove that $f'(a) = 0$.

5. *a)* Let A, a, and b be real numbers, and assume $a < b$. Let function f be a real-valued function on $[a, b]$. Define what it means for the number A to be the Riemann integral of f on the interval $[a, b]$. In your definition, you can take the following concepts granted (i.e., you do not have to explain or define them): partition, tagged partition, width of a partition, Riemann sum associated with a (tagged) partition.
   b) Let $a, b \in \mathbb{R}$ with $a < b$, and let $c \in [a, b]$. Assume $f(x) = 0$ for all $x \in [a, b]$ with $x \neq c$, and $f(c) = 1$. Prove that f is integrable on $[a, b]$ and $\int_a^b f = 0$.