SUPPLEMENTARY NOTES
ON INTRODUCTION TO ANALYSIS

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CONTENTS

Preface

These notes started as a set of handouts to the students while teaching a course on introductory analysis in the spring of 2002 at Brooklyn College of the City University of New York, using Rosenlicht's book [Ros].[1] While I find this book really excellent, there were some areas where I wanted to give an emphasis to the material different from that given in the book; for example, this being single-variable analysis, I wanted to give more emphasis to the real line than to abstract metric spaces. Hence the supplementary material enclosed here. The first draft of these notes were completed in April 2002. The date of the present revision is January 2013.

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[1]See the bibliography at the end of these notes for the full title and publication data.

1. Interchange of quantifiers

There are two kinds of quantifiers: universal: $\forall$, meaning "for all," and existential: $\exists$, meaning " there is" or "there exists." Within a quantifier, one may specify the kind of things the quantifier talks about, e.g.,

$$(\forall x : x \text{ is an integer}), \quad (\exists y : y \text{ is a real number}), \quad (\forall x \in \mathbb{Z}), \quad (\exists y \in \mathbb{R}),$$

etc.; here $\mathbb{Z}$ stands for the set of integers (positive, negative, or zero), and $\mathbb{R}$ stands for the set of reals. Two quantifiers of the same kinds mean either two universal quantifiers or two existential quantifiers, while two quantifiers of different kinds refer to one universal and one existential quantifier. Two quantifiers of the same kind are always interchangeable, but two quantifiers of different kinds are not. To see this, consider the following example:

$$(\forall x : x \text{ is licensed driver})(\exists y : y \text{ is a car}) \ (x \text{ has driven } y).$$

This sentence is entirely reasonable, since it only says that "every licensed driver has driven a car," or, more precisely, "every licensed driver has driven at least one car." In real life, there are exceptions even to such a reasonable statements, but if you restrict your attention to life in a small town, the statement is most likely true. If one interchanges the quantifiers, the result is totally absurd:

$$(\exists y : y \text{ is a car}) \ (\forall x : x \text{ is licensed driver}) \ (x \text{ has driven } y).$$

This says that "there is a single car that every licensed driver has driven," and this is unlikely to be true even in a small town, unless everyone has gone to the same driving school within the last few years, and got a chance of practising on the same car.

Another, more mathematical, example is the following. Let x and y run over integers. Then the formula

$$(\forall x)(\exists y)[x > y]$$

is true, while the formula

$$(\exists y)(\forall x)[x > y]$$

is false. Indeed, the first formula is true. Given an arbitrary integer x, we can pick $y = x - 1$ to ensure that $x > y$. On the other hand, the second formula is not true. To see this, pick an arbitrary number y. Then $(\forall x)[x > y]$ is certainly not true; for example, $x > y$ is not true with $x = y - 1$.

2. More on logic

A sentence is a statement that is either true or false.[2] Sentences can be connected by the logic operations, also called sentential connectives *and*, *or*, *if … then*, and *if and only if*, denoted in turn by $\&$, $\vee$, $\to$, and $\leftrightarrow$. Instead of $\&$, people often use the symbol $\wedge$. The logic operations only connect sentences (or, more generally, statements – see below). For

[2] That is, true or false in principle. If one talks about formal logic, then a sentence would be a declarative sentence. Some declarative sentences, however, cannot be considered sentences in the sense of logic, since they cannot be considered true or false even in principle. For example, the sentence "This sentence is false" cannot be taken as true (since it then says that it is false), and it cannot be taken as false (since then it says that it is true). In mathematical logic, what is a sentence or a statement is defined formally by describing the rules how a statement can be formed. The collection of these rules is called *syntax*.

example, the "or" in the sentence "You can have coffee or tea with your breakfast" is not a logic operation.[3]

The meaning of the sentential connectives can be illustrated by truth tables. Writing T for true and F for false (T and F are called *truth values*) we have for the operation $\&$, called *conjunction*:

A	B	$A \,\&\, B$
T	T	T
T	F	F
F	T	F
F	F	F

For the operation $\vee$, called *disjunction*, we have

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

As seen from the truth table, disjunction is always meant in the *inclusive* sense, that is $A \vee B$ is true unless both A and B are false. This differs from colloquial usage, where one often uses the word "or" in the exclusive sense, where "A or B in the exclusive sense" is true if exactly one of A and B is true.

For the operation $\to$, called *conditional*, we have

A	B	$A \to B$
T	T	T
T	F	F
F	T	T
F	F	T

The meaning of the conditional is often differs from its colloquial use, where the meaning of "if A then B" is unclear in case A is false. In mathematics, one strictly follows the truth table above. For example, the sentence "if 2 by 2 is 5 then the snow is black" is a true sentence in mathematics.[4] In colloquial speech one would consider this sentence meaningless, or at best pointless. But it illustrates an important point: in the conditional, there does not need to be a causal connection between the constituents.[5] For the operation $A \to B$, instead of saying "if A then B", it is often more convenient to say "A only if B". In case of this latter sentence, the colloquial meaning approaches more closely the mathematical meaning of the conditional. Namely, "A only B" means that A is allowed to be true only in case B is also true; indeed, when A is true and B is false, the truth table entry for $A \to B$ shows false.

One occasionally reverses the arrow in the conditional, using the symbol $A \leftarrow B$ meaning

[3]However, on can rewrite this as a logic operation by saying that "You can have coffee with your breakfast or you can have tea with your sentence" (although by doing this, one probably changes the meaning, since in logic, "or" is meant in the inclusive sense (allowing to have both coffee and tea), while on a restaurant menu the meaning is probably exclusive (not allowing to have both coffee and tea without paying extra).

[4]This example is due to the German mathematician David Hilbert. Note that the sentence "if 2 by 2 is 5 then the snow is white" is also true.

[5]That is, 2 by 2 being 5 does not cause the snow to be black. As for the constituents, or, with a more technical word, *operands*, in the conditional $A \to B$, A called the antecedent and B is called the consequence (the latter is a somewhat misleading name, since the name seems to imply a causal connection).

"A if B", or "if B then A":

A	B	$A \leftarrow B$
T	T	T
T	F	T
F	T	F
F	F	T

The conditional $A \leftarrow B$ (or $B \rightarrow A$) is usually called the *converse* of $A \rightarrow B$. Finally, the truth table of the operation $\leftrightarrow$, called *biconditional*, is

A	B	$A \leftrightarrow B$
T	T	T
T	F	F
F	T	F
F	F	T

$A \leftrightarrow B$ is expressed as "A if and only if B", or, sometimes, as 'A iff B" (but it is not clear how one should pronounce the word "iff"). The word "iff" was introduced by Paul Halmos. "A if and only if B" is short for saying "A if B and A only if B"; formally, for $(A \leftarrow B)\ \&\ (A \rightarrow B)$. It is easy to check that the truth table for this formula is the same as the one given for the biconditional above.

These logic operations are called *binary* operations, since they involve two constituents, called *operands*, A and B in the above truth tables. The letters A and B used in the above formulas are often called *sentential variables*, i.e., variables that can either be true or false.

Negation. The operation "not" is called *negation*. "Not A means that "it is not the case that A", or, more simply, "A is not true". This is called a *unary* operation, since it has only one operand. The truth table for negation is

A	$\neg A$
T	F
F	T

Tautologies. A tautology is a logic expression (an expression involving the logical operations just defined) that is always true, whether or not the sentential variables in it are true or false. Examples for tautologies are

$$(\neg(A\ \&\ B)) \leftrightarrow ((\neg A) \vee (\neg B)).$$

For better readability, one can drop several pairs of parentheses here, to write

$$\neg(A\ \&\ B) \leftrightarrow \neg A \vee \neg B.$$

To make sense of this way of writing the formula, one can assigns priority to the logic operations in the order ,/ & , $\vee$, $\leftarrow$, $\rightarrow$, $\leftrightarrow$, meaning that one first try to perform the operations with higher priority.[6] Most people consider the priority between & and $\vee$, and between $\leftarrow$ and $\rightarrow$ unclear, so it is best to use parentheses to avoid misunderstanding. To check that the above formula is indeed a tautology, one can use the truth table to evaluate

[6] This is similar to the rule in algebra that $\cdot$ (multiplication) has higher priority than $+$ (addition). That is, the formula $2 + 3 \cdot 5$ means $2 + (3 \cdot 5)$, and not $(2 + 3) \cdot 5$. $+$ and $-$ have equal priority, so in the expression $2 + 3 - 5 - 6 + 8$ one performs the operations from left to write, i.e., this expression means $(((2+3)-5)-6)+8$ In computer science, one often uses the word *precedence* instead of priority; computer scientists unambiguously assign higher precedence to to logical and than to logical or; in mathematics, this precedence is not always taken for granted.

it for each choice of A and B to find that the formula is always true. One interpretation of the above tautology is that $\neg(A \ \& \ B)$ and $\neg A \lor \neg B$ mean the same thing. This is because the biconditional is true exactly when the two sides have the same truth value; so the above formula being always true means that the two sides on the biconditional in it always have the same truth value. The above formula is one of the two *De Morgan identities.* The other De Morgan identity is the tautology

$$\neg(A \lor B) \leftrightarrow \neg A \ \& \ \neg B.$$

Another simple tautologies are

$$(A \to B) \leftrightarrow \neg A \lor B;$$

we used parentheses on the left here, since the sometimes $\to$ and $\leftrightarrow$ is considered to have equal priority. When one wants to establish the *implication*[7] $A \to B$, one often takes advantage of the tautology

$$(A \to B) \leftrightarrow (\neg B \to \neg A),$$

and proves the implication $\neg B \to \neg A$ instead. The conditional $\neg B \to \neg A$ is called the *contrapositive* of $A \to B$. A further tautology is

$$\neg(A \leftrightarrow B) \leftrightarrow (\neg A \leftrightarrow B).$$

One often writes $A \not\leftrightarrow B$ instead of $\neg(A \leftrightarrow B)$. The truth table of $A \not\leftrightarrow B$ is

A	B	$A \not\leftrightarrow B$
T	T	F
T	F	T
F	T	T
F	F	F

One might call the operation $A \not\leftrightarrow B$ *exclusive or*, since it reflects the meaning the word "or" often used colloquially. However, it is best to avoid this term "exclusive or", since in mathematics the word "or" is always used in the inclusive sense, as defined by the logic operation $A \lor B$. The expression $A \not\leftrightarrow B$ in mathematics is often given as "A if and only if not B", reflecting the fact that this this expression is equivalent to (i.e., true exactly the same time as) the expression $A \leftrightarrow \neg B$; that is, the fact that

$$(A \not\leftrightarrow B) \leftrightarrow (A \leftrightarrow \neg B)$$

is a tautology.

Open statements. An *open statement* is a statement that has (zero or more) variables in it; when one gives values to these variables. For example, to say that "x is greater than 2" (where x denotes an unspecified real number) is on open statement. One can tell its truth value only after one specifies what real number x is. In mathematical logic, one describes the rules how open statements are formed; the collection of these rules are called *syntax.* However, want to discuss matters somewhat informally, so we will avoid a detailed discussion of syntax. Occasionally, open statements will be denoted by script capital letters such as $\mathcal{A}$ or $\mathcal{B}$.

[7] One often calls a conditional an implication, especially in informal usage, since instead of "if A then B" one might say "A implies B". However, logical implication has a meaning separate from, though related to, that of the conditional. Therefore, some consider calling a conditional an implication objectionable.

Negating quantified statements. The formula $\neg(\forall x)\mathcal{A}$ is the same as $(\exists x)\neg\mathcal{A}$; here $\mathcal{A}$ is some open statement.[8] That is, saying that "it is not true that for all x $\mathcal{A}$ holds" means that "there is an x for which A does not hold". One writes this by saying that

$$\neg(\forall x) \equiv (\exists x)\neg;$$

here $\equiv$ means that what are written on the two sides are equivalent, i.e., they mean the same thing, i.e. they can replace each other.[9] Similarly, the formula $\neg(\exists x)\mathcal{A}$ is the same as $(\forall x)\neg\mathcal{A}$. That is, to say that "it is not true that there exists an x for which $\mathcal{A}$ holds" means that "for all x it is not true that $\mathcal{A}$ holds". This can be expressed by the rule

$$\neg(\exists x) \equiv (\forall x)\neg.$$

One can use these rules and the tautologies mentioned above to move negation inside a formula. For example,

$$\neg(\forall\epsilon>0)(\exists\delta>0)(\forall p\in(0,1))(\forall q\in(0,1))\left(|p-q|<\delta\to\left|\frac{1}{p}-\frac{1}{q}\right|<\epsilon\right). \tag{1}$$

Here p, q, ϵ, and δ denote real numbers,[10] and $(0,1)$ denotes the interval $\{t : 0 < t < 1\}$. What this formula expresses is irrelevant for our present purpose.[11] The quantifiers $(\forall\epsilon > 0)$ is called a *restricted quantifier*, since it says that ϵ ranges over positive real numbers instead of all real numbers (where unrestricted variables range in the present case). Similarly, $(\forall\delta > 0)$, $(\exists p \in (0, 1)$ are restricted quantifiers. The important point at present is that the above rules of interchanging negation and quantifiers are also true for restricted quantifiers.

The above formula can be transformed as follows:

$$\begin{aligned}
&\neg(\forall\epsilon>0)(\exists\delta>0)(\forall p\in(0,1))(\forall q\in(0,1))\left(|p-q|<\delta\to\left|\frac{1}{p}-\frac{1}{q}\right|<\epsilon\right)\\
&\quad\equiv\neg(\forall\epsilon>0)(\exists\delta>0)(\forall p\in(0,1))(\forall q\in(0,1))\left(|p-q|<\delta\to\left|\frac{1}{p}-\frac{1}{q}\right|<\epsilon\right)\\
&\quad\equiv(\exists\epsilon>0)\neg(\exists\delta>0)(\forall p\in(0,1))(\forall q\in(0,1))\left(|p-q|<\delta\to\left|\frac{1}{p}-\frac{1}{q}\right|<\epsilon\right)\\
&\quad\equiv(\exists\epsilon>0)(\forall\delta>0)\neg(\forall p\in(0,1))(\forall q\in(0,1))\left(|p-q|<\delta\to\left|\frac{1}{p}-\frac{1}{q}\right|<\epsilon\right)\\
&\quad\equiv(\exists\epsilon>0)(\forall\delta>0)(\exists p\in(0,1))\neg(\forall q\in(0,1))\left(|p-q|<\delta\to\left|\frac{1}{p}-\frac{1}{q}\right|<\epsilon\right)\\
&\quad\equiv(\exists\epsilon>0)(\forall\delta>0)(\exists p\in(0,1))(\exists q\in(0,1))\neg\left(|p-q|<\delta\to\left|\frac{1}{p}-\frac{1}{q}\right|<\epsilon\right)\\
&\quad\equiv(\exists\epsilon>0)(\forall\delta>0)(\exists p\in(0,1))(\exists q\in(0,1))\left(|p-q|<\delta\ \ \&\ \ \neg\left|\frac{1}{p}-\frac{1}{q}\right|<\epsilon\right)\\
&\quad\equiv(\exists\epsilon>0)(\forall\delta>0)(\exists p\in(0,1))(\exists q\in(0,1))\left(|p-q|<\delta\ \ \&\ \ \left|\frac{1}{p}-\frac{1}{q}\right|\ge\epsilon\right).
\end{aligned}$$

[8] Presumably, $\mathcal{A}$ depends on x, so one might be tempted to write $\mathcal{A}(x)$ instead. However, writing $\mathcal{A}(x)$ really does not make things clearer.

[9] We do not regard $\equiv$ as a logical connective here. It can be considered as a rule according to which we can change (or transform) formulas without changing their meanings. That is, it expresses a transformation rule; see below.

[10] One also says that these variables *range over* real numbers.

[11] It expresses the statement that the function $f(x) = 1/x$ is not uniformly continuous in the interval $(0, 1)$, discussed below in these notes.

Restricted quantifiers. If E is a set, and $\mathcal{A}$ is an open statement, quantifiers of the form $(\forall x \in E)$ and $(\exists x \in E)$ are called *restricted quantifiers.* More generally, given an open statement $\mathcal{A}$, one can consider the restricted quantifiers ($\forall x : \mathcal{A}$ and $\exists x : \mathcal{A}$). If $\mathcal{B}$ is another open statement, then the formula $(\forall x : \mathcal{A})\mathcal{B}$ says that "for all x such that $\mathcal{A}$ holds we (also) have $\mathcal{B}$, and for $(\exists x : \mathcal{A})\mathcal{B}$ means that "for all x for which $\mathcal{A}$ holds we also have $\mathcal{B}$". It is easy to see that

$$(\forall x : \mathcal{A})\mathcal{B} \equiv (\forall x)(\mathcal{A} \to \mathcal{B}),$$

and

$$(\exists x : \mathcal{A})\mathcal{B} \equiv (\exists x)(\mathcal{A} \ \ \& \ \ \mathcal{B}).$$

Therefore, restricted quantifiers are not strictly necessary, but they are often convenient to use. They frequently make formulas simpler, and, a very important point, the rules discussed above involving the interchange of negation and quantifiers are also true for restricted quantifiers. Finally, for a set E,

$$(\forall x \in E) \equiv (\forall x : x \in E)$$

and

$$(\exists x \in E) \equiv (\exists x : x \in E).$$

3. The Axiom of Completeness

A *cut* is a pair (A, B) such that A and B are nonempty subsets of the set $\mathbb{R}$ of real numbers with $A \cup B = \mathbb{R}$, and such that for every $x \in A$ and every $y \in B$ we have $x < y$. If (A, B) is a cut, it is clear that the sets A and B must be disjoint (since every element of A is less than every element of B). For a cut (A, B) and a real number t we say that the cut *determines* the number t if for every $x \in A$ we have $x \leq t$ and for every $y \in B$ we have $t \leq y$.

For example, a cut that determines the number 2 is the pair (A, B) with $A = \{t : t \leq 2\}$ and $B = \{t : t > 2\}$. Another cut that determines the number 2 is the pair (C, D) with $C = \{t : t < 2\}$ and $D = \{t : t \geq 2\}$.

It is clear that a cut cannot be determine more than one number. Assume, on the contrary, that the cut (A, B) determines the numbers t_1 and t_2 and $t_1 \neq t_2$. Without loss of generality, we may assume that $t_1 < t_2$. Then the number $c = (t_1 + t_2)/2$ cannot belong to A, since we cannot have $t_1 < x$ for any element of A, whereas $t_1 < c$. Similarly $c = (t_1 + t_2)/2$ cannot be an element of B, since we cannot have $y < t_2$ for any element of B, whereas $c < t_2$.

Furthermore, the above example is of a general character, in that for every real number u there are exactly two cuts, (A, B) and (C, D) that determine u, where $A = \{t : t \leq u\}$ and $B = \{t : t > u\}$. $C = \{t : t < u\}$ and $D = \{t : t \geq u\}$. The Axiom of Completeness is an important property of real numbers:

Axiom of Completeness. *Every cut determines a real number.*

Ordinarily, one does not expect to prove this statement, since axioms are basic statements that one does not prove. However, one can prove the Axiom of Completeness if one defines the real numbers as infinite decimals.[12]

[12] However, defining the reals numbers as infinite decimals is inelegant in that the decimal number system came about only by a historical accident. One could also define the reals, however, as infinite binary fractions; i.e., "decimal" fractions in the binary number system. The binary number system is somewhat more natural than the decimal system in that it is the simplest of all number systems.

Proof of the Axiom of Completeness. Let (A, B) be a cut. Assume first that A contains a positive number. Let n be a positive integer, and let $x^{(n)} = x_0^{(n)}.x_1^{(n)}x_2^{(n)}\dots x_n^{(n)}$ be the largest decimal fraction with n digits after the decimal point such that $x^{(n)}$ is less than or equal to every element of B; here $x_0^{(n)}$ is a nonnegative integer, and, for each i with $1 \le i \le n$, $x_i^{(n)}$ is a digit, i.e., one of the numbers 0, 1, ..., 9.[13] There is such an $x^{(n)}$, since the set of decimal fractions with n digits after the decimal point that are nonnegative and less than or equal to every element of B is finite.[14]

It is easy to see that if m and n are positive integers and $m < n$ then for each i with $0 \le i \le m$ we have $x_i^{(m)} = x_i^{(n)}$. Indeed, we cannot have $x_i^{(m)} < x_i^{(n)}$. for any i with $0 \le i \le m$ since then the number $x' = x_0^{(n)}.x_1^{(n)}x_2^{(n)}\dots x_m^{(n)}$ would be an m-digit decimal greater than $x^{(m)}$ and less than or equal to every element of B. Furthermore, we cannot have $x_i^{(m)} > x_i^{(n)}$. for any i with $0 \le i \le m$, since then, with this i, the number $x'' = x_0^{(m)}.x_1^{(m)}x_2^{(n)}\dots x_i^{(m)}00\dots0$ ($n - i$ zeros at the end) would be an n-digit decimal greater than $x^{(n)}$ and less than or equal to every element of B.

Now, the infinite decimal $r = x_0^{(0)}.x_1^{(1)}x_2^{(2)}\dots x_n^{(n)}\dots$ is the number determined by cut (A, B). Indeed, $r \ge x$ for every $x \in A$. Assume, on the contrary, that $r < x$ and let n be such that $10^{-n} < x - r$. Then noting that $r - x^{(n)} < 10^{-n}$, the decimal number $x^{(n)} + 10^{-n} \le r + 10^{-n} < x$ would be less than every element of B (since x is less than every element of B), contradiction the choice that $x^{(n)}$ was the largest such n-digit decimal fraction. Furthermore, $r \le y$ for every $y \in B$. Assume, on the contrary that $r > y$ for some $y \in B$, and let n be such that $10^{-n} < r - y$. Then noting that $r - x^{(n)} < 10^{-n}$, we have $x^{(n)} > r - 10^{-n} > y$, contradiction the assumption that $x^{(n)} \le y$ for every $y \in B$. The proof is complete in case the cut (A, B) is such that A contains a positive number.

If (A, B) is a cut such that B contains the negative number, writing $-A = \{-x : x \in A\}$ $-B = \{-y : y \in B\}$, the cut $(-B, -A)$ is such that $-B$ contains a positive number; hence, by the above argument, this cut determines a number r. Then the cut (A, B) determines the number $-r$.

Finally, if the cut (A, B) is such that A does not contain a positive number and B does not contain a negative number, then it is clear that the cut (A, B) determines the number 0. The proof is complete.

The Axiom of Completeness guarantees, for example, that the number $\sqrt{2}$ exists. Namely, the cut (A, B) with $A = \{x : x < 0 \text{ or } x^2 \le 2\}$ and $B = \{x : x > 0 \text{ and } x^2 > 2\}$ and determines the number t such that $t^2 = 2$.

To show this, we will first show that for every $\epsilon > 0$ we have $|t^2 - 2| < \epsilon$. In order to show this, we may assume that $\epsilon < 1$. First note note that there is an $x \in A$ and a $y \in B$ with $x \ge 0$ and $y \ge 0$ (the latter holds for every $y \in B$) such that $y - x \le \epsilon/6$. To see this, consider the set

$$S = \{n\epsilon/6 : n \ge 0 \text{ is an integer} \quad \text{and} \quad n\epsilon/6 \in A\}.$$

S is not empty, since $0 \in S$. Furthermore, it is clearly a finite set, and so it has a largest element. Now, choose x to be the largest element of this set and put $y = x + \epsilon/6$ (clearly, $y \in B$, since if we had $y \notin B$ then we would have $y \in A$, and so $y \in S$, and then x would not be the largest element of x.

Observe that $x < 2$ (because $x \in A$, and so $x^2 \le 2$ unless $x < 0$ by the definition of A) and $y = x + \epsilon/6 \le 2 + 1/6 < 3$ (this is where we used the assumption $\epsilon < 1$). Furthermore, $x \le t \le y$, since t is the number determined by the cut (A, B). We have

$$t^2 - 2 \le t^2 - x^2 = (t - x)(t + x) \le (y - x)(y + x) < \frac{\epsilon}{6} \cdot 5 < \epsilon;$$

[13] i.e., if $x^{(n)} = 345.84$ then $n = 2$, $x_0^{(n)} = 345$, $x_1^{(n)} = 8$, and $x_2^{(n)} = 4$.

[14] The set of these decimal fractions is nonempty, since it contains the number 0. It is finite, because if b is an arbitrary element of B, there are at most $10^n \cdot b + 1$ n-digit nonnegative decimal fractions less than or equal to b.

the third inequality holds because we have $y - x \le \epsilon/6$, $x < 2$, and $y < 3$. Similarly,

$$2 - t^2 \le y^2 - t^2 = (y-t)(y+t) \le (y-x)(y+y) < \frac{\epsilon}{6} \cdot 6 = \epsilon.$$

These two inequalities together show that

$$|t^2 - 2| < \epsilon.$$

as claimed.

