Exam 3, Mathematics 4201, Section TR11 28 copies
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Instructor: Attila Máté[1]

1. *a)* State the second criterion (the one involving step functions) of Riemann integrability.

The Second Criterion of Riemann Integrability. *The real-valued function f on the interval $[a,b]$ is Riemann-integrable if and only if for every $\epsilon > 0$ there are step functions f_1 and f_2 on the interval $[a,b]$ such that $f_1(x) \le f(x) \le f_2(x)$ whenever $x \in [a,b]$ and*

$$\int_a^b (f_2 - f_1) < \epsilon.$$

b) Let f be a continuous function on $[a,b]$ $(a < b)$. Prove that f is Riemann integrable on $[a,b]$.

Solution. As f is continuous on the closed interval $[a,b]$, it follows that f is uniformly continuous on $[a,b]$.[2] Let $\epsilon > 0$ be arbitrary, and let $\delta > 0$ such that for any $x, y \in [a,b]$ we have

$$(1) \qquad |f(x) - f(y)| < \frac{\epsilon}{2(b-a)} \quad \text{whenever} \quad |x - y| < \delta.$$

Let

$$P : a = x_0 < x_1 < \ldots < x_N = b$$

be a partition of $[a,b]$ of width $< \delta$. Let

$$m_i = \min\{f(x) : x \in [x_{i-1}, x_i]\} \qquad \text{and} \qquad M_i = \max\{f(x) : x \in [x_{i-1}, x_i]\}$$

for i with $1 \le i \le N$.[3] Observe that

$$(2) \qquad M_i - m_i < \frac{\epsilon}{2(b-a)} \qquad \text{for each } i \text{ with } 1 \le i \le N.$$

This is because, given i, $m_i = f(\xi)$ and $M_i = f(\eta)$ for some $\xi, \eta \in [x_{i-1}, x_i]$. Then $|\eta - \xi| \le x_i - x_{i-1} < \delta$, and so $M_i - m_i = f(\eta) - f(\xi) < \epsilon/(2(b-a))$ according to (1), as claimed.

For $x \in [a,b]$, let

$$f_1(x) = \begin{cases} m_i & \text{if } x \in (x_{i-1}, x_i) \text{ for some } i \text{ with } 1 \le i \le N, \\ f(x) & \text{if } x = x_i \text{ for some } i \text{ with } 0 \le i \le N \end{cases}$$

(note that the ranges of i are different in the two cases), and

$$f_2(x) = \begin{cases} M_i & \text{if } x \in (x_{i-1}, x_i) \text{ for some } i \text{ with } 1 \le i \le N, \\ f(x) & \text{if } x = x_i \text{ for some } i \text{ with } 0 \le i \le N. \end{cases}$$

Then $f_1(x)$ and $f_2(x)$ are step functions such that $f_1(x) \le f(x) \le f_2(x)$ for all $x \in [a,b]$. It follows from (2) that $f_2(x) - f_1(x) < \epsilon/(2(b-a))$ for all $x \in [a,b]$ Hence,

$$\int_a^b (f_2 - f_1) \le \int_a^b \frac{\epsilon}{2(b-a)}\, dx \le \frac{\epsilon}{2} < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, the step functions f_1 and f_2 witness the integrability of f, according to the Second Criterion of Riemann Integrability. The proof is complete.

2. *a)* State the Fundamental Theorem of Calculus. Make sure that the conditions are precisely stated.

[1] All computer processing for this manuscript was done under Fedora Linux. $\mathcal{AMS}$-TEX was used for typesetting. PICTEX was used for the diagrams, together with the programming language Perl used for generating the data points.

[2] A closed interval on the real line is compact, and a continuous function on a compact space is uniformly continuous.

[3] The definitions of m_i and M_i are meaningful, since a continuous function on a closed interval has both a minimum and a maximum, according to the Maximum Value Theorem.

The Fundamental Theorem of Calculus. *Let U be an open interval of $\mathbb{R}$, and let f be a continuous function on U. Further, let $a \in U$. Then, for every $x \in U$, we have*

$$\frac{d}{dx}\int_a^x f = f(x).$$

The existence of the integral in the theorem follows from the fact that continuous functions are integrable.

b) Prove the Fundamental Theorem of Calculus.

Solution. For $x \in U$ write $F(x) = \int_a^x f$. Let $x_0 \in U$ be arbitrary. We need to show that

$$F'(x_0) = \lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0). \tag{1}$$

Let $\epsilon > 0$ be arbitrary, and using the continuity of f at x_0, let $\delta > 0$ be such that $(x_0 - \delta, x_0 + \delta) \subset U$, and for any $x \in (x_0 - \delta, x_0 + \delta) \subset U$ we have $|f(x) - f(x_0)| < \epsilon/2$. Let $x \in (x_0 - \delta, x_0 + \delta) \subset U$ be such that $x \neq x_0$. We have

$$\begin{aligned}
&\left|\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right| = \left|\frac{1}{x - x_0}\left(\int_a^x f(t)\,dt - \int_a^{x_0} f(t)\,dt\right) - f(x_0)\right| \\
&\quad = \left|\frac{1}{x - x_0}\int_{x_0}^x f(t)\,dt - \frac{1}{x - x_0}\int_{x_0}^x f(x_0)\,dt\right| = \left|\frac{1}{x - x_0}\int_{x_0}^x \big(f(t) - f(x_0)\big)\,dt\right| \\
&\quad \leq \frac{1}{|x - x_0|}\int_{x_0}^x \frac{\epsilon}{2}\,dt \leq \frac{\epsilon}{2} < \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, (1) follows.