Since $\epsilon > 0$ was arbitrary, this inequality holds for every $\epsilon > 0$. Now, assume that $t^2 \ne 2$. Then the number $|t^2 - 2|$ is positive. Choosing $\epsilon = |t^2 - 2|$, the above inequality cannot hold. This is a contradiction, showing that we must have $t^2 = 2$.

Let S be a nonempty set of reals. A number t such that $t \ge x$ for every $x \in S$ is called an *upper bound* of S. The set A is called *bounded* from above if it has an upper bound. The number c that is an upper bound of S such that $c \le t$ for every upper bound t of S is called the *least upper bound* or *supremum* of S. The supremum of S is denoted as $\sup S$.

LEMMA. *Let S be a nonempty set S of reals that is bounded from above. Then S has a supremum.*

PROOF. Let B be the set of upper bounds of S, and let A be the set of those reals that are not upper bounds of S (i.e., $A = \mathbb{R} \setminus B$). Then it is clear that (A, B) is a cut. Indeed, if we have $x \ge y$ and $y \in B$ for the reals x and y, then x is also an upper bound of S (since it is at greater than or equal to another upper bound, namely y). This shows that for every $x \in A$ and $y \in B$ we must have $x < y$. A is not empty since S is not empty, so not every real is an upper bound of S; B is not empty, since S is bounded from above by assumption.

Let t be the real determined by the cut (A, B). Then t is an upper bound of S. Assume, on the contrary, that there is an $s \in S$ is such that $s > t$. Let $u = (s+t)/2$. Then $u > t$, so we must have $u \in B$ (since the cut (A, B) determines t). Yet $u < s$, so u is not an upper bound of S, contradicting the relation $u \in B$.

Furthermore, t is the least upper bound of S. Assume, on the contrary, that y is an upper bound of S such that $y < t$. Then $y \in B$ (since B contains every upper bound of S). But then we must have $y \ge t$ (since the cut (A, B) determines t). This contradicts the relation $y < t$. The proof is complete.

For the empty set, one usually writes that $\sup \emptyset = -\infty$, and for a set S that is not bounded from above, one writes that $\sup S = +\infty$. With this extension, the symbol $\sup S$ will be meaningful for any subset of S reals.

Let S be a nonempty set of reals. A number t such that $t \le x$ for every $x \in S$ is called a *lower bound* of S. The set A is called *bounded* from below if it has an lower bound. The number c that is an lower bound of S such that $c \ge t$ for every lower bound t of S is called the *greatest lower bound* or *infimum* of S. The infimum of S is denoted as $\inf S$. Every nonempty set that is bounded from below has an infimum. The proof of this statement is similar to the proof of the Lemma above. Instead of carrying out this proof, one can argue more simply that $\inf S = -\sup(-S)$, where $-S \overset{def}{=} \{-s : s \in S\}$. One usually writes $\inf \emptyset = +\infty$, and if S is not bounded from below, then one writes $\inf S = -\infty$.

4. Supremum and limits

LEMMA. *Let S be a nonempty set of reals that is bounded from above, and let $a = \sup S$. Let $\epsilon > 0$. Then the interval $(a - \epsilon, a]$ contains an element of S.*

Proof. As $a - \epsilon$ is not an upper bound of S (a being its least upper bound), there must be an $x \in S$ with $x > a - \epsilon$. We have $x \le a$, since a is an upper bound of S. Thus $x \in S \cap (a - \epsilon, a]$.

Definition. A set S in the metric space (E, d) is said to be open if for every $p \in S$ there is an $\epsilon > 0$ such that the open ball $\{q : d(p, q) < \epsilon\}$ is a subset of S.

What is meant in the above definition is "if and only if" (that is, "(E, d) is said to be open *if and only if* for every ..."). In mathematical definitions, it is customary to say *if* in similar cases when one means *if and only if*. In other situations in mathematics, one makes a very careful distinction between "if" and "if and only if."

Definition. A set S in the metric space (E, d) is said to be closed if its complement $\complement S$ is open.

Lemma. *Let S be a nonempty closed set of reals that is bounded from above. Then S has a maximum. In other words, there is a $u \in S$ such that $u \ge s$ for every $s \in S$.*

Proof. As S is nonempty and bounded from above, it has a supremum; write $u = \sup S$. We will prove that $u \in S$; then it will be clear that $u = \max S$ (i.e. that $u \in S$ and $x \le u$ for every $x \in S$).

Assume, on the contrary, that $u \notin S$. Then $u \in \complement S$. As S is closed, its complement $\complement S$ is open; thus, there is an open ball $(u - \epsilon, u + \epsilon)$ with center u that is included in $\complement S$ ($\epsilon > 0$; in $\mathbb{R}$, the open balls are open intervals). Hence S has no elements in this interval; since u is an upper bound of S (i.e., $x \le u$ for every $x \in S$), this means that $u - \epsilon$ is an upper bound of S (i.e., $x \le u - \epsilon$ for every $x \in S$). This contradicts the assumption that u is the least upper bound of S. Therefore the assumption $u \notin S$ must be wrong, completing the proof.

Lemma. *Let S be a set in a metric space (E, d) is not closed. Then there is a point $p \notin S$ of E and a sequence $\{p_n\}_{n=1}^{\infty}$ of elements of S that converges to p.*

Proof. As S is not closed, its complement $\complement S$ is not open. Therefore, there is a point $p \in \complement S$ such that no open ball with center p is included in $\complement S$. That is, for every $\epsilon > 0$ the set

$$\{q : d(p, q) < \epsilon\} \cap S$$

is not empty (if it were, the open ball on the left would be included in $\complement S$). Using this statement with $\epsilon = 1/n$, for each positive integer n we can find a p_n in this set; that is $p_n \in S$ and $d(p, p_n) < 1/n$. From the latter inequality we can conclude that $\lim p_n = p$. Thus the sequence $\{p_n\}_{n=1}^{\infty}$ has the desired properties.

Lemma. *Assume $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence of reals that is bounded from above. Then $\{a_n\}_{n=1}^{\infty}$ is convergent.*

Proof. The set $S = \{a_n : 1 \le n < \infty\}$ is bounded from above, so it has a supremum. Let a be this supremum. We will show that $\lim a_n = a$.

To this end, let $\epsilon > 0$ be arbitrary. Since $a - \epsilon$ is not an upper bound of S, the set S has an element, say a_N for some positive integer N, for which $a_N > a - \epsilon$. As the sequence $\{a_n\}_{n=1}^{\infty}$ is increasing, we have $a_n \ge a_N$ for every $n \ge N$. As we also have $a_n \le a$ (since $a_n \in S$ and a is an upper bound of S), it follows that $a_n \in (a - \epsilon, a + \epsilon)$ whenever $n \ge N$. Hence the sequence $\{a_n\}_{n=1}^{\infty}$ converges to a.

Lemma. *Let a be such that $0 < a < 1$. Then $\lim_{n\to\infty} a^n = 0$.*

Proof. The assumptions imply that $0 < a^{n+1} < a^n$; thus the sequence $\{a^n\}_{n=1}^{\infty}$ is decreasing and bounded from below; therefore it has a limit. Write $x = \lim a^n$. Clearly, we also have $x = \lim a^{n+1}$. This can easily be verified directly. Alternatively, one may observe that $\{a^{n+1}\}_{n=1}^{\infty}$ is a subsequence[15] of $\{a^n\}_{n=1}^{\infty}$; hence, the former sequence must have the same limit as the latter. Now,

$$ax = a \cdot \lim a^n = \lim a \cdot a^n = \lim a^{n+1} = x,$$

i.e., $ax = x$, or $(a-1)x = 0$. As $a \neq 0$ by our assumptions, this is only possible if $x = 0$. Thus, $\lim a^n = 0$, which is what we wanted to prove.

Second proof. The assertion $\lim a^n = 0$ can also be proved by using Bernoulli's Inequality, saying that $(1+x)^n \geq 1 + nx$ holds whenever $x \geq -1$, for every positive integer n. For $x \geq 0$, Bernoulli's Inequality is a direct consequence of the Binomial Theorem. The case $-1 \leq x < 0$ of Bernoulli's Inequality is harder to establish, but this case is not needed for proving $\lim a^n = 0$.

Writing $x = \frac{1}{a} - 1$, we have $x \geq 0$ and $a = 1/(1+x)$, and so $(1/a)^n = (1+x)^n \geq 1 + nx$, i.e., $a^n \leq 1/(1+nx)$. Given $\epsilon > 0$, we will have $1/(1+nx) < 1/(nx) \leq \epsilon$ whenever $n \geq N = 1/(x\epsilon)$. For such n we will have $-\epsilon < a^n - 0 < \epsilon$, showing that $\lim a^n = 0$.

Lemma. *Let A and B be two nonempty sets of reals that are bounded from above. Then*

$$\sup\{x + y : x \in A \text{ and } y \in B\} = \sup A + \sup B.$$

Proof. Writing $S = \sup\{x + y : x \in A \text{ and } y \in B\}$, $a = \sup A$, and $b = \sup B$, we will first show that $\sup S \leq a + b$. To this end, let $x + y$ be an element of S, where $x \in A$ and $y \in B$. Then $x \leq a$ (since a is an upper bound of A; in fact, it is its least upper bound) and $y \leq b$ (since b is an upper bound of B). Thus $x + y \leq a + b$. As $x + y$ was an arbitrary element of S, this shows that $a + b$ is an upper bound of S. Therefore $\sup S \leq a + b$, since $\sup S$ is the *least* upper bound of S.

Next we will show that $\sup S \geq a+b$. We will do this by showing that no number $c < a+b$ is an upper bound of S. So, let $c < a + b$ be arbitrary and write $\epsilon = (a + b - c)/2$; then $\epsilon > 0$. Now $a - \epsilon$ is not an upper bound of A (as a is its least upper bound), so there must be an $x \in A$ with $x > a - \epsilon$. Similarly, $b - \epsilon$ is not an upper bound of B, so there must be a $y \in B$ with $y > b - \epsilon$. Then

$$x + y > (a - \epsilon) + (b - \epsilon) = a + b - 2\epsilon = c.$$

As $x+y \in S$, this implies that c is not an upper bound of S, as we wanted to show. As both $\sup S \leq a + b$ and $\sup S \geq a + b$ hold, we must have $\sup S = a + b$. The proof is complete.

Second proof. Using the notation introduced in the first proof, we will give a second proof of the inequality $\sup S \geq a + b$ (we will not give another proof of the inequality $\sup S \leq a+b$; the proof of this latter inequality will have to be taken from the first solution). Let c be an upper bound of S; it will be enough to show that $c \geq a + b$.

Let $x \in A$ and $y \in B$ be arbitrary. Then $x + y \in S$, and so $x + y \leq c$, since c is an upper bound of S. That is, $x \leq c - y$. Now, consider this inequality for a fixed $y \in B$. Then we can see that, for every $x \in A$, the inequality $x \leq c - y$ holds. I.e., $c - y$ is an upper bound of A, and so $c - y \geq a$, the least upper bound of A.

The last inequality can also be written as $y \leq c-a$; this inequality holds for every element $y \in B$ (since $y \in B$ was arbitrary). Thus $c - a$ is an upper bound of B; i.e., $c - a \geq b$, since b is the least upper bound of this set. Thus $c \geq a + b$, as we wanted to show.

[15] The sequence $\{b_n\}_{n=1}^{\infty}$ is called a subsequence of $\{a_n\}_{n=1}^{\infty}$ if $b_n = a_{f(n)}$ for some strictly increasing function f that is defined for each positive integer and has positive integers as values. In the present case, one can put $a_n = a^n$ and $b_n = a^{n+1} = a_{n+1}$, that is, $f(n) = n + 1$.

5. Upper and lower limit

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. The *upper limit* or *limit superior* of this sequence is defined as

$$\limsup_{n\to\infty} a_n = \sup\{x : x \le a_n \quad \text{for infinitely many} \quad n\}.$$

As the supremum of a set was always defined to be either a real number or $+\infty$ (when the set is not bounded from above) or $-\infty$ (when the set is empty), it follows every sequence has an upper limit that is either a real number or is $+\infty$ or $-\infty$.

The *lower limit* or *limit inferior* of the above sequence is defined as

$$\liminf_{n\to\infty} a_n = \inf\{x : x \ge a_n \quad \text{for infinitely many} \quad n\}.$$

As the infimum of a set was always defined to be either a number or $-\infty$ (when the set is not bounded from below) or $+\infty$ (when the set is empty), it follows every sequence has an lower limit that is either a real number or is $-\infty$ or $+\infty$. We have

LEMMA 1. *Given a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers, we have*

$$\liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n.$$

PROOF. Write $s = \liminf_{n\to\infty} a_n$, $S = \limsup_{n\to\infty} a_n$ and assume $s > S$. Let c be a real number such that $S < c < s$.[16] Then, noting that $c > S$, it follows that there are only finitely many n's such that $c \le a_n$; that is, the set $\{n : c \le a_n\}$ is finite.[17] Similarly, noting that $c < s$, it follows that the set $\{n : c \ge a_n\}$ is finite. This is a contradiction, since the union of these two sets is the set of all positive integers.

LEMMA 2. *Given a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers, if $\lim_{n\to\infty} a_n$ exists then we have $\lim_{n\to\infty} a_n \ge \limsup_{n\to\infty} a_n$.*

When saying that $\lim_{n\to\infty} a_n$ exists we of course mean that $\lim_{n\to\infty} a_n$ is a real number. However, the reader may reflect that the assertion of the Lemma remains valid even if we assume $\lim_{n\to\infty} a_n = -\infty$, and it can be proved similarly (with some modifications, since infinite limits are defined differently from limits that are equal to real numbers). If we assume $\lim_{n\to\infty} a_n = +\infty$, then the statement is of course valid, but in that case the statement is uninteresting.

PROOF. Write $L = \lim_{n\to\infty} a_n$, $S = \limsup_{n\to\infty} a_n$, and assume that $L < S$. Let c be a real number such that $L < c < S$.[18] Write $\epsilon = c - L$; clearly, $\epsilon > 0$. Then, noting that $L = \lim_{n\to\infty} a_n$, there is an N such that $|a_n - L| < \epsilon$ whenever $n > N$, i.e., such that $-\epsilon < a_n - L < \epsilon$ whenever $n > N$. Hence the set $\{n : x \le a_n\}$ is finite for $x \ge c = L + \epsilon$; in fact, no n with $n > N$ belongs to this set. Therefore, the supremum of the set

$$\{x : x \le a_n \quad \text{for infinitely many} \quad n\}$$

is less than or equal to c. On the other hand, the supremum of this set is $S = \limsup_{n\to\infty} a_n$, according to the definition of upper limit. This is a contradiction, since we have $S > c$. The proof is complete.

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[16]If S and s are real numbers, one can take $c = (s + S)/2$, but this choice will not work if S or s is (positive or negative) infinity. It is, however, easy to see that an appropriate c can be found even in this case.

[17]Note that this argument works even if $S = -\infty$. Similarly, the argument next works even if $s = +\infty$. We cannot have $S = +\infty$ or $s = -\infty$ since $S < s$ according to our assumption.

[18]If S is a real number then we can take $c = (L + S)/2$. If $S = +\infty$ then we can take $c = L + 1$.

LEMMA 3. *Given a sequence* $\{a_n\}_{n=1}^{\infty}$ *of real numbers, if* $\lim_{n\to\infty} a_n$ *exists then we have* $\lim_{n\to\infty} a_n \le \liminf_{n\to\infty} a_n$.

For the proof, one can repeat the argument of Lemma 2, with appropriate changes. However, it is easier to note that $\lim_{n\to\infty} a_n = -\lim_{n\to\infty}(-a_n)$ and $\liminf_{n\to\infty} a_n = -\limsup_{n\to\infty}(-a_n)$, According the Lemma 2, we have

$$-\lim_{n\to\infty}(-a_n) \le -\limsup_{n\to\infty}(-a_n),$$

and so the conclusion follows.

Again, the reader may reflect that the assertion of the Lemma remains valid even if we assume $\lim_{n\to\infty} a_n = +\infty$. If we assume $\lim_{n\to\infty} a_n = -\infty$, then the statement is of course valid, but in that case the statement is uninteresting.

COROLLARY. *Given a sequence* $\{a_n\}_{n=1}^{\infty}$ *of real numbers, if* $\lim_{n\to\infty} a_n$ *exists then we have*

$$\lim_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n.$$

This is an immediate consequence of Lemmas 1, 2, and 3, since according to these we have

$$\lim_{n\to\infty} a_n \le \liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n \le \lim_{n\to\infty} a_n.$$

Again, the statement of the Corollary can be extended to the case when $\lim a_n$ is $+\infty$ or $-\infty$ by using similar extensions of Lemmas 2 and 3.[19] In the converse direction we have

LEMMA 4. *Given a sequence* $\{a_n\}_{n=1}^{\infty}$ *of real numbers, if* $\limsup_{n\to\infty} a_n$ *is a real number and*

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n,$$

then $\lim_{n\to\infty} a_n$ *exists.*

PROOF. Write $L = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$, and let $\epsilon > 0$ be arbitrary. Then noting that $L+\epsilon > \limsup_{n\to\infty} a_n$, it follows that there are only finitely many n's such that $L+\epsilon \le a_n$; that is, the set $\{n : L+\epsilon \le a_n\}$ is finite. Then there is an integer N_1 such that the set $\{n > N_1 : L+\epsilon \le a_n\}$ is empty.

Similarly noting that $L-\epsilon < \liminf_{n\to\infty} a_n$, it follows that there are only finitely many n's such that $L-\epsilon \ge a_n$; that is, the set $\{n : L+\epsilon \ge a_n\}$ is finite. Then there is an integer N_2 such that the set $\{n > N_2 : L-\epsilon \ge a_n\}$ is empty.

Writing $N = \max\{N_2, N_2\}$, both the sets $\{n > N : L+\epsilon \le a_n\}$ and $\{n > N : L-\epsilon \ge a_n\}$ are empty. That is, we have

$$L-\epsilon < a_n < L+\epsilon$$

for every $n > N$. Since $\epsilon > 0$ was arbitrary, it follows that $L = \lim_{n\to\infty} a_n$. The proof is complete.

By a modification of the argument used here, it is easy to show that if $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = -\infty$ then $\lim_{n\to\infty} a_n = -\infty$ and if $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = +\infty$ then $\lim_{n\to\infty} a_n = +\infty$.

The completeness of $\mathbb{R}$. An easy consequence of the above is the following

[19] Lemma 1 need not be extended, since in that Lemma we allowed $\liminf_{n\to\infty} a_n$ and $\limsup_{n\to\infty} a_n$ to be $+\infty$ or $-\infty$.

Lemma 5. *In $\mathbb{R}$, any Cauchy sequence is convergent.*

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be a Cauchy sequence. Let $\epsilon > 0$ be arbitrary, and let N be an integer such that $|a_m - a_n| < \epsilon/3$ whenever $m, n > N$. Fix $k > N$. Then the set

$$\{n \geq 1 : a_k + \epsilon/3 \leq a_n\}$$

is finite; in fact, it does not contain any integer $n > N$. Hence

$$\limsup_{n\to\infty} a_n = \sup\{x : x \leq a_n \quad \text{for infinitely many} \quad n\} \leq a_k + \epsilon/3.$$

Similarly, the set

$$\{n \geq 1 : a_k - \epsilon/3 \geq a_n\}$$

is finite; in fact, it does not contain any integer $n > N$. Hence

$$\liminf_{n\to\infty} a_n = \inf\{x : x \geq a_n \quad \text{for infinitely many} \quad n\} \geq a_k - \epsilon/3.$$

Thus

$$\limsup_{n\to\infty} a_n - \liminf_{n\to\infty} a_n \leq (a_k + \epsilon/3) - (a_k - \epsilon/3) = 2\epsilon/3 < \epsilon.$$

That is, we have

$$\limsup_{n\to\infty} a_n - \liminf_{n\to\infty} a_n < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, this implies that $\limsup_{n\to\infty} a_n - \liminf_{n\to\infty} a_n \leq 0$. Indeed, if we had $\limsup_{n\to\infty} a_n - \liminf_{n\to\infty} a_n > 0$, then the choice $\epsilon = \limsup_{n\to\infty} a_n - \liminf_{n\to\infty} a_n$ would mean that $\epsilon < \epsilon$ according to the last displayed inequality, a contradiction.

Thus, $\liminf_{n\to\infty} a_n \geq \limsup_{n\to\infty} a_n$. Since, according to Lemma 1 we also have $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$, the equality $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$ follows. Therefore, $\lim_{n\to\infty} a_n$ exists, according to Lemma 4. This completes the proof.

6. Subspaces and compact sets

Let $\mathcal{S}$ be a collection of sets ("collection" is just a synonym of "set," so a collection of sets is just a set of sets). The union of $\mathcal{S}$, also called the union of all elements of $\mathcal{S}$, is defined as

$$\bigcup \mathcal{S} \overset{def}{=} \{x : \exists y \in \mathcal{S}\ [x \in y]\}.$$

Given a metric space (E, d), a point $p \in E$, and a positive real number r, we will write $U_E(p, r)$ for the open ball of radius r centered at p. That is

$$U_E(p, r) \overset{def}{=} \{q \in E : d(p, q) < r\}.$$

If the space E is understood from the context, we may omit it, and write $U(p, r)$ instead of $U_E(p, r)$. The following is a simple characterization of open sets.

Lemma. *Let (E,d) be a metric space. The set $S \subset E$ is open if and only if*

$$S = \bigcup \{U_E(p,r) : p \in E \ \ \& \ \ r > 0 \ \ \& \ \ U_E(p,r) \subset S\}.$$

In plain language, the Lemma says that the set S is open if and only if it is the union of the open balls it includes. Sometimes one writes $\bigcup_{y \in \mathcal{S}} y$ instead of $\bigcup \mathcal{S}$, but the latter notation is clearly simpler. Using the former notation, the set on the right-hand side of the equation in the Lemma could be described as

$$S = \bigcup_{p \in E \ \& \ r>0 \ \& \ U_E(p,r) \subset S} U_E(p,r).$$

Proof of the Lemma. The set on the right-hand side of the equation of the Lemma, being a union of open balls, is an open set. Hence, if the equation does hold, then S is an open set. This proves the "*if*" part of the Lemma.

To prove the "*only if*" part, first note that

$$S \supset \bigcup \{U_E(p,r) : p \in E \ \ \& \ \ r > 0 \ \ \& \ \ U_E(p,r) \subset S\},$$

since a union of subsets of S is taken on the right-hand side. We will show that

$$S \subset \bigcup \{U_E(p,r) : p \in E \ \ \& \ \ r > 0 \ \ \& \ \ U_E(p,r) \subset S\}$$

also holds in case S is open. Indeed, assume S is open, and let $p \in S$ be arbitrary. Then there is an $r > 0$ such that $U_E(p,r) \subset S$; as $p \in U_E(p,r)$, this shows that p is also an element of the right-hand side. Since $p \in S$ was arbitrary, it follows that S is indeed a subset of the union on the right-hand side. Thus, both inclusions hold, and so S must be equal to the union on the right-hand side. This establishes the "*only if*" part. Thus, the proof of the Lemma is complete.

Now, let (E,d) be a metric space and let S be a subset of E. It is easy to see that then (S,d) is a metric space; this space has S as its set of points, and its distance function is the same d as is the distance function on E.[20] The space (S,d) is called a subspace of the space (E,d). The next lemma describes the open sets in the space (S,d) in terms of those in (E,d).

Lemma. *Let (E,d) be a metric space, and let $S \subset E$ be a set. A set $M \subset S$ is open in (S,d) if and only if there is a set U open in (E,d) such that $M = U \cap S$.*

Similarly, a set $M \subset S$ is closed in (S,d) if and only if there is a set F closed in (E,d) such that $M = F \cap S$.

Note that the fact that the set $M \subset S$ is open (or closed) in (S,d) does not mean that M is open (or closed) in (E,d). For example taking $E = \mathbb{R}$ ($\mathbb{R}$ is the set of real numbers, with the distance function $d(x,y) = |x-y|$), and taking S to be the semiclosed interval $[0,1)$, the set S is both open and closed in (S,d) (since the whole space is always both an open and a closed set), but S is neither open nor closed in (E,d).

[20] Technically, this is not quite correct; the distance function on S should be d' defined as

$$d'(p,q) = \begin{cases} d(p,q) & \text{if } p \in S \text{ and } q \in S, \\ \text{undefined otherwise (i.e., if } p \notin S \text{ or } q \notin S\text{)}, \end{cases}$$

since, clearly, the distance function on S should not be defined outside S. d' is called the restriction of d to S (or, more precisely, to $S \times S$). It will, however, be simpler to say, at the price of a minor inaccuracy, that the distance function on S is the same d as the distance function on E.

Proof of the Lemma. Assume that M is an open set in (S, d). Then, by the above Lemma,

$$M = \bigcup \{U_S(p, r) : p \in S \ \ \& \ \ r > 0 \ \ \& \ \ U_S(p, r) \subset M\}.$$

As $U_S(p, r) = S \cap U_E(p, r)$ provided $p \in S$,[21] the set on the right-hand side is equal to

$$\begin{aligned}\bigcup \{S \cap U_E(p, r) &: p \in S \ \ \& \ \ r > 0 \ \ \& \ \ U_S(p, r) \subset M\} \\ &= S \cap \bigcup \{U_E(p, r) : p \in S \ \ \& \ \ r > 0 \ \ \& \ \ U_S(p, r) \subset M\};\end{aligned}$$

the second equation here is true by a distributive rule for the union of sets.[22] The set

$$U \stackrel{def}{=} \bigcup \{U_E(p, r) : p \in S \ \ \& \ \ r > 0 \ \ \& \ \ U_S(p, r) \subset M\}$$

is an open set in the space (E, d), since it is union of open balls in E. Since we have $M = S \cap U$, this completes the proof of the "*only if*" part of the statement on open sets in the lemma.

As for the "*if*" part, assume $M = S \cap U$, where U is open in (E, d), and let $p \in M$ be arbitrary. Then we also have $p \in U$. Thus, by the openness of U in (E, d), there is an $r > 0$ such that $U_E(p, r) \subset U$. Then $U_S(p, r) = S \cap U_E(p, r) \subset S \cap U = M$. Since $p \in M$ was arbitrary, this shows that M is open in (S, d), completing proof of also the "*if*" part of the Lemma on open sets.

The second statement (i.e., the one about closed sets) is easy to verify by going to complements. That is, if M is closed in (S, d), then its complement $S \setminus M$ in S is open in (S, d). Thus by the statement about open sets, there is a set U open in (E, d) such that $S \setminus M = U \cap S$. Then, writing F for the complement in E of U (that is, $F = E \setminus U$), F is a set closed in (E, d) such that $M = F \cap S$.

Conversely, if $M = F \cap S$ where F is closed in (E, d), then $U \stackrel{def}{=} E \setminus F$ is open in (E, d), so $S \setminus M = U \cap S$ is open in (S, d) by the statement about open sets; thus M is closed in (S, d). The proof of the Lemma is complete.

The lemma just proved has important consequences for compact sets, defined next

Definition. Let (E, d) be a metric space and let S be a subset of E. The set S is said to be compact in (E, d) if the following holds: For any collection $\mathcal{U}$ of open sets in (E, d), if $S \subset \bigcup \mathcal{U}$, there is a finite subcollection $\mathcal{U}' \subset \mathcal{U}$ such that $S \subset \bigcup \mathcal{U}'$.

Further, the space (E, d) is said to be compact if E is a compact set in (E, d).

To say that $\mathcal{U}$ is a collection of open sets in (E, d) means that $\mathcal{U}$ is a collection every element of which is an open subset of (E, d). The inclusion $S \subset \bigcup \mathcal{U}$ is expressed by saying that $\mathcal{U}$ is on open cover of S. One also says that $\mathcal{U}$ covers S, or that the elements of $\mathcal{U}$ cover S. If $\mathcal{U}' \subset \mathcal{U}$ and $S \subset \bigcup \mathcal{U}'$, then $\mathcal{U}'$ is called a subcover of $\mathcal{U}$ (for S, if one wants to avoid repeating the preposition "of"). Thus, one can say shortly that a set is compact if and only if any of its open covers has a finite subcover.

Example. The interval $(0, 1)$ is not compact.

Indeed,

$$(0, 1) \subset \bigcup_{n=1}^{\infty} \left(\frac{1}{n}, 1\right),$$

[21] The left-hand side makes no sense if $p \notin S$.

[22] The distributive rule in question says that

$$X \cap \bigcup \mathcal{A} = \bigcup \{X \cap A : a \in \mathcal{A}\}$$

is valid for any set X and any collection $\mathcal{A}$ of sets. This equation is easy to verify with the aid of the definition of the union of sets.

and, for any positive integer N,

$$(0,1) \not\subset \bigcup_{n=1}^{N} \left(\frac{1}{n}, 1\right).$$

Compactness is an intrinsic property; that is, whether a set is compact does not depend on which larger space it is considered. That is, the second Lemma above has the following

Corollary. *Let (E,d) be a metric space and let S be a subset of E. Then the set S is compact in (E,d) if and only if (S,d) is a compact space.*

Proof. To establish the "*if*" part, assume that (S,d) is a compact space. Let $\mathcal{U} = \{U_\iota : \iota \in \mathcal{I}\}$ be a collection of sets U_ι, open in (E,d) such that $S \subset \bigcup \mathcal{U}$.[23] Then

$$\mathcal{V} \overset{def}{=} \{S \cap U_\iota : \iota \in \mathcal{I}\}$$

is, by the second Lemma above, a collection of sets that are open in (S,d). We have

$$S \subset S \cap \bigcup \mathcal{U} = \bigcup \mathcal{V};$$

that is $\mathcal{V}$ is an open cover, in the sense of (S,d) of S. As the space (S,d) is compact, $\mathcal{V}$ has a finite subcover, say $\mathcal{V}'$; i.e., $S \subset \bigcup \mathcal{V}'$. Since each element $\mathcal{V}$ has the form $S \cap U_\iota$ for some $\iota \in \mathcal{I}$, there is a finite subset $\mathcal{I}'$ of $\mathcal{I}$ such that

$$(*) \qquad \mathcal{V}' = \{S \cap U_\iota : \iota \in \mathcal{I}'\}$$

Then $S \subset \bigcup \mathcal{V}'$ implies that

$$S \subset \bigcup \{U_\iota : \iota \in \mathcal{I}'\}.$$

In other words,

$$(**) \qquad \mathcal{U}' \overset{def}{=} \{U_\iota : \iota \in \mathcal{I}'\}$$

is an finite open subcover, in the sense of (E,d), of $\mathcal{U}$, showing that S is compact in (E,d).

To establish the "*if*" part, assume that S is a compact set in (E,d), and let $\mathcal{V}$ be an open cover, in the sense of (S,d), of S; then $S \subset \bigcup \mathcal{V}$. Each element V of $\mathcal{V}$ is open in the space (S,d). Hence by the second Lemma above, each V in $\mathcal{V}$ has form $S \cap U$ with some U open in (E,d). Thus, taking

$$\mathcal{U} \overset{def}{=} \{U : U \text{ is open in } (E,d) \ \ \& \ \ S \cap U \in \mathcal{V}\},$$

$\mathcal{U}$ is an open cover, in the sense of (E,d), of S. As S is compact in (E,d), there is a finite subset $\mathcal{U}'$ of $\mathcal{U}$ that is an open cover, in the sense of (E,d), of S. Then

$$\mathcal{V}' \overset{def}{=} \{S \cap U : U \in \mathcal{U}'\}$$

is a finite subset of $\mathcal{V}$ that is an open cover, in the sense of (S,d), of S, showing that the space (S,d) is compact. The proof is complete.

Remark. In the first part of the proof, we indexed the elements of the set $\mathcal{U} = \{U_\iota : \iota \in \mathcal{I}\}$ with another set, $\mathcal{I}$. If one wants to use function notation instead of the customary,

[23]The letter ι is the Greek lower case iota, and it is used instead of i, since often i is used to name the elements of a countable set, and here it is not assumed that $\mathcal{I}$ is countable. We use the set $\mathcal{I}$ to index, or name, the elements of the set $\mathcal{U}$. The reason for this will be explained after the proof.

but perhaps confusing, subscript notation, one can say $f(\iota)$ instead of U_ι. Then the set U equals the range of the function f (assuming that the domain of f is $\mathcal{I}$, and not a larger set). A set described this way is sometimes called an indexed set. Any set can be described as an indexed set. For example, one can take $\mathcal{I} = \mathcal{U}$ and f as the identity function on $\mathcal{U}$. It is not required that the function f be one-to-one.

In the above proof, the purpose of the index set $\mathcal{I}$ was to describe the set $\mathcal{V}'$ given in $(*)$ in terms of a finite index set $\mathcal{I}'$, since the same index set $\mathcal{I}'$ can then be used to define the set $\mathcal{U}'$, as was done in $(**)$. Without using the index set $\mathcal{I}$, one might have been tempted to say that let $\mathcal{V}'$ be a finite subcover of $\mathcal{V}$, and then write

$$\mathcal{U}' \stackrel{def}{=} \{U : U \in \mathcal{U} \ \ \& \ \ \exists V \in \mathcal{V}' \, [V = S \cap U]\}.$$

This would not be correct though, since the set $\mathcal{U}'$ can be infinite. That is because, even though there are finitely many choices for $V \in \mathcal{V}'$, for a given V there may be infinitely many choices for $U \in \mathcal{U}$ such that $V = S \cap U$. That is, for each $V \in \mathcal{V}'$ one needs to pick a single $U = g(V)$ in $\mathcal{U}$ such that $V = S \cap U$. Then one can define $\mathcal{U}'$ as

$$\mathcal{U}' \stackrel{def}{=} \{g(V) : V \in \mathcal{V}'\}.$$

This definition of $\mathcal{U}'$ accomplishes the same thing as the definition given in $(**)$, but the present one is somewhat to messy to say. The use of the index set $\mathcal{I}$ in the above proof avoids the need for this complicated circumlocution. In any case, difference between the two approaches (one using indexed sets, the other not) is only a matter of style, and not of substance.

Next we show the following.

Theorem. *Let (E, d) be a metric space, and let $S \subset E$ be compact. Then S is closed.*

Proof. Assume S is not closed. Then its complement, $E \setminus S$, is not open. Therefore, there is a $q \in E \setminus S$ such that for no $r > 0$ is the open ball $U(q, r)$ a subset of $E \setminus S$. That is, we have $q \notin S$ for this q, and for every $r > 0$ we have

$$S \cap U(q, r) \neq \emptyset.$$

For each $p \in S$, let $r_p = d(p, q)/2$.[24] Then the open balls $U(p, r_p)$ and $U(q, r_p)$ are disjoint, as is easily verified with the aid of the triangle inequality. Then

$$S \subset \bigcup_{p \in S} U(p, r_p),$$

since each point p of S is an element of the union on the right-hand side (as $p \in U(p, r_p)$. By compactness, the open cover on the right-hand side has a finite subcover. That is, there is a finite subset S' of S such that[25]

$$S \subset \bigcup_{p \in S'} U(p, r_p).$$

Now, let

$$r = \min\{r_p : p \in S'\};$$

the set on the right-hand side is a finite set of positive reals. Thus, the indicated minimum exists, and is positive; i.e., $r > 0$. Since, as we remarked above, the open balls $U(p, r_p)$ and

[24] The dependence on q of r_p is not indicated, since q is fixed throughout the argument.

[25] See the Remark above about indexed sets. Here we chose S as the index set for the open cover $\{U(p, r_p) : p \in S\}$.

$U(q, r_p)$ are disjoint for $p \in S$, the open balls $U(p, r_p)$ and $U(q, r)$ are also disjoint whenever $p \in S'$, because $r \le r_p$ for such p. Thus,[26]

$$U(q, r) \cap \bigcup_{p \in S'} U(p, r_p) = \emptyset.$$

This is, however, a contradiction, since the set on the right-hand side of the intersection sign $\cap$ includes S, and, as we indicated above, the open ball on the left-hand side of the intersection sign is not disjoint to S. This contradiction completes the proof of the theorem.

7. Totally bounded spaces

Our goal here is to give a characterization of compact metric spaces. To this end we introduce the following concept.

Definition. *The metric space* (E, d) *is called* totally bounded *if for every* $\epsilon > 0$ *it can be covered by finitely many closed balls of radius* ϵ.

It will be convenient, but not essential, that we said closed balls above, since, clearly, if E can be covered by finitely many closed balls of radius $\epsilon/2$, then the open balls with the same center and radius ϵ will also cover E. One can also observe that if (E, d) and $S \subset E$, then the subspace (S, d) is also totally bounded. This is almost obvious, but there is the minor hitch: if we cover E by finitely many closed balls of radius ϵ, then we may not be able use these same closed balls to cover S, since the centers of these closed balls may not all be in S, and so they will not be closed balls in (S, d). Cover instead E by finitely many closed balls of radius $\epsilon/2$, then take only those balls B for which $B \cap S \neq \emptyset$, take $p \in B \cap S$, and replace B with a closed ball B' in S with center p and radius ϵ. As $B \cap S \subset B'$, it is clear that these new closed balls will cover S.

A space metric space is called *bounded* if it can be included in a single open ball. It is clear that a totally bounded space is also bounded, since finitely many closed balls can always be covered by a single open ball; the converse is, however, not true. For any integer $n \ge 1$ let $\mathbb{R}^n$ be the set $\{(x_1, \dots, x_n) : x_i \in \mathbb{R} \text{ for } 1 \le i \le n\}$; for $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{x} = (x_1, \dots, x_n)$ write $d(\mathbf{x}, \mathbf{y}) = \left(\sum_{i=1}^n (x_i - y_i)^2\right)^{1/2}$. Then one can show that $(\mathbb{R}^n, d)$ is a metric space (see Rosenlicht [1, pp. 34–36]); this space is called the *n-dimensional Euclidean space*. It is easy to show that any bounded subspace of the n-dimensional Euclidean space is totally bounded (see [1, p. 57]). Since this space is also complete (this is easy to conclude from the completeness of $\mathbb{R}$; cf. [1, p. 53]), it follows from the Theorem below that any closed and bounded subspace of the n-dimensional Euclidean space is compact.

The space l^2 is defined as the pair (E, d) where $E = \{(x_1, x_2, \dots) : x_i \in \mathbb{R} \text{ for } i \ge 1 \text{ and } \sum_{i=1}^\infty x_i^2 < +\infty\}$ and the distance function d is defined as $d(\mathbf{x}, \mathbf{y}) = \left(\sum_{i=1}^\infty (x_i - y_i)^2\right)^{1/2}$ for $\mathbf{x} = (x_1, x_2, \dots)$ and $\mathbf{y} = (y_1, y_2, \dots)$ in E. It can be shown that l^2 is a complete metric space, and that no closed ball of positive radius in l^2 is totally bounded.

We will need a few more concepts.

[26] For a formal verification of the next equality, one may use the distributive property mentioned above:

$$U(q, r) \cap \bigcup_{p \in S'} U(p, r_p) = \bigcup_{p \in S'} \bigl(U(q, r) \cap U(p, r_p)\bigr).$$

A common sense argument may admittedly be simpler.

Definition. *The metric space is called sequentially compact if every sequence in it has a convergent subsequence.*

Definition. *Let (E, d) be a metric space and let $p \in E$ and $S \subset E$. We say that p is a* cluster point *of S if every open ball with center p in E contains infinitely many elements of S.*

Observe that we can equivalently say that p is a cluster point of S if every open ball with center p contains at least one element of S different from p. Indeed, if the open ball with center p were to contain only finitely many elements of S, all these points, except p, wound be excluded from an open ball with center p and an appropriately smaller radius. The following result is simple.

Lemma. *Let (E, d) be a metric space, and let S and T be sets such that $T \subset S \subset E$, S is compact, and T is infinite. Then T has a cluster point in S.*

Proof. Assume, on the contrary, that no point $p \in S$ is a cluster point of T. For each $p \in S$ we can then take an open ball B_p with center p such that $B_p \cap T$ is finite. We have

$$S \subset \bigcup_{p \in S} B_p.$$

That is, these open balls cover S. By the compactness of S, there are finitely many among these open balls that also cover S. In other words, there is a finite set $S' \subset S$ such that

$$S \subset \bigcup_{p \in S'} B_p.$$

We then have

$$T \subset S \cap T \subset \Big(\bigcup_{p \in S'} B_p \Big) \cap T \subset \bigcup_{p \in S'} (B_p \cap T).$$

The sets $B_p \cap T$ for $p \in S'$ are finite; the union of finitely many of these is finite. This is, however, a contradiction, since T was assumed to be infinite, completing a proof.

Theorem. *Let (E, d) be a metric space. The following are equivalent:*

(i) *(E, d) is compact;*
(ii) *(E, d) is sequentially compact;*
(iii) *(E, d) is complete and totally bounded.*

Proof. We will prove this result by showing the implications (i) $\to$ (ii), (ii) $\to$ (iii), and (iii) $\to$ (i).

Proof of (i) $\to$ (ii). Assume (E, d) is compact. Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of points $p_n \in E$. If the set $\{p_n : n \in \mathbb{N}\}$ is finite, then there is a $p \in E$ such that $p = p_n$ for infinitely many n's. Then we can take a subsequence $\{p_{n_k}\}_{k=1}^{\infty}$ of $\{p_n\}_{n=1}^{\infty}$ such that $p_{n_k} = p$ for all $k \in \mathbb{N}$. This subsequence converges to p.

If the set $\{p_n : n \in \mathbb{N}\}$ is infinite, then it has a cluster point p by the above Lemma. For each $k \in \mathbb{N}$ the open ball $U(p, 1/k)$ contains infinitely elements of the set $\{p_n : n \in \mathbb{N}\}$. Let $n_k \in \mathbb{N}$ be such that $p_{n_k} \in U(p, 1/k)$ and, in case $k > 1$, we also have $n_k > n_{k-1}$. Then the subsequence $\{p_{n_k}\}_{k=1}^{\infty}$ converges to p.

Proof of (ii) $\to$ (iii). Assume (E, d) is sequentially compact. We will first show that (E, d) is complete. Let $\{p_n\}_{n=1}^{\infty}$ be a Cauchy sequence; then this sequence has a convergent subsequence. A Cauchy sequence that has a convergent subsequence is itself convergent; so $\{p_n\}_{n=1}^{\infty}$ itself is convergent, showing that (E, d) is complete.

Next we show that (E, d) is totally bounded. Assume, on the contrary that it is not totally bounded; let $\epsilon > 0$ be such that (E, d) cannot be covered by finitely many closed balls of radius ϵ. Select a sequence p_n of points $p_n \in E$ such that, writing $\bar{U}(p,\epsilon)$ for the closed ball of center p and radius ϵ, when selecting p_n we have

$$p_n \notin \bigcup_{k=1}^{n-1} \bar{U}(p_k,\epsilon);$$

since the closed balls on the right-hand side do not cover E according to our assumptions, it is possible to select p_n in such a way. Then $d(p_n,p_m) > \epsilon$ for every two distinct $n,m \in \mathbb{N}$, showing that the sequence $\{p_n\}_{n=1}^{\infty}$ is not Cauchy sequence, and so it does not converge. This contradicts the assumption that (E,d) is sequentially compact, showing that (E,d) is totally bounded.

(iii) $\to$ (i). Assume (E,d) is complete and totally bounded, and assume, on the contrary, that E is not compact. Let $\mathcal{U}$ be a collection of open sets such that $E \subset \bigcup \mathcal{U}$ (in this case we actually have $E = \bigcup \mathcal{U}$, since E is the whole space) and there is no finite $\mathcal{U}' \subset \mathcal{U}$ for which $E \subset \bigcup \mathcal{U}'$, i.e., that E cannot be covered by finitely many sets in $\mathcal{U}$. We will construct a sequence of closed balls B_n for $n \ge 0$ such that $B_0 = E$ (since E is totally bounded, it is also bounded, so E can be regarded as a closed ball), for $n \ge 0$ the closed ball B_n cannot be covered by finitely many elements of $\mathcal{U}$, i.e., for $n \ge 0$ there is no finite set $\mathcal{U}' \subset \mathcal{U}$ with $B_n \subset \bigcup \mathcal{U}'$, and such that, for each $n \ge 1$, the radius of B_n is $1/n$, and the center p_n of B_n is in B_{n-1}, i.e., $p_n \in B_{n-1}$,

Let $n \ge 1$ and assume that B_{n-1} has already been constructed in such a way that B_{n-1} cannot be covered by finitely many elements of $\mathcal{U}$ (this is certainly true if $n = 1$, since $B_0 = E$). To construct B_n for $n \ge 1$, cover B_{n-1} by finitely many closed balls of radius $1/n$ in such a way that the centers of each of these balls is in B_{n-1}; as we pointed out above, this is possible since (B_{n-1},d), being the subspace of a totally bounded space, is totally bounded. There must be among these closed balls one that cannot be covered by finitely many elements of $\mathcal{U}$; select one of these balls as B_n. Let p_n be the center of B_n.

The sequence $\{p_n\}_{n=1}^{\infty}$ is a Cauchy sequence; indeed, for $m > n$ we have $p_m \in B_n = \bar{U}(p_n,1/n)$, so $d(p_n,p_m) \le 1/n$. Let p be the limit of this sequence. As B_n, being a closed ball, is a closed set for each $n \in \mathbb{N}$ and $p_m \in B_n$ for all $m \ge n$, we have $p \in B_n$ for each $n \in \mathbb{N}$. That is, $d(p,p_n) \le 1/n$ for all $n \in \mathbb{N}$. Now, $p \in U$ holds for some $U \in \mathcal{U}$, since $\mathcal{U}$ covers E. As U is open, there is an $\epsilon > 0$ such that $U(p,\epsilon) \subset U$. Then, for $n \ge 3/\epsilon$ we have $3/n \le \epsilon$, so $U(p,3/n) \subset U(p,\epsilon) \subset U$. Then we also have $B_n = \bar{U}(p_n,1/n) \subset \bar{U}(p,2/n) \subset U(p,3/n) \subset U = \bigcup\{U\}$; the first inclusion here holds since for any $q \in B_n$ we have $d(p,q) \le d(p,p_n) + d(p_n,q) \le 1/n + 1/n = 2/n$. That is, B_n can be covered by a one-element subset of $\mathcal{U}$. This is a contradiction, since we assumed that B_n cannot be covered by finitely many elements of $\mathcal{U}$. This contradiction shows that the assumption that (E,d) is not compact was wrong, completing the proof.

Topological spaces are generalizations of metric spaces; in topological spaces, there are open and closed sets, but there is no distance function. There are topological spaces that are sequentially compact but not compact; so, maintaining the distinction between the notions of compact and sequentially compact is important.

8. Compactness of closed intervals

The Heine–Borel Theorem. *A closed interval* $[a,b]$ *of* $\mathbb{R}$ *(*$a,b \in \mathbb{R}$*,* $a \le b$*) is compact.*

Proof. Let $\mathcal{U}$ be a collection of open sets such that $[a,b] \subset \bigcup \mathcal{U}$; here $\bigcup \mathcal{U}$ is the modern notation for $\bigcup_{U \in \mathcal{U}} U$. We will say that $\mathcal{U}$ *covers* the interval $[a,b]$, or that the elements of $\mathcal{U}$

cover $[a, b]$. The word *collection* is a synonym for set; that is, we could have said that $\mathcal{U}$ is a set of open sets.

We must show that finitely many elements of $\mathcal{U}$ cover $[a, b]$, i.e., that there is a finite $\mathcal{U}' \subset \mathcal{U}$ such that $[a, b] \subset \bigcup \mathcal{U}'$. Let

$$\begin{aligned} A =& \{x \in \mathbb{R} : a \le x \le b + 1 \text{ and } x \text{ is such that} \\ & \text{finitely many elements of } \mathcal{U} \text{ cover } [a, x]\}. \end{aligned}$$

Clearly, A is not empty, since $a \in A$. In fact, we have $a \in U_0$ for some $U_0 \in \mathcal{U}$, and then the single open set U_0 covers the interval $[a, a] = \{a\}$ (here $\{a\}$ denotes the set with a as its only element). Further, A is bounded from above (by $b + 1$). Let

$$c = \sup A.$$

Then $c > a$. Indeed, with $U_0 \in \mathcal{U}$ as above, there is an $\epsilon > 0$ such that $(a - \epsilon, a + \epsilon) \subset U_0$, as U_0 is open. Hence $[a, a + \epsilon/2] \subset U_0$, i.e., the single set U_0 covers the closed interval $[a, a + \epsilon/2]$. Therefore, we have $a + \epsilon/2 \in A$. So we have $c \ge a + \epsilon/2$, and thus $c > a$.

It is easy to see that if $a \le x < c$ then $x \in A$, i.e., that finitely many elements of $\mathcal{U}$ cover $[a, x]$. That is,

$$[a, c) \subset A. \tag{2}$$

In fact, if $[a, x]$ could not be covered by finitely many elements of $\mathcal{U}$, then, for $x' > x$, $[a, x']$ could not be covered by finitely elements of $\mathcal{U}$, either; that is, we would have $x' \notin A$ for any such x', and so x would be an upper bound of A.