3. *a*) State Cauchy's criterion for the series $\sum_{n=1}^\infty a_n$ of real numbers to be convergent.

Cauchy's Convergence Criterion for Series. *The series $\sum_{n=1}^\infty a_n$ of real numbers converges if and only if for every $\epsilon > 0$ there is an N such that*

$$\left|\sum_{k=m+1}^{n} a_k\right| < \epsilon$$

whenever $n > m \geq N$.

Formally

$$\sum_{n=1}^\infty a_n \text{ converges } \leftrightarrow (\forall \epsilon > 0)\,(\exists N)\,(\forall m)\,(\forall n)\,\left(n > m \geq N \to \left|\sum_{k=m+1}^{n} a_k\right| < \epsilon\right)$$

b) Prove that the harmonic series $\sum_{n=1}^\infty 1/n$ is divergent. Use material covered in the course (that is, *do not* use the Integral Test for convergence).

Proof. We will show that the harmonic series does not satisfy the condition in Cauchy's Convergence Criterion. Let N be arbitrary, and let j be a nonnegative integer such that $m \geq N$. We have

$$\sum_{n=m+1}^{2m} \frac{1}{n} \geq \sum_{n=m+1}^{2m} \frac{1}{2m} = m \cdot \frac{1}{2m} = \frac{1}{2}.$$

Thus, Cauchy's convergence Criterion fails with $\epsilon = 1/2$ for the harmonic series, showing that the harmonic series diverges.

c) Assume that $\sum_{n=1}^\infty a_n$ of real numbers is absolutely convergent. Prove that it is also convergent.

Proof. Assume that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent; that is, that $\sum_{n=1}^{\infty} |a_n|$ is convergent. By Cauchy's Criterion, then, given an arbitrary $\epsilon > 0$, there is an N such that

$$\Big| \sum_{k=m+1}^{n} |a_k| \Big| < \epsilon$$

holds whenever $n > m \geq N$; the absolute value signs on the outside on the left-hand side have of course no effect, and they can be omitted. Thus

$$\Big| \sum_{k=m+1}^{n} a_k \Big| \leq \sum_{k=m+1}^{n} |a_k| < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, the inequality here between the extreme left and the extreme right implies that $\sum_{n=1}^{\infty} a_n$ is convergent in view of Cauchy's Convergence Criterion.

Instead of proving this result directly from the Cauchy Convergence Criterion, one can first establish the Comparison Test as an easy consequence of the Cauchy Convergence Criterion, and then use the Comparison Test to prove the result:

Comparison Test. *Let $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ be infinite series of real numbers such that $|c_n| \leq b_n$ for all positive integers n. If $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} c_n$ also converges.*

Using this with $b_n = |a_n|$ and $c_n = a_n$, we can see that if the series $\sum_{n=1}^{\infty} |a_n|$ converges, then the series $\sum_{n=1}^{\infty} a_n$ also converges.

4. *a*) State the theorem about the rearrangement of absolutely convergent series.

Theorem. *Let $\sum_{i=1}^{\infty}$ be absolutely convergent, and let $f : \mathbb{N} \to \mathbb{N}$ be one-to-one and onto $\mathbb{N}$, where $\mathbb{N} = \{1, 2, 3, \dots\}$. Then $\sum_{n=1}^{\infty} a_{f(n)}$ is absolutely convergent, and*

$$\sum_{n=1}^{\infty} a_{f(n)} = \sum_{n=1}^{\infty} a_n.$$

b) In proving the theorem described in part *a*), for a one-to-one function $f : \mathbb{N} \to \mathbb{N}$ onto $\mathbb{N}$, where $\mathbb{N} = \{1, 2, 3, \dots\}$, given a positive integer N, we start out with the equation

$$\sum_{n=1}^{N} a_{f(n)} - \sum_{n=1}^{N} a_n = \sum_{i \in S_1} a_i - \sum_{i \in S_2} a_i,$$

valid with certain sets $S_1, S_2 \subset \mathbb{N}$ such that $S_1 \cap S_2 = \emptyset$. Describe the sets S_1 and S_2.

Solution. We have

$$S_1 = \{f(i) : i \in \mathbb{N} \ \ \& \ \ 1 \leq i \leq N\} \setminus \{i \in \mathbb{N} : 1 \leq i \leq N\}$$

and

$$S_2 = \{(i \in \mathbb{N} : 1 \leq i \leq N\} \setminus \{f(i) : i \in \mathbb{N} \ \ \& \ \ 1 \leq i \leq N\}.$$

5. Let $\xi \neq 0$ be such that $\sum_{n=0}^{\infty} c_n \xi^n$ is convergent, and let x be such that $|x| < |\xi|$. Prove that $\sum_{n=0}^{\infty} c_n x^n$ is convergent.

Solution. As $\sum_{n=0}^{\infty} c_n \xi^n$ is convergent, we must have $\lim_{n\to\infty} c_n \xi^n = 0$. Since a convergent sequence is bounded, there is a real number M such that $|c_n \xi^n| \leq M$ for all $n \geq 0$. We then have

$$|c_n x^n| \leq |c_n \xi^n| \cdot \left| \frac{x}{\xi} \right|^n \leq M \left| \frac{x}{\xi} \right|^n$$

for all $n \geq 0$. As $|x/\xi| < 1$, the series $\sum_{n=0}^{\infty} M |x/\xi|^n$ is a convergent geometric series. Hence $\sum_{n=0}^{\infty} c_n x^n$ is convergent by the Comparison Test.