On the other hand, if $x > c$ then $x \notin A$ (since c is an upper bound of A). That is,

$$A \subset [a, c]. \tag{3}$$

The relations (2) and (3) together show that either $A = [a, c)$ or $A = [a, c]$, Which of these cases occur, i.e., whether or not we have $c \in A$, will not concern us here.

Claim. *$c > b$.*

Indeed, assume, on the contrary, that $c \le b$. Then $c \in [a, b]$, and so there must be a $U_1 \in \mathcal{U}$ such that $c \in U_1$. As U_1 is open, there is an $\epsilon > 0$ be such that

$$(c - \epsilon, c + \epsilon) \subset U_1;$$

as $c > a$, we may choose ϵ such that $c - \epsilon > a$. Choosing $\epsilon \le 1$ will also ensure that $c + \epsilon \le b + 1$. Let $\mathcal{U}' \subset \mathcal{U}$ be a finite collection that covers the closed interval $[a, c - \epsilon]$. There must be such a $\mathcal{U}'$, since $c - \epsilon \in A$ according to (2). Then the finite collection $\{U_1\} \cup \mathcal{U}'$ (i.e., the collection consisting of U_1 and the elements of $\mathcal{U}'$) covers the interval $[a, c + \epsilon)$. Hence it also covers the smaller closed interval $[a, c + \epsilon/2]$; thus $c + \epsilon/2 \in A$. This contradicts (3). Thus the Claim is established.

It is easy to derive the assertion of the Theorem from the Claim. In fact, according to (2), the Claim implies $b \in A$; that is, finitely many elements of $\mathcal{U}$ cover $[a, b]$. The proof of the Theorem is complete.

9. The method of interval-halving: The Heine–Borel Theorem

The Heine-Borel Theorem. *Every closed interval I of $\mathbb{R}$ is compact.*

The proof we will give of this result uses an argument that can be called the method of interval-halving. Other proofs can also be given.

Proof. Let $\mathcal{U}$ be a collection of open sets such that $I \subset \bigcup \mathcal{U}$; here $\bigcup \mathcal{U}$ is the modern notation for $\bigcup_{U\in\mathcal{U}} U$. We will say that $\mathcal{U}$ *covers* the interval I, or that the elements of $\mathcal{U}$ *cover* I. The word *collection* is a synonym for set; that is, we could have said that $\mathcal{U}$ is a set of open sets. We need to show that finitely many elements of $\mathcal{U}$ cover I, i.e., that there a finite subset $\mathcal{U}'$ of $\mathcal{U}$ such that $I \subset \bigcup \mathcal{U}'$. Assume that this is not the case. That is, assume that I cannot be covered by finitely many elements of $\mathcal{U}$.

Write $I_0 = [a_0, b_0]$ for the interval I, and divide I_0 into two equal parts, $I_{00} = [a_0, (a_0 + b_0)/2]$ and $I_{01} = [(a_0 + b_0)/2, b_0]$. If both these intervals can be covered by finitely many elements of $\mathcal{U}$, then so can I_0. Hence at least one of them cannot so be covered. Pick $I_1 = [a_1, b_1]$ as that one between I_{00} and I_{01} which cannot be covered by finitely elements of $\mathcal{U}$. If neither can be so covered, then pick I_1 arbitrarily as one of these.[27] Again, subdivide the interval $I_1 = [a_1, b_1]$ into two equal parts, $I_{10} = [a_1, (a_1+b_1)/2]$ and $I_{11} = [(a_1+b_1)/2, b_1]$. One of these intervals cannot be covered by finitely elements of $\mathcal{U}$. Pick I_2 as one that cannot be so covered.

Continue in this fashion indefinitely. That is, if $I_n = [a_n, b_n]$ has already been picked in such a way that cannot be covered by finitely elements of $\mathcal{U}$, then subdivide it into two equal parts, $I_{n0} = [a_n, (a_n + b_n)/2]$ and $I_{n1} = [(a_n + b_n)/2, b_n]$, and let $I_{n+1} = [a_{n+1}, b_{n+1}]$ be one of these intervals that cannot be covered by finitely elements of $\mathcal{U}$.

In this way, we obtain an infinite sequence of intervals

$$I_1 \supset I_2 \supset \ldots I_n \supset I_{n+1} \supset \ldots,$$

where $I_n = [a_n, b_n]$. The inclusion $I_{n+1} \subset I_n$ is equivalent to saying that $a_{n+1} \geq a_n$ and $b_{n+1} \leq b_n$. Therefore, the sequence $\{a_n\}_{n=0}^{\infty}$ is increasing,[28] and the sequence $\{b_n\}_{n=0}^{\infty}$ is decreasing. Of course these sequences are also bounded, since for every nonnegative integer n we have $a_0 \leq a_n < b_n \leq b_0$; therefore, these sequences are convergent. Let $a = \lim a_n$ and $b = \lim b_n$.

The length of the interval I_{n+1} is half the length of the interval I_n. Thus, the length of I_n equals $b_n - a_n = (b_0 - a_0)/2^n$, and so $\lim(b_n - a_n) = 0$. Therefore, $a = \lim a_n = \lim a_n + \lim(b_n - a_n) = \lim b_n = b$.

Since $\mathcal{U}$ covers I and $a \in I$, there is a set $U_0 \in \mathcal{U}$ such that $a \in U_0$. As U_0 is open, there is an $\epsilon > 0$ such that the interval $(a - \epsilon, a + \epsilon)$ is a subset of U_0. As $a = \lim a_n = \lim b_n$, there is a positive integer n such that both a_n and b_n belong to the interval $(a - \epsilon, a + \epsilon)$. Thus $I_n = [a_n, b_n] \subset (a - \epsilon, a + \epsilon)$. Hence the single open set $U_0 \in \mathcal{U}$ covers the interval I_n; yet, according the construction of I_n, the interval I_n cannot be covered by finitely many elements of $\mathcal{U}$. This is a contradiction, completing the proof.

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[27] Reasons that go beyond the scope of this course and involve the foundations of mathematics suggest that it is better to set up a definite rule as to which interval to select rather than make an arbitrary choice: e.g., one can stipulate to pick the interval on the left if both intervals qualify.

[28] in the wider sense, i.e., with equality allowed. Some authors would call such a sequence *nondecreasing.*

10. The Bolzano–Weierstrass Theorem

THE BOLZANO–WEIERSTRASS THEOREM. *Let S be an infinite bounded subset of $\mathbb{R}$. Then one can pick distinct elements of S that form a convergent sequence.*

Note that the Theorem follows from results we already know. In fact, one can take a closed interval $[a, b]$ including the set S. Since a closed interval in $\mathbb{R}$ is compact, and we know that out of every infinite subset of a compact set one can pick a convergent sequence of distinct elements, the assertion follows. The purpose here is to give a more direct proof.

PROOF. We will show that one can pick distinct elements of S forming an infinite monotone sequence. Since a monotone bounded sequence is always convergent, the result will follow.

To accomplish this, assume that there is no infinite increasing sequence of distinct elements of S. Then we will show the following

CLAIM. *For every real x, the set*

$$S(x) \overset{def}{=} \{s \in S : s < x\}$$

is either empty or has a maximum.

Indeed, assume that $S(x)$ is not empty and it has no maximum (i.e., largest element). Let $a_1 \in S(x)$ be arbitrary. Since a_1 is not the largest element of $S(x)$ (as $S(x)$ has no largest element), one can pick an element $a_2 \in S(x)$ with $a_1 < a_2$. Now a_2 is not the largest element of $S(x)$, so one can pick an $a_3 \in S(x)$ with $a_2 < a_3$. This can be continued for ever, and so one can pick elements of $S(x)$ with $a_1 < a_2 < \ldots < a_n < \ldots$. This would be an infinite increasing sequence of elements of S, in contradiction with our assumption that there is no such sequence. Hence the assumption that $S(x)$ is not empty and has no maximum cannot be correct. Thus the Claim is established.

Using the above Claim, it is easy to pick an infinite decreasing sequence of elements of S. Indeed, let b be an upper bound of S, and let b_1 be the maximum of $S(b + 1)$ (observe that $S(b + 1) = S$ is a nonempty set; in fact, it is infinite by our assumption). Let b_2 the maximum of $S(b_1)$ (again, observe that $S(b_1) = S \setminus \{b_1\}$ is still an infinite set). Let b_3 be the maximum of $S(b_2)$. In general, let b_{n+1} be the maximum of $S(b_n)$; noting that $S(b_n) = S \setminus \{b_k : 1 \le k \le n\}$ is an infinite set (the equality here holds in view of the way the b_k's were selected), $S(b_n)$ is an infinite set. Hence it is not empty; so it is possible to pick b_{n+1} as required, according to the above claim. Now, it is clear that the sequence $\{b_n\}_{n=1}^{\infty}$ is an infinite decreasing sequence of elements of S.

The above argument establishes the assertion that one can pick an infinite monotone sequence of distinct elements of S. Since a bounded monotone sequence is convergent, the assertion of the Theorem follows, namely, that can pick a convergent sequence of distinct elements of S. The proof is complete.

Note. Another way to prove the Theorem is to show that the number

$$c = \sup\{x : \text{there are infinitely many elements of } S \text{ less than } x\}$$

is an cluster point of S, and then one can show that there is a sequence of elements of S that converges to c. This approach is reminiscent of the discussion of the limit superior on p. 11 in Section 5.

11. The method of interval-halving: The Bolzano–Weierstrass Theorem

Recall the following

Definition. A real number r is said to be a cluster point of the subset S of $\mathbb{R}$ if for every $\epsilon > 0$ the interval $(r-\epsilon, r+\epsilon)$ contains infinitely many elements of S.

Note that it is not assumed that r belongs to S. We have

The Bolzano–Weierstrass Theorem. *Every infinite bounded subset S of $\mathbb{R}$ has a cluster point.*

Another version of this theorem says that every infinite bounded subset S of $\mathbb{R}$ includes a convergent sequence; it is easy to derive each version from the other. The theorem follows from results we already know. In fact, one can take a closed interval $[a, b]$ including the set S. A closed interval in $\mathbb{R}$ is compact, and we know that every infinite subset of a compact set has a cluster point. The purpose here is to give a more direct proof.

Proof. Let $I_0 = [a_0, b_0]$ be a closed interval that includes the set S. Divide I into two equal parts, $I_{00} = [a_0, (a_0 + b_0)/2]$ and $I_{01} = [(a_0 + b_0)/2, b_0]$. One of these two intervals must contain infinitely many elements of S; pick $I_1 = [a_1, b_1]$ as that one which contains infinitely many elements of S; if both I_{00} and I_{01} contain infinitely many elements of S, then pick I_1 arbitrarily as one of these.[29] Again, subdivide the interval $I_1 = [a_1, b_1]$ into two equal parts, $I_{10} = [a_1, (a_1 + b_1)/2]$ and $I_{11} = [(a_1 + b_1)/2, b_1]$. One of these intervals must contain infinitely many elements of S; pick $I_2 = [a_2, b_2]$ as one of these two intervals containing infinitely many elements of S.

Continue in this fashion indefinitely. That is, if $I_n = [a_n, b_n]$ has already been picked in such a way that it contains infinitely many elements of S, subdivide it into two equal parts, $I_{n0} = [a_n, (a_n + b_n)/2]$ and $I_{n1} = [(a_n + b_n)/2, b_n]$, and let $I_{n+1} = [a_{n+1}, b_{n+1}]$ be one of these intervals containing infinitely many elements of S.

In this way, we obtain an infinite sequence of intervals

$$I_1 \supset I_2 \supset \ldots I_n \supset I_{n+1} \supset \ldots,$$

where $I_n = [a_n, b_n]$. The inclusion $I_{n+1} \subset I_n$ is equivalent to saying that $a_{n+1} \geq a_n$ and $b_{n+1} \leq b_n$. Therefore, the sequence $\{a_n\}_{n=0}^{\infty}$ is increasing,[30] and the sequence $\{b_n\}_{n=0}^{\infty}$ is decreasing. Of course these sequences are also bounded, since for every nonnegative integer n we have $a_0 \leq a_n < b_n \leq b_0$; therefore, these sequences are convergent. Let $a = \lim a_n$ and $b = \lim b_n$.

The length of the interval I_{n+1} is half the length of the interval I_n. Thus, the length of I_n equals $b_n - a_n = (b_0 - a_0)/2^n$, so $\lim(b_n - a_n) = 0$. Therefore, $a = \lim a_n = \lim a_n + \lim(b_n - a_n) = \lim b_n = b$.

We claim that a is a cluster point of S. Indeed, let $\epsilon > 0$ be arbitrary. As $a = \lim a_n = \lim b_n$, there is a positive integer n such that both a_n and b_n belong to the interval $(a-\epsilon, a+\epsilon)$. Thus $I_n = [a_n, b_n] \subset (a-\epsilon, a+\epsilon)$. As I_n contains infinitely many elements of S, it follows that the interval $(a-\epsilon, a+\epsilon)$ also contains infinitely many elements of S. Since $\epsilon > 0$ was arbitrary, this shows that a is a cluster point of S. The proof is complete.

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[29] Reasons that go beyond the scope of this course and involve the foundations of mathematics suggest that it is better to set up a definite rule as to which interval to select rather than make an arbitrary choice: e.g., one can stipulate to pick the interval on the left if both intervals qualify.

[30] in the wider sense, i.e., with equality allowed. Some authors would call such a sequence *nondecreasing*.

12. Continuous functions. An example

DEFINITION. Let (E, d) and (E', d') be two metric spaces, and let f be a function from E into E'. Let $p \in E$. The function f is said to be *continuous* at p if for every $\epsilon > 0$ there is a $\delta > 0$ such that we have $d'(f(p), f(q)) < \epsilon$ whenever $d(p, q) < \delta$.

Note that $f(p)$ and $f(q)$ are in E', so their distance is measured by d', while p and q are in E, so their distance is measured by d. The definition can be formally restated as

$$f \text{ is continuous at } p \leftrightarrow (\forall \epsilon > 0)(\exists \delta > 0)(\forall q \in E)\ [d(p, q) < \delta \to d'(f(p), f(q)) < \epsilon].$$

There are two observation to be made when comparing the formal definition to the definition above. In the verbal definition we say that "f is said to be continuous *if*," while the formal definition uses $\leftrightarrow$ at the same place. Correctly, the verbal definition should say "if and only if," but it is common to say that "something is called something if" or "something is said to be something if" in definitions when "if and only if" would be the correct phrase. In mathematics, one is very careful to say what one means, and so one would virtually never say "if" when one means "if and only if." Definitions containing the above phrase is an exception, where, by tradition, one says "if" and means "if and only if."

Another comment is that the phrase "whenever" in the verbal definition introduces the quantifier $\forall q$, and the context also implies that q is in E (since $d(p, q)$ makes no sense otherwise). It is understood that the variable q needs to be quantified, since it does not occur in the text earlier;[31] the word "whenever" is a cue that q (pun intended) is to be quantified universally. This is why the formal definition contains the quantifier $\forall q \in E$. The clause "such that we have $d'(f(p), f(q)) < \epsilon$ whenever $d(p, q) < \delta$" could have had a clumsier, but more precise phrasing such as "such that for every $q \in E$ we have $d'(f(p), f(q)) < \epsilon$ provided $d(p, q) < \delta$."

In the next example, $\mathbb{R}$ denotes the set of real numbers. $\mathbb{R}$ is a metric space with the distance function $d(x, y) = |x - y|$ for $x, y \in \mathbb{R}$. We will use $\mathbb{Z}$ to denote, as usual, the set of all integers, and $\mathbb{Z}_+$, the set of positive integers.

EXAMPLE. Define the function $f : \mathbb{R} \to \mathbb{R}$ as

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x \text{ is rational and } n \in \mathbb{Z}_+ \text{ is the least possible} \\ & \qquad \text{such that } x = \frac{m}{n} \text{ for some } m \in \mathbb{Z}, \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Then f is continuous at $x \in \mathbb{R}$ if x is irrational, and f is not continuous at x if x is rational.

We will give a proof of the claims in this example. If x is rational then $f(x) \neq 0$ according to the definition. Put $\epsilon = |f(x)|$. Given an arbitrary $\delta > 0$, there is an irrational number y with $|x - y| < \delta$. As $f(y) = 0$, we will not have $|f(x) - f(y)| < \epsilon$ for such a y. That is, no $\delta > 0$ can be found for which the definition of continuity can be satisfied (with $p = x$ and the given ϵ), showing that f is not continuous at x.

Now, assume that x is irrational, and let ϵ with $0 < \epsilon < 1$ be arbitrary. Put

$$S = \left\{ \frac{m}{n} : m \in \mathbb{Z} \ \& \ n \in \mathbb{Z} \ \& \ 0 < n \leq \frac{1}{\epsilon} \ \& \ x - 1 < \frac{m}{n} < x + 1 \right\}.$$

[31] If we do not quantify q, its presence in the text is totally unexplained. In mathematics, the meanings of letters are explained when the letters are first mentioned. So, an unexplained occurrence of a new letter would be puzzling without some understanding how such an occurrence should be interpreted. Quantifying such letters is often a natural thing to do.

Clearly, S is a finite set. This is because, first, the inequality $0 < n \le \frac{1}{\epsilon}$ allows only finitely any choices for n. Second, for a fixed n, the inequality $x - 1 < \frac{m}{n} < x + 1$ is satisfied only for finitely many choices for m. Further, we have $x \notin S$, since all elements of S are rational. Finally, the assumption $\epsilon < 1$ guarantees that S is not empty. Put

$$\delta = \min\{|x - y| : y \in S\}.$$

Then, clearly $0 < \delta < 1$; so $(x - \delta, x + \delta) \subset (x - 1, x + 1)$. Let $y \in (x - \delta, x + \delta)$ be arbitrary. If y is irrational, they $f(y) = 0$. If y is rational, then $f(y) = 1/n$ for some positive integer n. We must have $n > 1/\epsilon$, since otherwise the definition of S would imply $y \in S$, but we have $(x - \delta, x + \delta) \cap S = \emptyset$ by the definition of δ. Thus, we have $0 \le f(y) < \epsilon$ whether or not y is rational. Therefore the inequality $|f(x) - f(y)| < \epsilon$ holds (whenever $|x - y| < \delta$). As $\epsilon > 0$ was arbitrary (except for the harmless restriction that $\epsilon < 1$), it follows that f is continuous at x.

13. Continuous functions

LEMMA. *Let f be a continuous function from the metric space (E, d) to the metric space (E', d'), and let $V \subset E'$ be and open set. Then the set $f^{-1}[V] \stackrel{def}{=} \{p \in E : f(p) \in V\}$ is open.*[32]

PROOF. Consider an arbitrary $p \in f^{-1}[V]$. We need to show that there is an open ball with center p that is included in $f^{-1}[V]$. In other words, we need to show that there is a $\delta > 0$ such that $\{q \in E : d(p, q) < \delta\} \subset f^{-1}[V]$. As V is open and $f(p) \in V$ (since $p \in f^{-1}[V]$), there is an $\epsilon > 0$ such that the open ball $\{u \in E' : d'(f(p), u) < \epsilon\}$ is a subset of V. By the continuity of f, there is a $\delta > 0$ such that $d'(f(p), f(q)) < \epsilon$ holds whenever $d(p, q) < \delta$; in other words,

$$\{f(q) : q \in E \text{ and } d(p, q) < \delta\} \subset \{u \in E' : d'(f(p), u) < \epsilon\}.$$

Since the latter set is a subset of V, this implies that $\{q \in E : d(p, q) < \delta\} \subset f^{-1}[V]$, showing that $f^{-1}[V]$ is indeed open. The proof is complete.

LEMMA. *Let f be a continuous function from the metric space (E, d) to the metric space (E', d'). Assume E is compact. Then the set $f[E] \stackrel{def}{=} \{f(p) : p \in E\}$ is compact.*

PROOF. Let $\mathcal{U}$ be a collection of open subsets of E' such that

$$f[E] \subset \bigcup \mathcal{U}, \tag{4}$$

i.e., such that $\mathcal{U}$ cover $f[E]$. We need to show that there is a finite subcollection of $\mathcal{U}$ that covers $f[E]$. (Recall that the word *collection* is just a synonym for set; so we could have described $\mathcal{U}$ as a set of open subsets of E'.)

First observe that for any collection of $\mathcal{Z}$ of subsets of E',

$$f[E] \subset \bigcup \mathcal{Z} \qquad \text{if and only if} \qquad E \subset \bigcup \{f^{-1}[Z] : Z \in \mathcal{Z}\}. \tag{5}$$

[32]The notation $f^{-1}(V)$ is also used; however, f^{-1} is often used to denote the inverse of the function f, and the square bracket is used to avoid the confusion this may lead to.

Indeed, the formula on the left says that for every $p \in E$ there is a $Z \in \mathcal{Z}$ such that $f(p) \in Z$, while the formula on the right says that for every $p \in E$ there is a $Z \in \mathcal{Z}$ such that $p \in f^{-1}[Z]$. These two statements are, however, identical, since $f(p) \in Z$ if and only if $p \in f^{-1}[Z]$ (in view of the definition of the latter set).

In virtue of (5), formula (4) means that

$$E \subset \bigcup \{f^{-1}[U] : U \in \mathcal{U}\}.$$

By the continuity of f and the openness of the sets $U \in \mathcal{U}$, the sets $f^{-1}[U]$ are open (see the preceding Lemma), i.e., the set on the right-hand side is an open cover of E. Thus, by the compactness of E, it has a finite subset that is also an open cover; every subset of $\{f^{-1}[U] : U \in \mathcal{U}\}$ has the form $\{f^{-1}[U] : U \in \mathcal{U}'\}$ for some $\mathcal{U}' \subset \mathcal{U}$; a finite subset of the former set has this form with a finite $\mathcal{U}'$. Hence, what we said about the existence of a finite subset that is an open cover can be rephrased as follows: there is a finite $\mathcal{U}' \subset \mathcal{U}$ such that

$$E \subset \bigcup \{f^{-1}[U] : U \in \mathcal{U}'\}.$$

By (5) means that

$$f[E] \subset \bigcup \mathcal{U}';$$

i.e., $\mathcal{U}'$ is a finite open cover of $f[E]$. Since our aim was to show the existence of such a finite open cover, the proof is complete.

14. The Intermediate-Value Theorem

THE INTERMEDIATE-VALUE THEOREM. *Let $I = [a, b]$ be a closed interval in $\mathbb{R}$, and let f be a continuous real-valued function on I. Then f assumes every value between $f(a)$ and $f(b)$.*

Note that the theorem follows from results we already know. Namely, the range of a continuous function defined on a connected metric space is connected. As the interval I is a connected metric space, it follows that $S = \{f(x) : x \in I\}$ is a connected set. If there were a real $c \notin S$ with $f(a) < c < f(b)$ (in case $f(a) < f(b)$) or $f(a) > c > f(b)$ (in case $f(a) > f(b)$), then the set S would not be connected, so such a c cannot exist. The purpose here is to give a more direct proof.

PROOF. Assume $f(a) < f(b)$; the case $f(a) > f(b)$ can be handled in a similar way, or else, once the result in the former case has been established, one can prove it in the latter case by using the result for the function $-f$ (since $-f$ is continuous on I and $-f(a) < -f(b)$ in the second case). Note that the Theorem also allows the possibility that $f(a) = f(b)$, but in this case the assertion of the Theorem (that $f(\xi) = f(a) = f(b)$ for some $\xi \in I$) is trivial (namely, one can take $\xi = a$ or $\xi = b$).

Now, let c be an arbitrary real number with $f(a) < c < f(b)$, and write

$$M = \{x \in I : f(x) < c\}.$$

Note that M is bounded (since $M \subset I$) and nonempty (since $a \in M$). Let $\xi = \sup M$.

Claim. *We have $f(\xi) = c$.*

This Claim will of course establish the Theorem. To show the Claim, let $\epsilon > 0$ be arbitrary. Then, by the continuity of f, there is a $\delta > 0$ such that for every $x \in I$ with $\xi - \delta < x < \xi + \delta$ we have

$$f(\xi) - \epsilon < f(x) < f(\xi) + \epsilon. \tag{6}$$

Write $I_0 = (\xi - \delta, \xi]$, and pick x_0 be as an (arbitrary) element of the set $I_0 \cap M$; this is possible, because the set $I_0 \cap M$ is not empty. Indeed, this is so in case $a \in I_0$, since we also have $a \in M$ (as $f(a) < c$ by our assumptions). In case $a \notin I_0$ we have $\xi - \delta \geq a$. As ξ is an upper bound of M, if the set $I_0 \cap M$ were empty, then $\xi - \delta$ would also be an upper bound of M, contradicting the choice $\xi = \sup M$. Note that we have $f(x_0) < c$ in view of $x_0 \in M$. As $x_0 \in I_0 \cap I$ (note that $M \subset I$), (6) holds with x_0 replacing x. Thus, using only the left-hand side of (6), we obtain

$$f(\xi) - \epsilon < f(x_0) < c. \tag{7}$$

Next, write $I_1 = [\xi, \xi + \delta)$, and pick x_1 as an (arbitrary) element of the set $(I_1 \cap I) \setminus M$. This is possible, since this set is not empty. Indeed, in case $b \in I_1$ we have $b \in (I_1 \cap I) \setminus M$, since $b \notin M$ according to the assumption $c < f(b)$. In case $b \notin I_1$ (when $\xi + \delta \leq b$, i.e., $I_1 \subset I$), $x \notin M$ is true for every $x \in I_1 \setminus \{\xi\}$, since ξ is an upper bound of M (and all elements of $I_1 \setminus \{\xi\}$ are greater than ξ). Note that we have $c \leq f(x_1)$, since $x_1 \notin M$. As $x_1 \in I_1 \cap I$, (6) holds with x_1 replacing x. Thus, using only the right-hand side of (6), we obtain

$$c \leq f(x_1) < f(\xi) + \epsilon.$$

Together with (7), this shows that

$$f(\xi) - \epsilon < c < f(\xi) + \epsilon.$$

Since $\epsilon > 0$ was arbitrary, this implies that $f(\xi) = c$ (since in case $f(\xi) \neq c$ this inequality would be contradicted by the choice $\epsilon = |f(\xi) - c| > 0$), establishing the Claim.

This shows that there is a $\xi \in I$ with $f(\xi) = c$, completing the proof.

15. The method of interval-halving: the Intermediate-Value Theorem

The Intermediate-Value Theorem. *Let $I = [a, b]$ be a closed interval in $\mathbb{R}$, and let f be a continuous real-valued function on I. Then f assumes every value between $f(a)$ and $f(b)$.*

The proof we will give of this result uses an argument that can be called the method of interval-halving. Other proofs can also be given.

Proof. For the sake of simplicity, assume $f(a) < 0 < f(b)$; we will prove that there is a $\xi \in (a, b)$ such that $f(\xi) = 0$. This is not a serious restriction of generality, since the argument can easily be reformulated so as to apply to the general case. Alternatively, if f is continuous on $[a, b]$ and c is such $f(a) < c < f(b)$ (or $f(a) > c > f(b)$), then the function g defined by $g(x) = f(x) - c$ (or $g(x) = c - f(x)$) for $x \in [a, b]$ is continuous on $[a, b]$, and $g(a) < 0$ and $g(b) > 0$. So, the assertion we intend to prove, with g replacing f, shows that there is a $\xi \in (a, b)$ for which $g(\xi) = 0$; then $f(\xi) = c$.

Write $a_0 = a$, $b_0 = b$, and write $I = I_0$. I.e., I_0 is the interval $[a_0, b_0]$. Divide I_0 into two equal parts, $I_{00} = [a_0, (a_0 + b_0)/2]$ and $I_{01} = [(a_0 + b_0)/2, b_0]$. If $f((a_0 + b_0)/2) = 0$, then $\xi = (a_0+b_0)/2$ fulfills the requirements, and there nothing more to do. If $f((a_0+b_0)/2)) > 0$ then pick I_{00} as I_1; if $f((a_0+b_0)/2)) < 0$ then pick I_{01} as I_1. Write $I_1 = [a_1, b_1]$; then, clearly, $f(a_1) < 0$ and $f(b_1) > 0$. Again, subdivide the interval $I_1 = [a_1, b_1]$ into two equal parts, $I_{10} = [a_1, (a_1 + b_1)/2]$ and $I_{11} = [(a_1 + b_1)/2, b_1]$. If $f((a_1 + b_1)/2) = 0$, then $\xi = (a_1 + b_1)/2$ fulfills the requirements, and there nothing more to do. If $f((a_1 + b_1)/2)) > 0$ then pick I_{10} as I_2; if $f((a_1 + b_1)/2)) < 0$ then pick I_{11} as I_2. Write $I_2 = [a_2, b_2]$; then, clearly, $f(a_2) < 0$ and $f(b_2) > 0$.

Continue in this fashion indefinitely. That is, if $I_n = [a_n, b_n]$ has already been picked in such a way that $f(a_n) < 0$ and $f(b_n) > 0$, then subdivide it into two equal parts, $I_{n0} = [a_n, (a_n+b_n)/2]$ and $I_{n1} = [(a_n+b_n)/2, b_n]$. If $f((a_n+b_n)/2) = 0$, then $\xi = (a_n+b_n)/2$ fulfills the requirements, and there nothing more to do. If $f((a_n + b_n)/2)) > 0$ then pick I_{n0} as I_{n+1}; if $f((a_n + b_n)/2)) < 0$ then pick I_{n1} as I_{n+1}. Write $I_{n+1} = [a_{n+1}, b_{n+1}]$; then, clearly, $f(a_{n+1}) < 0$ and $f(b_{n+1}) > 0$.

In this way, either in finitely many steps we end up with a ξ such that $f(\xi) = 0$, or the procedure goes on indefinitely. In the latter case we obtain an infinite sequence of intervals

$$I_1 \supset I_2 \supset \dots I_n \supset I_{n+1} \supset \dots ,$$

where $I_n = [a_n, b_n]$. The inclusion $I_{n+1} \subset I_n$ is equivalent to saying that $a_{n+1} \geq a_n$ and $b_{n+1} \leq b_n$. Therefore, the sequence $\{a_n\}_{n=0}^{\infty}$ is increasing,[33] and the sequence $\{b_n\}_{n=0}^{\infty}$ is decreasing. Of course these sequences are also bounded, since for every nonnegative integer n we have $a_0 \leq a_n < b_n \leq b_0$; therefore, these sequences are convergent. Let $\xi = \lim a_n$ and $\eta = \lim b_n$.

The length of the interval I_{n+1} is half the length of the interval I_n. Thus, the length of I_n equals $b_n - a_n = (b_0 - a_0)/2^n$, and so $\lim(b_n - a_n) = 0$. Therefore, $\xi = \lim a_n = \lim a_n + \lim(b_n - a_n) = \lim b_n = \eta$.

Since $0 > f(a_n)$ for all positive integers n, using the continuity of f we obtain[34]

$$0 \geq \lim f(a_n) = f(\lim a_n) = f(\xi).$$

Similarly, using $0 < f(b_n)$ we obtain

$$0 \leq \lim f(b_n) = f(\lim b_n) = f(\eta) = f(\xi).$$

Thus $f(\xi) = 0$. From the construction it is clear that $\xi \in [a, b]$. However, $\xi = a$ is not possible since $f(a) < 0$, and $\xi = b$ is not possible since $f(b) > 0$. Thus $\xi \in (a, b)$. The proof is complete.

16. The Maximum-Value Theorem.

The Maximum-Value Theorem. *Let $I = [a, b]$ be a closed interval in $\mathbb{R}$, and let f be a continuous real-valued function on I. Then f assumes its maximum on $[a, b]$. That is, there is a $\xi \in [a, b]$ such that $f(\xi) \geq f(x)$ for every $x \in [a, b]$.*

Here f being continuous on $[a, b]$ means that f a continuous function on the metric space $[a, b]$. Note that this does not mean that f, considered as a function defined at some points

[33] in the wider sense, i.e., with equality allowed. Some authors would call such a sequence *nondecreasing.*

[34] For the last equality next, see Rosenlicht, the Proposition on p. 74.

of $\mathbb{R}$, is continuous at every point $x \in [a, b]$. The reason for this is that an open set of the metric space $[a, b]$ does not need to be an open set in $\mathbb{R}$. For example, for every $c \in (a, b)$, the semi-closed interval $[a, c)$ is an open subset of the metric space $[a, b]$, but it is not an open subset of $\mathbb{R}$.

One can reinterpret continuity of f on $[a, b]$ in terms of pointwise continuity of f in $\mathbb{R}$ as follows: f is continuous on $[a, b]$ if f is and continuous at every $t \in (a, b)$ (i.e., $f(t) = \lim_{x\to t} f(x)$), continuous from the right at a (i.e., $f(a) = \lim_{x\to a^+} f(x) \overset{def}{=} \lim_{\substack{x\to a\\ x>a}} f(x)$), continuous from the left at b (i.e., $f(b) = \lim_{x\to b^-} f(x) \overset{def}{=} \lim_{\substack{x\to b\\ x<b}} f(x)$).

Note that the theorem follows from results we already know. Namely, $[a, b]$ is compact, and the range of a continuous function defined on a compact subset is compact; hence, the range of f, being a compact set of real numbers, i.e., a closed and bounded subset of $\mathbb{R}$, has a maximum. Here we will give a proof based on the Bolzano–Weierstrass Theorem.

Proof. For each positive integer n define $x_n \in [a, b]$ as follows: pick x_n such that $f(x_n) > n$ if possible. If there is no such x_n, in which case the set

$$S = \{f(x) : x \in [a, b]\}$$

is bounded from above (by n), write $M = \sup S$; pick x_n such that $f(x_n) > M - 1/n$ (this is possible, since the interval $(M - 1/n, M]$ must contain an element of S). If the set is $A = \{x_n : 1 \le n < \infty\}$ is finite, then the same number, say ξ, must occur infinitely many times among the numbers x_n. If the set is A is infinite, then it has a cluster point ξ by the Bolzano-Weierstrass theorem. In either case, the sequence $\{x_n\}_{n=1}^{\infty}$ has a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ that converges to ξ. (Pick an increasing[35] sequence of positive integers n_k such that, in the first case, $x_{n_k} = \xi$, and, in the second case, $x_{n_k} \in (\xi - 1/k, \xi + 1/k)$ for every positive integer k.)

As $x_{n_k} \in [a, b]$ for every positive integer k, the limit ξ of the sequence $\{x_{n_k}\}_{k=1}^{\infty}$ must belong to (the closed set) $[a, b]$. As f is continuous on $[a, b]$, we have[36]

$$f(\xi) = f(\lim_{k\to\infty} x_{n_k}) = \lim_{k\to\infty} f(x_{n_k}). \tag{8}$$

As an immediate consequence of this is that the set S is bounded. Indeed, assuming S is not bounded, we must have made the choice $f(x_n) > n$ for every integer n. Then

$$f(x_{n_k}) > n_k \ge k$$

(the second inequality holds since the sequence $\{n_k\}_{k=1}^{\infty}$ of integers is increasing); hence the sequence $\{f(x_{n_k})\}_{k=1}^{\infty}$ cannot be convergent, contradicting (8).

The set S is bounded, so for $n > M = \sup S$ we must have chosen x_n such that $f(x_n) > M - 1/n$. As $f(x) \le M$ for every $x \in [a, b]$ by the definition of M, we have

$$M \ge f(x_{n_k}) > M - \frac{1}{n_k} \ge M - \frac{1}{k}$$

(the last inequality holds since $n_k \ge k$); hence, from (8) we can conclude that $f(\xi) = M$. As M is the least upper bound of the set $S = \{f(x) : x \in [a, b]\}$, we have $M = f(\xi) \ge f(x)$ for every $x \in [a, b]$. That is, f assumes its maximum in the interval $[a, b]$ at ξ. The proof is complete.

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[35] strictly increasing, i.e., $n_k < n_{k+1}$.

[36] For the second equality next, see Rosenlicht, the Proposition on p. 74.

17. Uniform continuity, uniform convergence

Let (E,d) and (E',d') be metric spaces.

Definition. The function $f : E \to E'$ is said to be continuous on E if for every $p \in E$ and for every $\epsilon > 0$ there is a $\delta > 0$ such that for every $q \in E$, if $d(p,q) < \delta$ then $d'(f(p), f(q)) < \epsilon$.

Note. The above definition says that f is continuous at p for every $p \in E$. Using logic notation, the function $f : E \to E'$ is said to be continuous on E if

$$(\forall p \in E)(\forall \epsilon > 0)(\exists \delta > 0)(\forall q \in E)\big(d(p,q) < \delta \to d'(f(p), f(q)) < \epsilon\big).$$

The first two quantifiers are interchangeable here, since two quantifiers of the same type (i.e., two universal quantifiers, or two existential quantifiers) are interchangeable, so we can write this also as

$$(9) \qquad (\forall \epsilon > 0)(\forall p \in E)(\exists \delta > 0)(\forall q \in E)\big(d(p,q) < \delta \to d'(f(p), f(q)) < \epsilon\big).$$

Definition. The function $f : E \to E'$ is said to be uniformly continuous on E if for every $\epsilon > 0$ there is a $\delta > 0$ such that for every $p \in E$ and for every $q \in E$, if $d(p,q) < \delta$ then $d'(f(p), f(q)) < \epsilon$.

Note. Using logic notation, the function $f : E \to E'$ is said to be uniformly continuous on E if

$$(10) \qquad (\forall \epsilon > 0)(\exists \delta > 0)(\forall p \in E)(\forall q \in E)\big(d(p,q) < \delta \to d'(f(p), f(q)) < \epsilon\big).$$

The formal difference between continuity and uniform continuity is indicated by the different order of the second and third quantifiers in formulas (9) and (10); aside from this difference, the two formulas are identical. However, these quantifiers are of different type (one is a universal quantifier, the other is an existential quantifier), and so they are not interchangeable. Hence the meanings of these two formulas are different.

To explain the difference in a less formal way, in case of continuity, δ depends on the choice of p as well as on ϵ (and, of course, on the function f itself), whereas in case of uniform continuity, δ depends only on ϵ but not on p (but it does depend on the function f itself). We will illustrate the situation with an

Example. The function $f(x) = 1/x$ is continuous but not uniformly continuous nn the interval $(0,1)$.

Proof. For every $x_0 \in (0,1)$ we have

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} \frac{1}{x} = \frac{\lim_{x \to x_0} 1}{\lim_{x \to x_0} x} = \frac{1}{x_0} = f(x_0),$$

showing that, indeed, f is continuous at every $x_0 \in (0,1)$. As for uniform continuity, we will show that formula (10) is not valid in the present example, that is, its negation is true:

$$\neg(\forall \epsilon > 0)(\exists \delta > 0)(\forall p \in E)(\forall q \in E)\big(d(p,q) < \delta \to d'(f(p), f(q)) < \epsilon\big).$$

In other words, we have

$$(\exists \epsilon > 0)(\forall \delta > 0)(\exists p \in E)(\exists q \in E)\big(d(p,q) < \delta \ \ \& \ \ d'(f(p), f(q)) > \epsilon\big).$$

This latter formula can be written down by following simple formal rules in moving the negation sign $\neg$ inside the formula (only rules (i), (ii), and (v) are needed for now); see also the discussion moving the negation inside in formula (1) in Section 2 on p. 5 for more details.

(i) $\neg(\forall x)$ can be replaced by $(\exists x)\neg$,
(ii) $\neg(\exists x)$ can be replaced by $(\forall x)\neg$,
(iii) $\neg(A \ \& \ B)$ can be replaced by $\neg A \vee \neg B$,
(iv) $\neg(A \vee B)$ can be replaced by $\neg A \ \& \ \neg B$,
(v) $\neg(A \to B)$ can be replaced by $A \ \& \ \neg B$.
(vi) $\neg(A \leftrightarrow B)$ can be replaced by $A \leftrightarrow \neg B$,
(vii) $\neg(\neg A)$ can be replaced by A.

In showing that the last displayed formula is true, note that we have $f(x) = 1/x$, $d(p,q) = |p-q|$, and $d'(p',q') = |p'-q'|$ and $E = (0,1)$ in the present case; the formula above becomes

$$(\exists\epsilon > 0)(\forall\delta > 0)\big(\exists p \in (0,1)\big)\big(\exists q \in (0,1)\big)\left(|p-q| < \delta \ \& \ \left|\frac{1}{p} - \frac{1}{q}\right| \geq \epsilon\right).$$

To show that this formula above is true, simply pick $\epsilon = 1$, then pick an arbitrary $\delta > 0$, then pick a $p \in (0,1)$ with $p < \delta$, then pick $q = p/2$. We have

$$\left|\frac{1}{p} - \frac{1}{q}\right| = \left|\frac{1}{p} - \frac{1}{p/2}\right| = \left|\frac{1}{p} - \frac{2}{p}\right| = \left|-\frac{1}{p}\right| = \frac{1}{p} > 1;$$

the last equality and the inequality holds because $p \in (0,1)$. As $\epsilon = 1$, this shows that we certainly have

$$\left|\frac{1}{p} - \frac{1}{q}\right| \geq \epsilon$$

As

$$|p-q| = \left|p - \frac{p}{2}\right| = \frac{p}{2} < \delta,$$

this shows that the above formula is indeed true, confirming that $f(x) = 1/x$ is not uniformly continuous in $(0,1)$.

Definition. The sequence $\{f_n\}_{n=1}^\infty$ of functions $f_n : E \to E'$ is said to converge to f if for every $p \in E$ and for every $\epsilon > 0$ there is an $N \in \mathbb{R}$ such that for every integer $n > N$ we have $d'(f_n(p), f(p)) < \epsilon$.

Note. The convergence described is called *pointwise* convergence. Pointwise convergence means that there is convergence at every point. In other words, for every $p \in E$,

$$\lim_{n\to\infty} f_n(p) = f(p).$$

Using logic notation, the sequence of functions $\{f_n\}_{n=1}^\infty$ is said to converge to the function f if

$$(\forall p \in E)(\forall\epsilon > 0)(\exists N \in \mathbb{R})(\forall n > N)\ d'(f_n(p), f(p)) < \epsilon.$$

The first two quantifiers are interchangeable here, so we can write this also as

$$(\forall\epsilon > 0)(\forall p \in E)(\exists N \in \mathbb{R})(\forall n > N)\ d'(f_n(p), f(p)) < \epsilon. \tag{11}$$

Definition. The sequence $\{f_n\}_{n=1}^\infty$ of functions $\{f_n\}_{n=1}^\infty$ is said to converge to f uniformly if for every $\epsilon > 0$ there is an $N \in \mathbb{R}$ such that for every integer $n > N$ for every $p \in E$ we have $d'(f_n(p), f(p)) < \epsilon$.

Note. Formally, the sequence $\{f_n\}_{n=1}^{\infty}$ of functions is said to converge to f uniformly if

$$(\forall\epsilon > 0)(\exists N \in \mathbb{R})(\forall n > N)\ (\forall p \in E)d'(f_n(p), f(p)) < \epsilon.$$

The third and fourth quantifiers here are of the same type, and so they can be interchanged:

$$(\forall\epsilon > 0)(\exists N \in \mathbb{R})(\forall p \in E)(\forall n > N)\ d'(f_n(p), f(p)) < \epsilon. \tag{12}$$

The difference between convergence and uniform convergence can most easily be appreciated by noting the difference in the orders of the second and third quantifiers in formulas (11) and (12). In other words, in case of pointwise convergence, N depends on p and ϵ (and, of course, on the sequence $\{f_n\}_{n=1}^{\infty}$ of functions), whereas in case of uniform convergence, N depends only on ϵ (and, of course, on the sequence $\{f_n\}_{n=1}^{\infty}$ of functions).

18. Differentiability and continuity

We will show that differentiability implies continuity. A typical result is the following. A similar result can be formulated to say that one-sided differentiability implies one-sided continuity.

Lemma. *Let f be a real-valued function on an open subset U of $\mathbb{R}$, and let $x_0 \in U$. Assume that f is differentiable at x_0. Then f is continuous at x_0.*

We will give two proofs. The first proof relies on the property that the limit of a product equals the product of the limits.

First proof. Write $f'(x_0)$ for the derivative of f at x_0. We have

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}. \tag{13}$$

Therefore,

$$\begin{aligned}\lim_{x\to x_0} (f(x) - f(x_0)) &= \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0)\\ &= \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x\to x_0} (x - x_0) = f'(x_0) \cdot 0 = 0.\end{aligned}$$

Hence

$$\begin{aligned}\lim_{x\to x_0} f(x) &= \lim_{x\to x_0} \big((f(x) - f(x_0) + f(x_0)\big)\\ &= \lim_{x\to x_0} (f(x) - f(x_0)) + \lim_{x\to x_0} f(x_0) = 0 + f(x_0) = f(x_0).\end{aligned}$$

This shows that f is indeed continuous at x_0, completing the proof.

The second proof proceeds directly from the definition of continuity.

Second proof. Write $f'(x_0)$ for the derivative of f at x_0. As before, we then have (13); that is, for every $\epsilon_0 > 0$ there is a $\delta_0 > 0$ such that

$$\left|\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0)\right| < \epsilon_0$$

for every $x \in (x_0 - \delta_0, x_0 + \delta_0)$ with $x \neq x_0$. Pick an $\epsilon_0 > 0$ (any such ϵ_0 will do; e.g., we can simply pick $\epsilon_0 = 1$), and let $\delta_0 > 0$ be a number such that the above inequality holds for every $x \in (x_0 - \delta_0, x_0 + \delta_0)$ with $x \neq x_0$ (by choosing δ_0 small enough, this interval will be a subset of U, as U is open). Multiplying both sides by $|x - x_0|$, we obtain

$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| \leq \epsilon_0 |x - x_0|$$

for every $x \in (x_0 - \delta_0, x_0 + \delta_0)$. We relaxed the inequality from $\leq$ to $<$; as a result, we can now allow $x = x_0$, since we obviously have equality here in that case.

Let $\epsilon > 0$ be arbitrary, and pick any x in the interval $(x_0 - \delta_0, x_0 + \delta_0)$. First using the triangle inequality, and then using the last displayed inequality above, we obtain that

$$\begin{aligned} |f(x) - f(x_0)| &= \big|\big(f(x) - f(x_0) - f'(x_0)(x - x_0)\big) + f'(x_0)(x - x_0)\big| \\ &\leq |f(x) - f(x_0) - f'(x_0)(x - x_0)| + |f'(x_0)(x - x_0)| \\ &\leq \epsilon_0 |x - x_0| + |f'(x_0)| \cdot |x - x_0| = (\epsilon_0 + |f'(x_0)|) \cdot |x - x_0|. \end{aligned}$$

If we choose $\delta = \min\left\{\delta_0, \frac{\epsilon}{\epsilon_0 + |f'(x_0)|}\right\}$, then, noting that $|x - x_0| < \delta$, the right-hand side here is less than ϵ, ensuring that the left-hand side, i.e., $|f(x) - f(x_0)|$, is also less than ϵ.

To summarize, given an arbitrary $\epsilon > 0$, we found a $\delta > 0$ such that for every x with $|x - x_0| < \delta$ we have $|f(x) - f(x_0)| < \epsilon$. This means that f is indeed continuous at x_0. The proof is complete

19. An Intermediate-Value Theorem for derivatives

Sometimes it is useful to consider one-sided limits.

Definition. Let f be a real-valued function on $\mathbb{R}$. Given $t \in \mathbb{R}$ and $L \in \mathbb{R}$, we say that $\lim_{x \to t+} f(x) = L$, or that the limit on the right (or that the right-sided limit) of $f(x)$ at t is L, if for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $t < x < t + \delta$. Formally,

$$\lim_{x \to t+} f(x) = L \leftrightarrow (\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})\ (t < x < t + \delta \to |f(x) - L| < \epsilon).$$

Limits on the left, written as $\lim_{x \to t-} f(x)$, can be defined similarly. These are used in defining continuity of a function on a closed interval.

Definition. Let $[a, b]$ be a closed interval in $\mathbb{R}$, and let $f : [a, b] \to \mathbb{R}$ be a function. f is said to be continuous on $[a, b]$ if $f(t) = \lim_{x \to t} f(x)$ for every $t \in (a, b)$, $f(a) = \lim_{x \to a+} f(x)$, and $f(b) = \lim_{x \to a-} f(x)$

This definition is natural in the following sense: if $[a, b]$ is considered a metric space with the metric it *inherits* from $\mathbb{R}$ (i.e. $d(x, y) = |x - y|$ for $x, y \in [a, b]$), then f is a continuous function in the sense of being a continuous function *from one metric space into another* if and only if it is continuous on $[a, b]$ in the sense of the above definition. We will leave the verification of this observation to the reader.

With the aid of one-sided limits, one can define one-sided derivatives:

$$f'_+(x_0) = \lim_{x \to x_0+} \frac{f(x) - f(x_0)}{x - x_0}, \qquad f'_-(x_0) = \lim_{x \to x_0-} \frac{f(x) - f(x_0)}{x - x_0};$$

We call these derivative *from the right*, and derivative *from the left*, respectively. Above it was established that differentiability implies continuity; in an entirely similar way, it can be established that one-sided differentiability implies one-sided continuity (on the same side; that is differentiability on the left implies continuity on the left, for example). They can be used to define differentiability on a closed interval.

Definition. Let $[a, b]$ be a closed interval in $\mathbb{R}$, and let $f : [a, b] \to \mathbb{R}$ be a function. f is said to be differentiable on $[a, b]$ if $f'(x)$ exists for every $x \in (a, b)$, and $f'_+(a)$ and $f'_-(b)$ exist.

We have the following

Intermediate-Value Theorem for Derivatives. *Assume f is a real-valued differentiable function on $[a, b]$, and let c be a number that is strictly between $f'_+(a)$ and $f'_-(b)$; that is, $f'_+(a) < c < f'_-(b)$ in case $f'_+(a) < f'_-(b)$, and $f'_+(a) > c > f'_-(b)$ in case $f'_+(a) > f'_-(b)$. Then there is a $\xi \in (a, b)$ such that $f'(\xi) = c$.*

The point is that the conclusion holds in spite of the fact that f' is not assumed to be, and indeed, need not be, continuous. If f' were continuous, then the conclusion could be obtained from the Intermediate-Value Theorem for continuous functions. Note that if $f'_+(a) = f'_-(b) = c$, we cannot conclude the existence of a $\xi \in (a, b)$ with $f'(x) = c$, as shown by the example of $f(x) = \cos x$ on $[0, \pi]$.

It is sufficient to prove the following special case:

Lemma. *Assume f is a real-valued differentiable function on $[a, b]$, and $f'_+(a) > 0$ and $f'_-(b) < 0$. Then there is a $\xi \in (a, b)$ such that $f'(\xi) = 0$.*

Indeed, if f and c satisfies the assumptions of the above theorem, we can use the Lemma for $F(x) = f(x) - cx$ (in case $f'_+(a) > c$) or $F(x) = cx - f(x)$ (in case $f'_+(a) < c$) to obtain the desired conclusion.

Proof of the Lemma. From $f'_+(a) > 0$ it follows that $f(x) > f(a)$ for some $x \in (a, b)$. Indeed, otherwise the quantity

$$\frac{f(x) - f(a)}{x - a}$$

would be ≤ 0 for every $x \in (a, b)$, so its limit could not be positive as x approaches a from the right. Similarly, from $f'_-(b) < 0$ it follows that $f(x) > f(b)$ for some $x \in (a, b)$.

Now, the differentiability of f on $[a, b]$ implies that the continuity of f on $[a, b]$; hence, by the Maximum-Value Theorem, f assumes its maximum on $[a, b]$; that is, there is a $\xi \in [a, b]$ such that $f(\xi) \ge f(x)$ for every $x \in [a, b]$. By what we said in the first paragraph of this proof, $\xi \ne a$ and $\xi \ne b$; that is, $\xi \in (a, b)$. Then $f'(\xi) = 0$ follows by Fermat's Theorem saying that the derivative at a local maximum is zero (if it exists).[37]

We will also give a different proof of the above Theorem. The proof relies on the fact that the a triangle in the plane (with its inside and some sides included) is a connected set.

Second proof of the Theorem. Consider the set

$$A = \{(x, y) : a \le x < y \le b\}$$

as a metric space with the distance function given by the distance in the Euclidean place. Then A is a connected metric space.[38] The function $g(x, y)$ defined on A by[39]

$$g(x, y) = \frac{f(y) - f(x)}{y - x}$$

is continuous. We have $\lim_{y \to a+} g(a, y) = f'_+(a)$ and $\lim_{x \to b-} g(x, b) = f'_-(b)$. Assuming, for the sake of simplicity that $f'_+(a) < c < f'_-(b)$, it follows that there are $x_0 \in [a, b)$ and

[37] Fermat's Theorem is given in Rosenlicht [Ros, p. 103], as the Proposition at the bottom of the page.

[38] This is an easy consequence of the second Proposition in Rosenlicht [Ros, p. 60]. However, since this proposition is often omitted in a discussion of one-dimensional analysis, we will indicate how to avoid using the connectedness of this set.

[39] Since the elements of A consist of pairs of form (x, y), the overly pedantic may write $g((x, y))$ instead of $g(x, y)$. Also, note that (x, y) can denote an element of the set A, and it can also denote the open interval (x, y); the fastidious may write $\langle x, y \rangle$ for the former, to avoid this notational ambiguity.

$y_0 \in (a, b]$ such that $g(a, y_0) < c$ and $c < g(x_0, b)$.[40] Then, by the form of the Intermediate-Value Theorem for continuous functions on connected spaces,[41] we find that there is a point $(x_1, y_1) \in A$ such that $g(x_1, y_1) = c$. Then, using the Mean-Value Theorem of differentiation on the interval (x_1, y_1) with the function f, we can conclude that there is a $\xi \in (x_1, y_1)$ such that

$$f'(\xi) = \frac{f(y_1) - f(x_1)}{y_1 - x_1} = g(x_1, y_1) = c.$$

The proof is complete.

Note. The connectedness of the set A was used to establish the existence of a point $(x_1, y_1) \in A$ such that $g(x_1, y_1) = c$. This can be established also by repeated uses of the Intermediate-Value Theorem stated only for functions on real intervals as follows.

Starting out the same way as in the above proof, and making the assumption $f'_+(a) < c < f'_-(b)$ as before, we can conclude the existence of numbers $x_0 \in [a, b)$ and $y_0 \in (a, b]$ such that $g(a, y_0) < c$ and $c < g(x_0, b)$. Now, if $g(a, b) > c$, then, using the Intermediate-Value Theorem for the function $h(t) = g(a, t)$ and noting that $h(y_0) = g(a, y_0) < c$ and $h(b) = g(a, b) > c$, we can see that there is a $t_0 \in (y_0, b)$ such that $h(t_0) = c$. Then we can take $(x_1, y_1) = (a, t_0)$ in the above proof, since $g(a, t_0) = h(t_0) = c$.

If, on the other hand, we have $g(a, b) < c$, then we can use the Intermediate-Value Theorem for the function $k(t) = g(t, b)$. Noting that $k(a) = g(a, b) < c$ and $k(x_0) = g(x_0, b) > c$ in this case, it follows that there is a $t_1 \in (a, x_0)$ such that $k(t_1) = c$. Then we can take $(x_1, y_1) = (t_1, b)$ in the above proof, since $g(t_1, b) = k(t_1) = c$.

Finally, if $g(a, b) = c$, we can take $(x_1, y_1) = (a, b)$ in the above proof.

20. The chain rule for differentiation

The composition $g \circ f$ of the functions g and f, is defined by stipulating that $(g \circ f)(x) = g(f(x))$ for every x for which the right-hand side is meaningful. More precisely,

Definition. Given any two functions f and g, we define the function $g \circ f$ by writing

$$\mathrm{dom}(g \circ f) \overset{def}{=} \{x \in \mathrm{dom}(f) : f(x) \in \mathrm{dom}(g)\},$$

and putting $(g \circ f)(x) \overset{def}{=} g(f(x))$ whenever $x \in \mathrm{dom}(g \circ f)$.

There is a simple rule, called the *Chain Rule*, for differentiating the composition of two functions saying that

$$(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0).$$

In a naive derivation of the chain rule, one would write

$$\begin{aligned}(g \circ f)'(x_0) &= \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} \\ &= \lim_{x \to x_0} \left(\frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \cdot \frac{f(x) - f(x_0)}{x - x_0} \right),\end{aligned}$$

but this is not correct, since it cannot be guaranteed that there is an $\eta > 0$ such that $f(x) - f(x_0) \neq 0$ for every $x \in (x_0 - \eta, x_0 + \eta) \setminus \{x_0\}$; i.e., it might not be possible to find an $\eta > 0$ such that the expression after the limit on the right-hand side is defined for every $x \in (x_0 - \eta, x_0 + \eta) \setminus \{x_0\}$.

To avoid this difficulty, one has to take a somewhat more refined approach. We start with giving a precise statement of the result.

[40] Observe that the points (a, y_0) and (x_0, b) belong to the set A.

[41] See the Theorem on p. 82 in Rosenlicht [Ros]; also see the discussion of the theorem on p. 83.

The Chain Rule for Differentiation. *Let U and V be open subsets of $\mathbb{R}$, and let f and g be real-valued functions such that $U \subset \operatorname{dom}(f) \subset \mathbb{R}$ and $V \subset \operatorname{dom}(g) \subset \mathbb{R}$. Let $x_0 \in U$ and assume $f(x_0) \in V$. Assume, further, that f is differentiable at x_0 and g is differentiable at $f(x_0)$. Then*

$$(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0). \tag{14}$$

Proof. To begin with, notice that there is an open set $U' \subset \mathbb{R}$ such that $x_0 \in U'$ and $U' \subset \operatorname{dom}(g \circ f)$. Indeed, write $y_0 = f(x_0)$, and let $\epsilon > 0$ be such that the interval $V' = (y_0 - \epsilon, y_0 + \epsilon)$ is a subset of V; there is such an ϵ since the set V is open and $y_0 \in V$. Observing that f is continuous at x_0 since, by our assumptions, it is differentiable there, there is a $\delta > 0$ such that $f(x) \in V'$ for every $x \in (x_0 - \delta, x_0 + \delta)$. Writing $U' = (x_0 - \delta, x_0 + \delta)$, it is clear that U' is an open set such that

$$x_0 \in U' \subset \operatorname{dom}(g \circ f),$$

as claimed.

Define the function h on V by putting

$$h(y) = \begin{cases} \frac{g(y) - g(y_0)}{y - y_0} & \text{for } y \in V \text{ with } y \neq y_0, \\ g'(y_0) & \text{for } y = y_0. \end{cases} \tag{15}$$

Observe that h is continuous at $y_0 = f(x_0)$, since $\lim_{y \to y_0} h(y) = g'(y_0) = h(y_0)$ by the definition of the derivative (together with the definition of h and the assumption that g is differentiable at y_0).

Now,

$$(g \circ f)'(x_0) = \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} = \lim_{x \to x_0} h(f(x)) \frac{f(x) - f(x_0)}{x - x_0}.$$

This equation is correct, since it is easy to see from (15) that the two expressions after the lim signs are equal for every $x \in U' \setminus \{x_0\}$ (if $f(x) = f(x_0)$ then both expressions are 0). Using the product rule for limits, the right-hand side here equals

$$\lim_{x \to x_0} h(f(x)) \cdot \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

The first limit here is equal to $h(f(x_0))$. Indeed, as noted above, f is continuous at x_0, and h is continuous at $y_0 = f(x_0)$; thus, their composition $h \circ f$ is continuous at x_0.[42] The second limit equals $f'(x_0)$ by the definition of the derivative, since f was assumed to be differentiable at x_0. By noting that $h(f(x_0)) = g'(f(x_0))$ by the definition (15) of h, equation (14) follows. The proof is complete.

[42] see Rosenlicht, the Proposition on p. 71.

21. The inverse of a monotone function

Definition. Given a f function, its inverse f^{-1} is defined as follows. The domain of f^{-1} is the set of those elements x of the range of f for which there is a unique y in the domain of f such that $f(x) = y$. For such an x, we put $f^{-1}(x) = y$.

Formally,

$$\begin{aligned}\mathrm{dom}(f^{-1}) = &\{x \in \mathrm{ra}(f) : (\forall y \in \mathrm{dom}(f))(\forall z \in \mathrm{dom}(f)) \\ &((f(y) = x \ \ \& \ \ f(z) = x) \to (y = z))\},\end{aligned}$$

and, for every $x \in \mathrm{dom}(f^{-1})$,

$$y = f^{-1}(x) \leftrightarrow (y \in \mathrm{dom}(f) \ \ \& \ \ f(y) = x).$$

Next we will define the concept of monotone functions.

Definition. A real-valued function f defined on a set $D \subset \mathbb{R}$ is said to be *increasing*[43] if for every $x, y \in D$ with $x < y$ we have $f(x) < f(y)$. f is said to be *decreasing* if for every $x, y \in D$ with $x < y$ we have $f(x) > f(y)$. f is said to be *monotone* if it is either increasing or decreasing.

The inverse of a monotone continuous function has important properties.

Theorem. *Let $[a, b]$ be an interval, and assume that f is an increasing real-valued continuous function on the interval $[a, b]$. Then the domain of its inverse, f^{-1}, is the interval $[f(a), f(b)]$. Furthermore, the function f^{-1} is continuous.*

An analogous result can be formulated for decreasing functions. A similar result can be formulated for other types of intervals (such as open, semi-closed, or infinite intervals), but there are slight complications. For example, if f is an increasing function on the open interval (a, b), then the range of f cannot be described as $(f(a), f(b))$, since $f(a)$ and $f(b)$ are not defined.[44]

Proof. Be the Intermediate-Value Theorem, for any x in the interval $[f(a), f(b)]$ there is a $y \in [a, b]$ such that $f(y) = x$; thus, the range of f is the interval $[f(a), f(b)]$. Observe that, for such an x, there cannot be two distinct numbers $y, z \in [a, b]$ such that $f(y) = f(z) = x$, since we must have either $y < z$, in which case $f(y) < f(z)$ as f is increasing, or $z < y$, in which case $f(z) < f(y)$. Thus the domain of the inverse of f is $[f(a), f(b)]$.

Write g for the inverse of f. We are going to show that g is continuous. Let $V \subset [a, b]$ be an arbitrary set open in $[a, b]$.[45] We need to show that the set

$$g^{-1}[V] = \{x \in \mathrm{dom}(g) : g(x) \in V\}$$

[43] We use the terms increasing, decreasing, and monotone in the strict sense; some authors use these terms in the wider sense (with equality of the function values allowed). For *increasing in the wider sense* can can use the term *nondecreasing*, and For *decreasing in the wider sense* can can use the term *nonincreasing.*

[44] The existence of the inverse is easily established in the same way as below by the Intermediate-Value Theorem for intervals that are not closed. However, the technique used for establishing the continuity of the inverse does not work for unbounded intervals, since bounded open sets in this interval cannot be represented as complements of compact sets. The continuity of the inverse in this case can be established with the aid of the above theorem (as formulated with closed intervals). For example, if a monotone continuous function f is defined on an unbounded interval I, then for every closed interval $J \subset I$, f restricted to the interval J has a continuous inverse by the above theorem. From this it can easily be concluded that the inverse of f is continuous.

[45] I.e., "open in the sense of $[a, b]$." Thus, for a number $c \in (a, b)$, the interval $[a, c)$ is an open subset of $[a, b]$, but it is not an open subset of $\mathbb{R}$. To express this distinction, one sometimes says that $[a, c)$ is open *relative to* $[a, b]$.

is open in $\operatorname{dom}(g) = [g(a), g(b)]$[46] (but it need not be open in $\mathbb{R}$). Noting that $g(x) = y$ if and only if $f(y) = x$, we have

$$g^{-1}[V] = f[V] = \{f(y) : y \in V\}.$$

Hence,

$$g^{-1}[V] = \operatorname{dom}(g) \setminus g^{-1}[[a,b] \setminus V] = \operatorname{dom}(g) \setminus f[[a,b] \setminus V].$$

The set $[a,b] \setminus V$ is compact, so its image $f[[a,b] \setminus V]$ under the continuous function f is compact; thus its complement $g^{-1}[V]$ in $\operatorname{dom}(g)$ is open in $\operatorname{dom}(g)$. As V was an arbitrary open set in $[a,b]$, this shows that g^{-1} is continuous, completing the proof.

Note. As far as the continuity of the inverse is concerned, from the proof it is clear that we established the following, somewhat more general statement.

THEOREM. *Let f be a function defined on a subset of the metric space (E,d) with values in another metric space (E',d'). Assume f is continuous and one-to-one, and* $\operatorname{dom}(f)$ *is a compact set. Then its inverse, f^{-1}, is continuous.*

As the domain of f is a compact set, the argument used in the above proof can be used without change, since for any set $V \subset \operatorname{dom}(f)$ that is open in $\operatorname{dom}(f)$, the set $\operatorname{dom}(f) \setminus V$ is compact.

Next we present an important result about the derivative of the inverse function.

THEOREM. *Let f be a real-valued differentiable function on an open interval I, and assume $f'(x) \neq 0$ for every $x \in I$. Then f^{-1} is differentiable for every $x \in \operatorname{dom}(f^{-1})$, and*

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

for any such x.

PROOF. As a derivative always satisfies the Intermediate-Value Theorem (even if it is not continuous), we must have either $f'(x) > 0$ for all $x \in I$ or $f'(x) < 0$ for all $x \in I$; thus, f is monotone on I. For the sake of simplicity, assume that f is increasing. Note that f is continuous on I, since differentiability implies continuity. Let $a, b \in I$ with $a < b$, and restrict f to the interval $[a,b]$; this is a change of notation: from now on, $\operatorname{dom}(f)$ is $[a,b]$, and not $\operatorname{dom}(f) = I$. This change of notation is harmless, and is done so that we can use the first Theorem in this section about the inverse of a continuous function without change.

Write g for the inverse of f; then $\operatorname{dom}(g) = [f(a), f(b)]$. Let $x_0 \in (f(a), f(b))$ be arbitrary (note that we want to avoid choosing x_0 as an endpoint of this interval), and write $y_0 = g(x_0)$. For $y \in (a,b)$ define the function h by

$$h(y) = \begin{cases} \frac{y - y_0}{f(y) - f(y_0)} & \text{for } y \neq y_0, \\ \frac{1}{f'(y_0)} & \text{for } y = y_0. \end{cases}$$

First, note that this definition is sound, since $f(y) < f(y_0)$ for $y < y_0$, and $f(y) > f(y_0)$ for $y > y_0$; second, $f'(y_0) \neq 0$ by our assumption, so we do not have zero in the denominator here. Second, this function is continuous at y_0, since

$$\lim_{y \to y_0} \frac{y - y_0}{f(y) - f(y_0)} = \lim_{y \to y_0} \frac{1}{\frac{f(y) - f(y_0)}{y - y_0}} = \frac{1}{\lim_{y \to y_0} \frac{f(y) - f(y_0)}{y - y_0}} = \frac{1}{f'(y_0)}.$$

[46]The notation here can cause some confusion. Recall that earlier we defined $h[V]$ as $\{h(x) : x \in V\}$, and we could have means $g^{-1}[V]$ as $h[V]$ with $h = g^{-1}$. One can avoid this confusion by writing $(g^{-1})[V]$. In the present situation one can show that $(g^{-1})[V] = g^{-1}[V]$, but this need not always be the case; this equation is true for *one-to-one* functions g, that is, for functions g such that for each x there is at most one y such that $g(y) = x$.

Observe that we have $f(g(x)) = x$ for any $x \in [f(a), f(b)]$ in view of the definition of the inverse function; recalling that $y_0 = g(x_0)$, we obtain that

$$\frac{g(x) - g(x_0)}{x - x_0} = h(g(x))$$

for every $x \neq x_0$ in view of the definition of h; note that only the first alternative in the definition of h is used for this, since $g(x) \neq g(x_0)$ for $x \neq x_0$. Thus

$$g'(x_0) = \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = \lim_{x \to x_0} h(g(x)) = h(g(x_0)) = \frac{1}{f'(g(x_0))}.$$

The third equality here holds because g is continuous at x_0 (and everywhere else) and h is continuous at $y_0 = g(x_0)$; the fourth equality follows from the definition of h. This equality completes the proof of the theorem.

Note. The argument used in the calculation of $g'(x_0)$ above is reminiscent of the proof of the Chain Rule. This is no accident, since writing i for the identity function on $[g(a), g(b)]$ (i.e., $i(x) = x$ for every $x \in [g(a), g(b)]$), we have, $i(x) = f(g(x))$, that is, $i = g \circ f$, and so

$$1 = i'(x_0) = (f \circ g)'(x_0) = f'(g(x_0)) \cdot g'(x_0)$$

according to the Chain Rule. Calculating $g'(x_0)$ from this equation, we obtain the above result.

The trouble with this argument is that, in order to use the Chain Rule, we need to know that g' is differentiable at x_0, and this needs to be established. There are ways to do this, but it is easier to entirely avoid having to do this, and use the argument presented in the above proof, without referring to the Chain Rule.

Note that the argument used here (and in the proof of the Chain Rule) looks like a substitution rule for limits:

$$\lim_{x \to c} \phi(\psi(x)) = \lim_{t \to \phi(c)} \phi(t).$$

In an imprecise, intuitive argument, it is useful to think in terms of such a substitution rule. However, the conditions under which such a rule is valid are somewhat complicated and artificial, and in a rigorous argument it is usually better to avoid using such a rule, in a way this was done in the proof of the above theorem and in the proof of the Chain Rule.

To illustrate why this is so, one can formulate the following rule. Assume ψ is a real valued function, c is a real number, and $\lim_{x \to c} \psi(x) = L$, and ϕ is a real-valued function on $\mathbb{R} \setminus \{L\}$. Assume, further, that there is an $\delta > 0$ such that $\psi(x) \neq L$ for all x with $0 \neq |x - c| < \delta$. Then

$$\lim_{x \to c} \phi(\psi(x)) = \lim_{t \to L} \phi(t), \tag{16}$$

provided the right-hand side exists. There are two problems here. First, this would have worked above, in finding the derivative of the inverse function (with $c = x_0$, $\psi = g$, and $\phi(y) = \frac{y - L}{f(y) - f(y_0)}$), but it would not have been sufficient to establish the chain rule. Second, the simplest way to establish this rule is to write

$$\psi_0(x) = \begin{cases} \psi(x) & \text{for } x \neq c, \\ L & \text{for } x = c, \end{cases} \quad \text{and} \quad \phi_0(y) = \begin{cases} \phi(y) & \text{for } y \neq L, \\ \lim_{y \to L} \phi(y) & \text{for } y = L. \end{cases}$$

Then, assuming that the limit on the right-hand side of (16) exists, the function ϕ_0 is continuous at c and ψ_0 is continuous at $\phi_0(c)$, and the equality in (16) can be shown to hold by noting that it is is equivalent to the continuity of $\phi_0 \circ \psi_0$ at c. All this is too much effort to expend in order to establish a rule of limited usability.

22. l'Hospital's rule

Rolle's Theorem. *Let $[a,b] \subset \mathbb{R}$ be a closed interval, and let f be a function that is continuous on $[a,b]$ and differentiable in (a,b). Assume that $f(a)=f(b)=0$. Then there is a number $\xi \in (a,b)$ such that $f'(\xi)=0$.*

Cauchy's Mean-Value Theorem. *Let f and g be continuous real-valued functions on the interval $[a,b]$ that are differentiable on (a,b). Then there is a number $\xi \in (a,b)$ such that*

$$f'(\xi)(g(b)-g(a)) = g'(\xi)(f(b)-f(a)).$$

Proof. Consider the function $F(x) = (f(x)-f(a))(g(b)-g(a)) - (g(x)-g(a))(f(b)-f(a))$. It is easy to see that F is continuous on $[a,b]$, differentiable in (a,b), and $F(a) = F(b) = 0$; thus, we can use Rolle's Theorem for the function F. We can conclude that there is a $\xi \in (a,b)$ such that $F'(\xi)=0$. That is,

$$0 = F'(\xi) = f'(\xi)(g(b)-g(a)) - g'(\xi)(f(b)-f(a)).$$

The desired conclusion easily follows from this equality. The proof is complete

Notes. If $g'(x) \neq 0$ holds for every $x \in (a,b)$ then $g(b)-g(a) \neq 0$, since by the usual (i.e., not Cauchy's) Mean-Value Theorem there is an $\eta \in (a,b)$ such that

$$0 \neq g'(\eta) = g(b) - g(a).$$

That is, in this case we can divide both sides of the equation expressing Cauchy's Mean-Value Theorem by $g'(\xi)(g(b)-g(a)$ to obtain a modified version of

Cauchy's Mean-Value Theorem (modified version). *Let f and g continuous real-valued functions on the interval $[a,b]$ that are differentiable on (a,b). Assume, further, that $g'(x) \neq 0$ for every $x \in (a,b)$. Then there is a $\xi \in (a,b)$ such that*

$$\frac{f(b)-f(a)}{g(a)-g(b)} = \frac{f'(\xi)}{g'(\xi)} \tag{17}$$

The principal use of Cauchy's Mean-Value Theorem is to derive l'Hospital's rule. Before we state l'Hospital's rule, we would like to state some definitions involving limits at infinity and infinite limits.

Definition. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Let $t \in \mathbb{R}$. We say that

$$\lim_{x \to t} f(x) = \pm\infty$$

if for every $M \in \mathbb{R}$ there is an $\delta > 0$ such that for every x with $0 \neq |x-t| < \delta$ we have $f(x) > M$. Formally,

$$\begin{aligned}\lim_{x \to t} f(x) = \pm\infty \leftrightarrow & (\forall M \in \mathbb{R})(\exists \delta > 0)(\forall x \in \mathbb{R}) \\ & \qquad (0 \neq |x-t| < \delta \to |f(x)| > M).\end{aligned}$$

Let $L \in \mathbb{R}$. We say that $\lim_{x \to +\infty} = L$ if for every $\epsilon > 0$ there is an $N \in \mathbb{R}$ such that for all $x > N$ we have $|f(x) - L| < \epsilon$. Formally,

$$\lim_{x \to +\infty} f(x) = L \leftrightarrow (\forall \epsilon > 0)(\exists N \in \mathbb{R})(\forall x > N)\ |f(x) - L| < \epsilon.$$

Similar definitions can be given for $\lim_{x\to t} f(x) = +\infty$, $\lim_{x\to t} f(x) = -\infty$, $\lim_{x\to-\infty} = L$, and $\lim_{x\to t-} f(x) = L$, and various combinations of limits or one-sided limits being infinite. It is a good exercise to write out these definitions. It is easy to show that $\lim_{x\to t} f(x) = \pm\infty$ if and only if $\lim_{x\to t} \frac{1}{f(x)} = 0$ (it is instructive to carry out a detailed verification of this fact). It is easy to see that these definitions make sense even when f is only defined on an appropriate part of the real line, not on the whole real line (it is a good exercise to state the exact requirements). An important note is that when a limit is $\pm\infty$, $+\infty$, or $-\infty$, one says that the limit does not exist. That is, one only says that a limit exists when that limit is a number. When the limit does not exist, then the limit may or may not be $\pm\infty$, $+\infty$, or $-\infty$; to say that it is one of these amounts to giving more precise information than just saying that the limit does not exist. Also note that when the limit is $+\infty$, it is also correct to say that the limit is $\pm\infty$.

L'HOSPITAL'S RULE.

a) Let f and g be real-valued functions differentiable in an open interval $I = (a, b)$, and assume that $\lim_{x\to a+} f(x) = \lim_{x\to a+} g(x) = 0$. Assume, further, that the limit $\lim_{x\to a+} \frac{f'(x)}{g'(x)}$ exists. Then the limit $\lim_{x\to a+} \frac{f(x)}{g(x)}$ also exists and

$$\lim_{x\to a+} \frac{f(x)}{g(x)} = \lim_{x\to a+} \frac{f'(x)}{g'(x)}. \tag{18}$$

b) The same conclusion holds if $\lim_{x\to a+} f(x)$ and $\lim_{x\to a+} g(x)$ are both $\pm\infty$ rather than 0 and the rest of the assumptions hold.

c) The same conclusion holds if $I = (a, +\infty)$ and the limits are taken at $+\infty$ instead of $a+$.

PROOF. There are other variants of L'Hospital's rule, such as the case of limits at $-\infty$, and two-sided limits (i.e., limits such as $\lim_{x\to c}$) and limits from the left. Finally, the result remains true when the limit on the right-hand side of (18) is assumed to be $+\infty$, $-\infty$, or $\pm\infty$; it needs to be realized, however, that in this case the equality in equation (18) does not refer to equality of numbers ($+\infty$, $-\infty$, or $\pm\infty$ are not numbers) but to the similar behavior of two functions after the lim symbols on the two sides of the equation. It is a good exercise to formulate these cases and prove the corresponding assertions. The cases presented are typical, and the proofs in these cases can easily be adapted to the remaining cases.

Case a) We may extend the definition of f and g to the point a by putting $f(a) = g(a) = 0$; then the functions f and g will be continuous on the interval $[a, b)$. Continuity at every $x \in (a, b)$ follows from differentiability, and right-handed continuity at a will hold since $\lim_{x\to a+} f(x) = \lim_{x\to a+} g(x) = 0$.

Write

$$\lim_{x\to a+} \frac{f'(x)}{g'(x)} = L.$$

Let $\epsilon > 0$ be arbitrary, and let δ with $0 < \delta < b - a$ (so that $(a, a+\delta) \subset (a, b)$) be such that

$$\left| \frac{f'(t)}{g'(t)} - L \right| < \epsilon$$

for every $t \in (a, a+\delta)$. Note that, for this inequality to hold, the left-hand side must be meaningful, and so we must have $g'(t) \neq 0$ for every $t \in (a, a+\delta)$. Let $x \in (a, a+\delta)$ be arbitrary. We claim that there is a $\xi \in (a, x)$ such that

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(\xi)}{g'(\xi)}. \tag{19}$$

Indeed, the first equality here holds since we made the stipulation $f(a) = g(a) = 0$, and the second equality holds by the (modified version of) Cauchy's Mean-Value Theorem (cf. (17)).

Thus, we have
$$\left|\frac{f(x)}{g(x)} - L\right| = \left|\frac{f'(\xi)}{g'(\xi)} - L\right| < \epsilon.$$
As $\epsilon > 0$ was arbitrary,
$$\lim_{x\to a+} \frac{f(x)}{g(x)} = L$$
follows. This establishes (18) in Case (a).

Note. It should be clear from this proof that in (18) we cannot conclude the existence of the limit on the right-hand side from that of the left-hand side. The reason for this is that equation (19) does not talk about every number $\xi > a$ close to a. It only talks about those values of ξ that come up in the application of Cauchy's Mean-Value Theorem; it is instructive to think through a situation when these values of ξ do not comprise all values of $\xi > a$ close to a. For example, given $f(x) = x^2 \sin\frac{1}{x}$ and $g(x) = \sin x$, it is quite easy to see that
$$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{x^2 \sin\frac{1}{x}}{\sin x} = 0,$$
while an attempt at applying l'Hospital's rule leads to
$$\lim_{x\to 0+} \frac{f'(x)}{g'(x)} = \lim_{x\to 0+} \frac{2x\sin\frac{1}{x} - \cos\frac{1}{x}}{\cos x},$$
and it is not hard to see that this limit does not exist.

Case b) The idea for the proof is the following. Let $x_0 \in (a, b)$ be fixed. Then we *ought to have*
$$\lim_{x\to a+} \frac{f(x)}{g(x)} = \lim_{x\to a+} \frac{f(x) - f(x_0)}{g(x) - g(x_0)},$$
since, as x approaches a, the values $f(x)$ and $g(x)$ will be dominate over (i.e., will be much larger in absolute value than) $f(x_0)$ and $g(x_0)$; this is so because $\lim_{x\to a+} f(x) = \pm\infty$ and $\lim_{x\to a+} g(x) = \pm\infty$ by our assumptions. The expression on the right-hand side will allow us to apply Cauchy's Mean-Value Theorem. We said that the above equality *ought to* rather than it *does* hold since the existence of the limits in this equality has not been established, so it is not clear at this point whether this equality makes any sense.

To make this idea precise, assume
$$L = \lim_{x\to a+} \frac{f'(x)}{g'(x)}.$$
Let $\{x_n\}_{n=1}^{\infty}$ an arbitrary sequence of numbers $x_n \in (a, b)$ such that $\lim_{n\to\infty} x_n = a$. It is then enough to show that
$$\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = L.$$
This is so in view of the Proposition on p. 74 in Rosenlicht. In applying this Proposition, we need to take the metric space E to be the interval $(a, +\infty)$ (or (a, b)) rather than $\mathbb{R}$, because we are considering $\lim_{x\to a+}$ rather than $\lim_{x\to a}$; the space E' can (and should) be taken as $\mathbb{R}$.

Assume, on the contrary, that L is not the limit of the sequence $\left\{\frac{f(x_n)}{g(x_n)}\right\}_{n=1}^{\infty}$ (i.e., the assumption is that either this sequence does not have a limit, or that it does have a limit but that limit is not equal to L). Then there must be an $\epsilon' > 0$ such that there are infinitely many elements of this sequence outside the interval $(L - \epsilon', L + \epsilon')$. Let $\{y_n\}_{n=1}^{\infty}$ be a subsequence of $\{x_n\}_{n=1}^{\infty}$ such that all the numbers $\left\{\frac{f(y_n)}{g(y_n)}\right\}_{n=1}^{\infty}$ lie outside the interval $(L - \epsilon', L + \epsilon')$, Now, one of two things must happen. First, it is possible that the sequence $\left\{\frac{f(y_n)}{g(y_n)}\right\}_{n=1}^{\infty}$ is not bounded, in which case there is a subsequence $\{z_n\}_{n=1}^{\infty}$ of $\{y_n\}_{n=1}^{\infty}$ for which $\lim_{n\to\infty}\left|\frac{f(z_n)}{g(z_n)}\right| = +\infty$, i.e.,

$$\lim_{n\to\infty}\frac{g(z_n)}{f(z_n)} = 0; \tag{20}$$

second, it is possible that this sequence is bounded, in which case, according to the Bolzano–Weierstrass Theorem, there is a subsequence $\{z_n\}_{n=1}^{\infty}$ of $\{y_n\}_{n=1}^{\infty}$ and a number K such that

$$\lim_{n\to\infty}\frac{f(z_n)}{g(z_n)} = K; \tag{21}$$

further, we must have $K \ne L$ by our assumption. Indeed, all elements of this sequence belong to the closed set $\{x \in \mathbb{R} : x \le L - \epsilon' \text{ or } x \ge L + \epsilon'\}$, so its limit K must also belong there.

We are going to show that neither (20) nor (21) is possible.

As $\lim_{x\to a+} f(x) = \pm\infty$, i.e., $\lim_{x\to a+}\frac{1}{f(x)} = 0$, in this case, and $\lim_{n\to\infty} z_n = a$ ($\{z_n\}_{n=1}^{\infty}$ is a subsequence of the sequence $\{x_n\}_{n=1}^{\infty}$, and this latter sequence was assumed to converge to a), we can conclude from the Proposition on p. 74 in Rosenlicht that $\lim_{n\to\infty}\frac{1}{f(z_n)} = 0$, i.e., $\lim_{n\to\infty} f(z_n) = \pm\infty$; similarly, $\lim_{n\to\infty} g(z_n) = \pm\infty$. Thus, there is a positive integer N_0 such that $f(z_n) \ne 0$ and $g(z_n) \ne 0$ for $n \ge N_0$.

Let $\epsilon > 0$ be arbitrary, and, recalling that L was defined a the limit $\lim_{x\to a+}\frac{f'(x)}{g'(x)}$, let δ with $0 < \delta < b - a$ be such that

$$\left|\frac{f'(x)}{g'(x)} - L\right| < \epsilon$$

for every $x \in (a, a + \delta_0)$. Note that for this inequality to hold, we must have $g'(x) \ne 0$ for every $x \in (a, a + \delta)$. Let $u \in (a, a + \delta)$ be fixed, and let $N \ge N_0$ be an integer such that $z_n < u$ for $n \ge N$; there is such an N, since the sequence $\{z_n\}_{n=1}^{\infty}$ converges to a. Using (the modified version of) Cauchy's Mean-Value Theorem, for every $n \ge N$ there is a $\xi_n \in (z_n, u)$ such that

$$\frac{f(u) - f(z_n)}{g(u) - g(z_n)} = \frac{f'(\xi_n)}{g'(\xi_n)}$$

(cf. (17) with z_n and u replacing a and b; recall that we noted above that $g'(x) \ne 0$ for every $x \in (a, a + \delta)$.) Thus, for any $n \ge N$ we have

$$\left|\frac{\frac{f(u)}{f(z_n)} - 1}{\frac{g(u)}{g(z_n)} - 1}\cdot\frac{f(z_n)}{g(z_n)} - L\right| = \left|\frac{f(u) - f(z_n)}{g(u) - g(z_n)} - L\right| = \left|\frac{f'(\xi_n)}{g'(\xi_n)} - L\right| < \epsilon. \tag{22}$$

Now

$$\lim_{n\to\infty}\frac{\frac{f(u)}{f(z_n)} - 1}{\frac{g(u)}{g(z_n)} - 1} = \frac{\lim_{n\to\infty}\left(\frac{f(u)}{f(z_n)} - 1\right)}{\lim_{n\to\infty}\left(\frac{g(u)}{g(z_n)} - 1\right)} = 1; \tag{23}$$

to see the second equality here note that $\lim_{n\to\infty} f(z_n) = \pm\infty$ and $\lim_{n\to\infty} g(z_n) = \pm\infty$; hence, for example,

$$\lim_{n\to\infty}\frac{f(u)}{f(z_n)} = \lim_{n\to\infty}\left(f(u)\cdot\frac{1}{f(z_n)}\right) = f(u)\cdot 0 = 0$$

(note that u is fixed here).

Assume now that (21) holds. Making $n \to \infty$ in (22) we obtain $|K - L| \le \epsilon$. This is obviously a contradiction, since $\epsilon > 0$ was arbitrary, and this inequality cannot hold with $\epsilon = |K - L|/2$ (recall that $K \ne L$). Thus (21) is impossible.

Note. To appreciate why we get $|K - L| \le \epsilon$ rather than $|K - L| < \epsilon$ in the preceding argument argument, it is worth inspecting how this conclusion was obtained. Namely, the set

$$S = \{x : |x - L| \le \epsilon\} = \{x : L - \epsilon \le x \le L + \epsilon\}$$

is closed; by (22), the elements of the sequence

$$\frac{\frac{f(u)}{f(z_n)} - 1}{\frac{g(u)}{f(g_n)} - 1} \cdot \frac{f(z_n)}{g(z_n)}$$

belong to S for $n \ge N$, hence its limit $1 \cdot K = K$ also belongs to S.

Finally, we will show that (20) is also impossible. To this end, assume that (20) holds, and multiply (22) by $\left|\frac{g(y_n)}{f(z_n)}\right|$ to obtain

$$\left| \frac{\frac{f(u)}{f(z_n)} - 1}{\frac{g(u)}{g(z_n)} - 1} - \frac{g(z_n)}{f(z_n)} \cdot L \right| < \left| \frac{g(z_n)}{f(z_n)} \right| \cdot \epsilon$$

for $n \ge N$. Note here that the integer N_0 was chosen above such that that $f(z_n) \ne 0$ for $n \ge N_0$, and we picked $N \ge N_0$; so multiplying by $\left|\frac{g(z_n)}{f(z_n)}\right|$ was allowed for $n \ge N$. Making $n \to \infty$ and using (20) and (23), we obtain $|1 - 0 \cdot L| \le 0 \cdot \epsilon$, i.e., $1 < 0$. This is the contradiction, showing that (20) is also impossible. Having shown that both (20) and (21) are impossible, the proof is complete in Case (b).

Case c) This case can be reduced to the previous two cases by noting that

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \lim_{t \to 0+} \frac{f\left(\frac{1}{t}\right)}{g\left(\frac{1}{t}\right)} = \lim_{t \to 0+} \frac{-\frac{1}{t^2} f'\left(\frac{1}{t}\right)}{-\frac{1}{t^2} g'\left(\frac{1}{t}\right)} = \lim_{x \to +\infty} \frac{f'(x)}{g'(x)},$$

where the second equality is obtained by using the result for the already established Cases (a) or (b).

Note. The proof of Case *(b)* was the most difficult among the cases above. We presented a proof that was "conceptual." It illustrated a useful technique of using subsequences so as to use theorems about limits in a situation when the limit either does not exist or is not known to exist. Another way to go around the nonexistence of limits is to use limit superior and limit inferior; for sequences, this is described in Problems 18 and 19 on p. 63 in Rosenlicht. For a bounded function f on $\mathbb{R}$ and $a \in \mathbb{R}$, we define the limit superior as

$$\limsup_{x \to a+} f(x) \overset{def}{=} \lim_{x \to a+} \sup\{f(y) : a < y < x\};$$

the existence of the limit on the right-hand side is not hard to prove. The limit inferior can be defined similarly by interchanging sup and inf. The definition in cases of left-sided limits, two-sided limits, and limits at infinity can be stated similarly. These concepts can be extended to unbounded functions if we allow $+\infty$ or $-\infty$ as the values of lim inf or lim sup.

Next we present a proof that is based on similar ideas, but rather than being conceptual, it is based on calculations that look somewhat arbitrary. Note the implicit re-proving of an instance of the product rule for limits in the argument (but we cannot explicitly invoke the product rule, since the existence of the limits in question is not guaranteed).

Alternative proof in Case b). Assume, as before, that

$$L = \lim_{t \to a+} \frac{f'(t)}{g'(t)}.$$

Let ϵ be arbitrary with $0 < \epsilon < 1$, and let δ_0 with $0 < \delta_0 < b - a$ be such that

$$\left| \frac{f'(x)}{g'(x)} - L \right| < \frac{\epsilon}{4}$$

for every $x \in (a, a + \delta_0)$. For this inequality to hold, we must have $g'(x) \neq 0$ for every $x \in (a, a+\delta_0)$. Thus, by (the modified version of) Cauchy's Mean-Value Theorem, for every u and v with $a < u < v < a + \delta_0$ there is a $\xi \in (u, v)$ such that

$$\frac{f(v) - f(u)}{g(v) - g(u)} = \frac{f'(\xi)}{g'(\xi)}$$

(cf. (17) with u and v replacing a and b). Thus, for any such u and v we have

$$\left| \frac{f(v) - f(u)}{g(v) - g(u)} - L \right| = \left| \frac{f'(\xi)}{g'(\xi)} - L \right| < \frac{\epsilon}{4}.$$

Now, let $v \in (a, a + \delta_0)$ be fixed. We have

$$\lim_{u \to a+} \frac{\frac{g(v)}{g(u)} - 1}{\frac{f(v)}{f(u)} - 1} = \frac{\lim_{u \to a+} \left(\frac{g(v)}{g(u)} - 1 \right)}{\lim_{u \to a+} \left(\frac{f(v)}{f(u)} - 1 \right)} = 1;$$

the second equality here holds since $\lim_{u \to a+} f(u) = \pm\infty$ and $\lim_{u \to a+} g(u) = \pm\infty$.

Pick a δ with $0 < \delta < v - a$ (note that $v - a < \delta_0$, so $\delta < \delta_0$) such that

$$\left| \frac{\frac{g(v)}{g(u)} - 1}{\frac{f(v)}{f(u)} - 1} - 1 \right| < \frac{\epsilon}{2(|L| + 1)}$$

holds for every $u \in (a, a + \delta)$. As

$$\frac{f(u)}{g(u)} = \frac{f(v) - f(u)}{g(v) - g(u)} \cdot \frac{\frac{g(v)}{g(u)} - 1}{\frac{f(v)}{f(u)} - 1},$$

we then have

$$\begin{aligned} \left| \frac{f(u)}{g(u)} - L \right| &= \left| \left(\frac{f(v) - f(u)}{g(v) - g(u)} - L \right) \cdot \frac{\frac{g(v)}{g(u)} - 1}{\frac{f(v)}{f(u)} - 1} + L \cdot \left(\frac{\frac{g(v)}{g(u)} - 1}{\frac{f(v)}{f(u)} - 1} - 1 \right) \right| \\ &\leq \left| \frac{f(v) - f(u)}{g(v) - g(u)} - L \right| \cdot \left| \frac{\frac{g(v)}{g(u)} - 1}{\frac{f(v)}{f(u)} - 1} \right| + |L| \cdot \left| \frac{\frac{g(v)}{g(u)} - 1}{\frac{f(v)}{f(u)} - 1} - 1 \right| \end{aligned}$$

Noting that we have

$$\left| \frac{\frac{g(v)}{g(u)} - 1}{\frac{f(v)}{f(u)} - 1} \right| \leq 1 + \left| \frac{\frac{g(v)}{g(u)} - 1}{\frac{f(v)}{f(u)} - 1} - 1 \right|,$$

we can see that the first term (i.e., first summand, that is, the first product) on the right-hand side above less than

$$\frac{\epsilon}{4}\left(1+\frac{\epsilon}{2(|L|+1)}\right)<\frac{\epsilon}{2};$$

the inequality here holds since we assumed $\epsilon<1$. The second term is less than

$$|L|\cdot\frac{\epsilon}{2(|L|+1)}<\frac{\epsilon}{2}.$$

Thus

$$\left|\frac{f(u)}{g(u)}-L\right|<\epsilon$$

holds for every $u\in(a,a+\delta)$. As ϵ was arbitrary with $0<\epsilon<1$, $\lim_{u\to a+}\frac{f(u)}{g(u)}=L$ follows. Thus (18) is established in this case as well.

23. The remainder term in Taylor's formula

In Taylor's formula, a function $f(x)$ is approximated by a polynomial

$$\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k.$$

The goodness of this approximation can be measured by the remainder term $R_n(x,a)$, defined as

$$R_n(x,a)\overset{def}{=}f(x)-\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k.$$

To estimate $R_n(x,a)$, we need the following lemma.

Lemma. *Let $n\ge 0$ be an integer. Let U be an open interval in $\mathbb{R}$ and let $f:U\to\mathbb{R}$ be a function that is $n+1$ times differentiable. Given any $b\in U$, we have*

$$\frac{d}{dx}R_n(b,x)=-\frac{f^{(n+1)}(x)(b-x)^n}{n!}\tag{24}$$

for every $x\in U$.

Proof. The case $n=0$ is obvious, since $R_0(b,x)=f(b)-f(x)$. For $n\ge 1$ we have

$$R_n(b,x)=f(b)-\sum_{k=0}^{n}f^{(k)}(x)\frac{(b-x)^k}{k!}=f(b)-f(x)-\sum_{k=1}^{n}f^{(k)}(x)\frac{(b-x)^k}{k!}.$$

We separated out the term for $k=0$ since we are going to use the product rule for differentiation, and the term for $k=0$ involves no product. We have

$$\begin{aligned}\frac{d}{dx}R_n(b,x)&=\frac{d}{dx}f(b)-\frac{d}{dx}f(x)\\&\quad-\sum_{k=1}^{n}\left(\frac{df^{(k)}(x)}{dx}\frac{(b-x)^k}{k!}+f^{(k)}(x)\frac{d}{dx}\frac{(b-x)^k}{k!}\right)\\&=-f'(x)-\sum_{k=1}^{n}\left(f^{(k+1)}(x)\frac{(b-x)^k}{k!}+f^{(k)}(x)\frac{-k(b-x)^{k-1}}{k!}\right)\\&=-f'(x)-\sum_{k=1}^{n}\left(f^{(k+1)}(x)\frac{(b-x)^k}{k!}-f^{(k)}(x)\frac{(b-x)^{k-1}}{(k-1)!}\right).\end{aligned}$$

Writing

$$A_k = f^{(k+1)}(x)\frac{(b-x)^k}{k!}$$

for k with $0 \le k \le n$, the sum (i.e., the expression described by $\sum_{k=1}^{n}$) on the right-hand side equals[47]

$$\begin{aligned}\sum_{k=1}^{n}(A_k - A_{k-1}) &= (A_1 - A_0) + (A_2 - A_1) + \ldots (A_n - A_{n-1}) \\ &= A_n - A_0 = f^{(n+1)}(x)\frac{(b-x)^n}{n!} - f'(x).\end{aligned}$$

Substituting this in the above equation, we obtain

$$\frac{d}{dx}R_n(b,x) = -f'(x) - \left(f^{(n+1)}(x)\frac{(b-x)^n}{n!} - f'(x)\right) = -f^{(n+1)}(x)\frac{(b-x)^n}{n!},$$

as we wanted to show.

Corollary 1. *Let $n \ge 0$ be an integer. Let U be an open interval in $\mathbb{R}$ and let $f : U \to \mathbb{R}$ be a function that is $n+1$ times differentiable. For any any $a, b \in U$ with $a \ne b$, there is a $\xi \in (a,b)$ (if $a < b$) or $\xi \in (b,a)$ (if $a > b$) such that*

$$R_n(b,a) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(b-a)^{n+1}. \tag{25}$$

Proof. For the sake of simplicity, we will assume that $a < b$.[48] We have $b - a \ne 0$, and so the equation

$$R_n(b,a) = K \cdot \frac{(b-a)^{n+1}}{(n+1)!} \tag{26}$$

can be solved for K. Let K be the real number for which this equation is satisfied, and write

$$\phi(x) = R_n(b,x) - K \cdot \frac{(b-x)^{n+1}}{(n+1)!}$$

Then ϕ is differentiable in U; as differentiability implies continuity, it follows that f is continuous on the interval $[a,b]$ and differentiable in (a,b).[49] As $\phi(a) = 0$ by the choice of K and $\phi(b) = 0$ trivially, we can use Rolle's Theorem to obtain the existence of a $\xi \in (a,b)$ such that $\phi'(\xi) = 0$. Using (24), we can see that

$$0 = \phi'(\xi) = -\frac{f^{(n+1)}(\xi)(b-\xi)^n}{n!} - K \cdot \frac{-(n+1)(b-\xi)^n}{(n+1)!}.$$

Noting that $\frac{(n+1)}{(n+1)!} = \frac{1}{n!}$ and keeping in mind that $\xi \ne b$, we obtain $K = f^{n+1}(\xi)$ from here. Thus the result follows from (26).

[47] A sum that exhibits the cancelation that follows is called a *telescoping sum*.

[48] This assumption is never really used except that it helps us avoid circumlocutions such as $\xi \in (a,b)$ if $a < b$ or $\xi \in (b,a)$ if $b < a$.

[49] Naturally, ϕ is differentiable also at a and b, but this is not needed for the rest of the argument.

Note. The above argument can be carried out in somewhat more generality. Let $g(x)$ be a function that is differentiable on (a, b), $g'(x) \neq 0$ for $x \in (a, b)$, and $g(b) = 0$. Note that, by the Mean-Value Theorem,

$$-\frac{g(a)}{b-a} = \frac{g(b) - g(a)}{b-a} = g'(\eta)$$

for some $\eta \in (a, b)$. Since we have $g'(\eta) \neq 0$ by our assumptions, it follows that $g(a) \neq 0$. Instead of (26), now determine K such that

$$R_n(b, a) = Kg(a). \tag{27}$$

As $g(a) \neq 0$, it is possible to find such a K. Write

$$\phi(x) = R_n(b, x) - Kg(x).$$

We have $\phi(a) = 0$ by the choice of K. We have $\phi(b) = 0$ since $R_n(b, b) = 0$ and $g(b) = 0$ (the latter by our assumptions). As ϕ is differentiable on (a, b), there is a $\xi \in (a, b)$ such that $\phi'(\xi) = 0$. Thus, by (24) we have

$$0 = \phi'(\xi) = -\frac{f^{(n+1)}(\xi)(b-\xi)^n}{n!} - Kg'(\xi).$$

As $g'(\xi) \neq 0$ by our assumptions, we can determine K from this equation. Substituting the value of K so obtained into (27), we can see that

$$R_n(b, a) = -\frac{f^{(n+1)}(\xi)(b-\xi)^n}{n!} \cdot \frac{g(a)}{g'(\xi)}.$$

Note that in the argument we again assumed that $a < b$, but this was unessential. Further, note that the function g can depend on a and b. We can restate the result just obtained in the following

Corollary 2. *Let $n \geq 0$ be an integer. Let U be an open interval in $\mathbb{R}$ and let $f : U \to \mathbb{R}$ be a function that is $n+1$ times differentiable. For any any $a, b \in U$ with $a < b$, let $g_{a,b}(x)$ be a function such that $g_{a,b}$ is continuous on $[a, b]$ and differentiable on (a, b). Assume, further, that $g_{a,b}(b) = 0$ and $g'_{a,b}(x) \neq 0$ for $x \in (a, b)$. Then there is a $\xi \in (a, b)$ such that*

$$R_n(b, a) = -\frac{f^{(n+1)}(\xi)(b-\xi)^n}{n!} \cdot \frac{g_{a,b}(a)}{g'_{a,b}(\xi)}.$$

If $b < a$, then the same result holds, except that one needs to write (b, a) instead of (a, b) and $[b, a]$ instead of $[a, b]$ for the intervals mentioned (but the roles of a and b should not otherwise be interchanged).

This result is given in [B]; see also [M]. The Wikipedia entry [Wiki] also discusses the result (especially under the subheading *Mean value theorem*), without giving attributions. Taking $g_{a,b}(x) = (b - x)^r$ for an integer r with $0 < r \leq n + 1$, we obtain that

$$R_n(b, a) = \frac{f^{(n+1)}(\xi)}{rn!}(b-\xi)^{n-r+1}(b-a)^r.$$

This is called the Roche–Schlömilch Remainder Term of the Taylor Series. Here ξ is some number in the interval (a, b) or (b, a); it is important to realize that the value of ξ depends on r. Taking $r = n+1$ here, we get formula (25); this is called Lagrange's Remainder Term of the Taylor Series. Taking $r = 1$, we obtain

$$R_n(b,a) = \frac{f^{(n+1)}(\xi)}{n!}(b-\xi)^n(b-a);$$

this is called Cauchy's Remainder Term of the Taylor Series.

The different forms of the remainder term of Taylor's series are useful in different situations. For example, Lagrange's Remainder Term is convenient for establishing the convergence of the Taylor series of e^x, $\sin x$, and $\cos x$ on the whole real line, but it is not suited to establish the convergence of the Taylor series of $(1+x)^\alpha$, where α is an arbitrary real number. The Taylor series of this last function is convergent on the interval $(-1,1)$, and on this interval it does converge to the function $(1+x)^\alpha$ (this series is called the *Binomial Series*). This can be established by using Cauchy's Remainder Term.

24. The binomial series

For $k \ge 0$ an integer and α a real, define the binomial coefficient $\binom{\alpha}{k}$ as

$$\binom{\alpha}{k} \overset{def}{=} \prod_{j=0}^{k-1} \frac{\alpha-j}{k-j} = \frac{\prod_{j=0}^{k-1}(\alpha-j)}{k!}.$$

For $k = 0$ this gives $\binom{\alpha}{k} = 1$, as the empty product is defined to be 1. For $k < 0$ it would also give 1. However, for $k < 0$, if one defines $\binom{\alpha}{k}$ at all, one should write $\binom{\alpha}{k} \overset{def}{=} 0$ so as to preserve known identities involving binomial coefficients.[50] If α is a positive integer, this definition agrees with the usual definition of binomial coefficients.

Theorem. *For any real α and every real x with $|x| < 1$ we have*

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n. \tag{28}$$

First Proof. The result is obviously correct if $x = 0$. In what follows, assume that $x \ne 0$. Writing $f(x) = (1+x)^\alpha$, we find for its nth derivative

$$f^{(n)}(x) = (1+x)^{\alpha-n} \prod_{j=0}^{n-1}(\alpha-j) = (1+x)^{\alpha-n} n! \cdot \binom{\alpha}{n}.$$

Thus, according to Taylor's Formula the equation

$$(1+x)^\alpha = \sum_{k=0}^{n} \binom{\alpha}{k} x^k + R_n(x,0)$$

[50] See equation (34) below. As mentioned there, to preserve this equation for $n = 0$, one needs to stipulate the $\binom{\alpha}{-1} = 0$.

holds, where, using Cauchy's Remainder Term, we have[51]

$$
\begin{aligned}
R_n(x,0) &= \frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n x = \binom{\alpha}{n+1} n(1+\xi)^{\alpha-n-1}(x-\xi)^n x \\
&= \binom{\alpha}{n+1} n(1+\xi)^{\alpha-1} x \left(\frac{x-\xi}{1+\xi}\right)^n
\end{aligned} \tag{29}
$$

for some ξ between 0 and x (i.e., $0<\xi<x$ if $x>0$, and $x<\xi<0$ if $x<0$). We claim that

$$\left|\frac{x-\xi}{1+\xi}\right| < |x|.$$

This is clear if $0<x<1$, since then the numerator on the left is less than x and the denominator is greater than 1. In case $-1<x<0$ we have

$$x < \frac{x-\xi}{1+\xi} < 0.$$

Indeed, the inequality on the right holds since the denominator is positive and the numerator is negative. The inequality on the left is equivalent to $x + x\xi < x - \xi$, i.e., to $(1+x)\xi < 0$; the latter inequality holds since the first factor on the left is positive, and the second one is negative.

Further, it is easy to estimate the factor $(1+\xi)^{\alpha-1}$ occurring on the right-hand side (29). If $\alpha - 1 \ge 0$ then we can use the inequality $0 < 1+\xi < 2$, and so[52]

$$(1+\xi)^{\alpha-1} \le 2^{\alpha-1}.$$

If $\alpha - 1 < 0$ then we can observe that $1+\xi > 1-|x| > 0$, and so

$$(1+\xi)^{\alpha-1} = \left(\frac{1}{1+\xi}\right)^{1-\alpha} < \left(\frac{1}{1-|x|}\right)^{1-\alpha}.$$

In any case, there is a number M that may depend on x and α, but that does not depend on ξ (and it certainly does not depend on n)[53] such that

$$(1+\xi)^{\alpha-1} \le M$$

holds whether or not $\alpha - 1 \ge 0$. Using these estimates we obtain from (29) that

$$|R_n(x,0)| \le \left|\binom{\alpha}{n+1}\right| nM|x|^{n+1},$$

where M does not depend on n (but it may depend on x and α). To obtain the desired result, first note that if α is a positive integer, then $\binom{\alpha}{k} = 0$ for $k > \alpha$, since the product defining $\binom{\alpha}{k}$ contains a zero factor in the numerator in this case. In this case $R_n(x,0) = 0$ for large n, so (28) certainly holds.[54]

[51] Lagrange's Remainder Term is not suitable to establish the result claimed in this theorem, at least not for negative values of x.

[52] Equality in the next formula will only hold in case $\alpha - 1 = 0$.

[53] Dependence of M on n is not as silly as it sounds. In fact, the only thing we know about ξ is that it is between 0 and x; so ξ may depend on n. Thus, if M were to depend on ξ, then it could, indirectly through ξ, depend on M. However, as M does not depend on ξ, it cannot depend on n, either.

[54] In this case, the nonzero terms on the right-hand side of (28) constitute a finite sum, and the assertion described in (28) is the well-known Binomial Theorem.

Assuming α is not a positive integer, $\binom{\alpha}{k} \neq 0$ holds for any positive integer k; we have

$$\lim_{n\to+\infty} \left| \frac{\binom{\alpha}{n+1}}{\binom{\alpha}{n}} \right| = \left| \frac{\alpha - n}{n+1} \right| = 1. \tag{30}$$

Hence, writing

$$A_n = \left| \binom{\alpha}{n+1} \right| nM|x|^{n+1},$$

we have

$$\lim_{n\to+\infty} \frac{A_{n+1}}{A_n} = |x| < 1.$$

From this it is easy to conclude that

$$\lim_{n\to+\infty} A_n = 0. \tag{31}$$

One way to do so is to note that the series $\sum_{n=1}^{\infty} A_n$ converges by the Ratio Test, and so we must have (31). As $|R_n(x,0)| \le A_n$, it follows from (31) that

$$\lim_{n\to+\infty} R_n(x,0) = 0.$$

Hence (28) also follows.

Remark. A more direct way to do this is to note if ρ is any (fixed) number with $|x| < \rho < 1$, then there is a positive integer N such that $\frac{A_{k+1}}{A_k} \le \rho$ whenever $k \ge N$. Hence, for any $n > N$ we have

$$A_n = A_N \cdot \prod_{k=N}^{n-1} \frac{A_{k+1}}{A_k} \le A_N \rho^{n-N};$$

thus (31) holds.[55]

Second Proof. First note that the only differentiable function $f(x)$ that satisfies the equations

$$(1+x)f'(x) = \alpha f(x) \quad \text{and} \quad f(0) = 1 \qquad (-1 < x < 1). \tag{32}$$

is $f(x) - (1+x)^\alpha = 0$. Indeed, let any $I \subset (-1,1)$ be an open interval with $0 \in I$ such that $f(x) \neq 0$ for $x \in I$. Then we have

$$\frac{f'(x)}{f(x)} - \frac{\alpha}{x+1} = 0,$$

i.e.,

$$(\log f(x) - \alpha \log(x+1))' = 0.$$

Thus

$$\log f(x) - \alpha \log(x+1) = C.$$

for some constant C. As $f(0) = 1$, substituting $x = 0$ here gives $C = 0$. Thus we have

$$f(x) = (x+1)^\alpha \qquad (x \in I). \tag{33}$$

[55] This second way of establishing (31) is not very different from the first way. In fact, the second way amounts to re-proving the Ratio Test, since the last displayed line just says that the series $\sum_{n=N+1}^{\infty} A_n$ is dominated by a convergent geometric series.

In this argument, one can choose the interval I as large as possible. That is, writing $I = (a, b)$ with

$$a = \inf\{t \colon t \ge -1 \ \&\ f(x) \ne 0 \quad \text{for} \quad x \in (t, 0)\}$$

and[56]

$$b = \sup\{t \colon t \le 1 \ \&\ f(x) \ne 0 \quad \text{for} \quad x \in (0, t)\},$$

we must have either $a = -1$ or $f(a) = 0$, and, similarly, $b = 1$ or $f(b) = 0$ in view of the continuity of f. Since $f(a) = 0$ (with $a > -1$) and $f(b) = 0$ is not possible in view of (33) (indeed, (33) must hold also at the endpoints of I, in view of the continuity of f), we must have $a = -1$ and $b = 1$. Thus (33) holds on all of $I = (-1, 1)$.

Hence, to establish (28), we need to show only that the series on the right-hand side of (28) satisfies (32). First, note that the radius of convergence of the series on the right-hand side of (28) is 1 in view of (30). Thus, the series on the right-hand side of (28) can be differentiated termwise for x with $-1 < x < 1$. Writing $f(x)$ for the function represented by this series, and noting $\binom{\alpha}{n} = \frac{\alpha}{n}\binom{\alpha-1}{n-1}$ for $n \ge 1$, we have

$$\begin{aligned}
(1+x)f'(x) &= (1+x)\sum_{n=1}^{\infty}\binom{\alpha}{n} n x^{n-1} = (1+x)\sum_{n=1}^{\infty}\alpha\binom{\alpha-1}{n-1}x^{n-1} \\
&= \alpha\sum_{n=1}^{\infty}\binom{\alpha-1}{n-1}x^{n-1} + \alpha\sum_{n=1}^{\infty}\binom{\alpha-1}{n-1}x^{n} = \alpha\sum_{n=0}^{\infty}\binom{\alpha-1}{n}x^{n} + \alpha\sum_{n=1}^{\infty}\binom{\alpha-1}{n-1}x^{n} \\
&= \alpha\sum_{n=0}^{\infty}\left(\binom{\alpha-1}{n} + \binom{\alpha-1}{n-1}\right)x^{n},
\end{aligned}$$

where, to justify the last equality, we stipulated that $\binom{\alpha-1}{-1} = 0$. Hence, to show the first equality in (32), we only need to observe that

$$\binom{\alpha-1}{n} + \binom{\alpha-1}{n-1} = \binom{\alpha}{n}. \tag{34}$$

Indeed, both sides equal 1 for $n = 0$. To handle the case $n \ge 1$, note that

$$\frac{\beta-k+1}{k}\binom{\beta}{k-1} = \binom{\beta}{k} = \frac{\beta}{k}\binom{\beta-1}{k-1} \qquad (k \ge 1)$$

in view of the defining equation of $\binom{\beta}{k}$. Thus,

$$\binom{\alpha-1}{n} + \binom{\alpha-1}{n-1} = \frac{\alpha-n}{n}\binom{\alpha-1}{n-1} + \binom{\alpha-1}{n-1} = \frac{\alpha}{n}\binom{\alpha-1}{n-1} = \binom{\alpha}{n},$$

establishing (34). Hence (28) follows.

[56]There is not much reason to require that $t \le 1$ in the definition of the set on the right-hand side next. Requiring this ensures that this set is bounded; instead of $t \le 1$ we could have required $t \le c$ (with any $c \ge 1$). However, in the argument below there is no use in extending $f(x)$ beyond $x = 1$, so we might as well require that $t \le 1$.

25. The Riemann integral

Definition. A *partition* P of the interval $[a,b]$ is a finite set of points $\{x_0, x_1, \dots, x_N\}$ such that

$$a = x_0 < x_1 < x_2 < \ldots < x_N = b.$$

The *width* or *norm* of a partition is

$$\|P\| \stackrel{def}{=} \max\{x_i - x_{i-1} : 1 \le i \le N\}.$$

That is, a partition P of $[a,b]$ is just a finite subset of $[a,b]$ such that $a, b \in P$. The rest of the above definition sets up the notation in a way that is helpful for introducing related concepts.[57] The numbers x_i, $x_1, \dots, x_N$ are called *partition points*, or *points of the partition*, and the intervals $I_i = [x_{i-1}, x_i]$ are called the *intervals of the partition.* One often writes Δx_i or $|I_i|$ for the length $x_i - x_{i-1}$ of the ith interval.

Definition. Given a partition

$$P : a = x_0 < x_1 < x_2 < \ldots < x_N = b$$

of the interval $[a,b]$, a *tag* for the interval $[x_{i-1}, x_i]$ with $1 \le i \le N$ is a number $\xi_i \in [x_{i-1}, x_i]$ for each i. A partition with a tag for each interval $[x_{i-1}, x_i]$ is called a *tagged* partition. Given a tagged partition as described, and given a real-valued function f on $[a,b]$, the corresponding Riemann sum is

$$S = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}).$$

Definition. If there is a real number A such that for every $\epsilon > 0$ there is a $\delta > 0$ such that for any Riemann sum S for f associated with a partition of width $< \delta$ of $[a,b]$ we have $|A - S| < \epsilon$, then we call A the *Riemann integral* of f on $[a,b]$, and we write $A = \int_a^b f$. In this case we call f *Riemann integrable* on $[a,b]$.

The definition of Riemann sums above immediately implies the following

Boundedness Lemma. *Assume there is a partition P of the interval $[a,b]$ such that the set values of Riemann sums for f associated with P is bounded. Then f is bounded on $[a,b]$.*

Proof. Let M be such that for any Riemann sum S for the partition

$$P : a = x_0 < x_1 < \cdots < x_N = b$$

we have $|S| \le M$. Let the numbers $\xi_i, \eta_i \in [x_{i-1}, x_i]$, to be specified later, for i with $1 \le i \le N$ form two sets of tags for the partition P, and let S_1 and S_2 be the corresponding Riemann sums. Then we have $|S_1| \le M$ and $|S_2| \le M$, and so

$$|S_1 - S_2| < 2M.$$

[57] Other sources define partition as a sequence of points; yet others, as a set of nonoverlapping intervals covering $[a,b]$. While formally different, these definitions all describe the same concept. The one we chose allows some technical simplifications in the concept of *refinement* below.

Let k be a fixed integer with $1 \le k \le N$ and pick ξ_i and η_i such that $\xi_i = \eta_i$ for all i with $i \ne k$ and $1 \le i \le N$. Then all but one term in the difference $S_1 - S_2$ cancel, so the above inequality becomes

$$|S_1 - S_2| = |(f(\xi_k) - f(\eta_k))(x_k - x_{k-1})| < 2M.$$

That is,

$$|f(\xi_k) - f(\eta_k)| < \frac{2M}{x_k - x_{k-1}}.$$

Hence

$$|f(\xi_k)| = |f(\eta_k) + (f(\xi_k) - f(\eta_k))| \le |f(\eta_k)| + |f(\xi_k) - f(\eta_k)| < |f(\eta_k)| + \frac{2M}{x_k - x_{k-1}},$$

i.e.,

$$|f(\xi_k)| < |f(\eta_k)| + \frac{2M}{x_k - x_{k-1}}.$$

Picking a fixed $\eta_k \in [x_{k-1}, x_k]$ and letting $\xi_k \in [x_{k-1}, x_k]$ be arbitrary, this inequality shows that f is bounded on the interval $[x_{k-1}, x_k]$. Since this is true for each k with $1 \le k \le N$, and the union of these intervals is $[a, b]$, it follows that f is bounded on $[a, b]$.

Corollary. *If the real-valued function f is Riemann integrable on the interval $[a, b]$, then f is bounded on $[a, b]$.*

Proof. Given an arbitrary $\epsilon > 0$ ($\epsilon = 1$ will do), let $\delta > 0$ such that $|\int_a^b f - S| < \epsilon$ for every Riemann sum S for f associated with a partition of width $< \delta$ of $[a, b]$. Then, given a partition P of width $< \delta$ of $[a, b]$, the set of values of Riemann sums for f associated with P is bounded. Thus f is bounded on $[a, b]$ by the Boundedness Lemma.

First Criterion for Riemann Integrability. *A real-valued function f on $[a, b]$ is Riemann integrable if and only if for every $\epsilon > 0$ there is a $\delta > 0$ such that for any two Riemann sums S_1 and S_2 for f associated with partitions of $[a, b]$ we have $|S_1 - S_2| < \epsilon$.*

It is important to note that it is not assumed that S_1 and S_2 are associated with the same partition. More about this soon. The proof of this criterion is given in Rosenlicht [Ros, p. 118]; here we only present a

Proof of Necessity. Assume f is Riemann integrable on f and write $A = \int_a^b f$. Let $\epsilon > 0$ be arbitrary. According to the definition of the Riemann integral, there is a $\delta > 0$ such that $|A - S| < \epsilon/2$ for any Riemann sum S for f associated with a partition of width $< \delta$ of $[a, b]$. Thus S belongs to the interval $(A - \epsilon/2, A + \epsilon/2)$. Hence, if S and S' are Riemann sums for f associated with partitions of width $< \delta$ of $[a, b]$, then both S and S' belong to this interval; thus $|S' - S| < \epsilon$.

Note. There are several modified versions of the above criterion. For example

Modified First Criterion for Riemann Integrability. *A real valued function f on $[a, b]$ is Riemann integrable if and only if for every $\epsilon > 0$ there is a partition P of $[a, b]$ such that for any two Riemann sums S_1 and S_2 for f associated with the partition P we have $|S_1 - S_2| < \epsilon$.*

Note two differences here with the above criterion. First, no condition is imposed on the width of the partition P, second, here the two Riemann sums S_1 and S_2 are associated with the same partition, and in the earlier version this was not assumed.

The necessity of the present condition is clear from the earlier criterion, since if the condition of the earlier criterion is satisfied, then the condition in the present criterion is also satisfied (by taking P to be any partition of width $< \delta$ with the δ of the earlier criterion). It is somewhat harder to show that the criterion is also sufficient.

To prove the sufficiency, we need to discuss *refinements* of a partition. A partition P' is called a refinement of the partition P of the interval $[a, b]$ if $P \subset P'$. The following is an important result concerning refinements:

First Refinement Lemma. *Let f be a function defined on the interval $[a, b]$, let P be a partition of $[a, b]$ and let the partition P' be a refinement of P. Let S' be a Riemann sum for f associated with the partition P'. Then there are Riemann sums S_1 and S_2 for f associated with the partition P such that $S_1 \le S' \le S_2$.*

Note that the integrability of f is not assumed.

Proof. Assume first that P' has just one additional point not in P. That is, write

$$P : a = x_0 < x_1 < \ldots < x_N = b,$$

and assume that in addition to these points, P' contains an additional partition point c in the interval (x_{k-1}, x_k) for some k with $1 \le k \le N$.

Now, consider the set of tags giving rise to the Riemann sum S' associated with the partition P'. Write $\xi_i \in [x_{i-1}, x_i]$ for $i \ne k$ with $1 \le i \le N$ and $\eta_1 \in [x_{i-1}, c]$, and $\eta_2 \in [c, x_i]$ for these tags. For the partition P, pick the same tags ξ_i for $i \ne k$ with $1 \le i \le N$, and pick the tag $\xi_k \in [x_{k-1}, x_k]$ in a way about to be determined. Write S for the Riemann sum for f corresponding to the partition P that is associated with these tags. When forming the difference $S' - S$, all terms in S corresponding to the intervals $[x_{i-1}, x_i]$ with $i \ne k$ will cancel:

$$S' - S = f(\eta_1)(c - x_{k-1}) + f(\eta_2)(x_k - c) - f(\xi_k)(x_k - x_{k-1}).$$

If we define ξ_k by

$$\xi_k = \begin{cases} \eta_1 & \text{if } f(\eta_1) \le f(\eta_2), \\ \eta_2 & \text{if } f(\eta_1) > f(\eta_2), \end{cases}$$

then we have $S' - S \ge 0$. That is, if we define S_1 as S with this choice of ξ_k, we have $S_1 \le S'$. If we define ξ_k by

$$\xi_k = \begin{cases} \eta_1 & \text{if } f(\eta_1) \ge f(\eta_2), \\ \eta_2 & \text{if } f(\eta_1) < f(\eta_2), \end{cases}$$

then we have $S' - S \le 0$. That is, if we define S_2 as S with the latter choice of ξ_k, we have $S_2 \ge S'$. This establishes the assertion of the lemma if the partition P' contains just one point not in P.

To prove the assertion in the general case, let $\{c_j : 1 \le j \le m\}$ be the set of points in P' that are not in P, and for each integer l with $0 \le l \le m$ consider the partition

$$P(l) = P \cup \{c_j : 1 \le j \le l\}.$$

Then $P(0) = P$, $P(m) = P'$, and for each l with $1 \le l \le m$ the partition $P(l)$ is obtained by adding the single point c_l to the partition $P(l-1)$. Thus, writing $S_1(m) = S_2(m)$ for the Riemann sum S' associated with the partition $P' = P(m)$, the special case of the assertion of the lemma already proven allows us, for each $l = m-1, m-2 \ldots, 1, 0$, to find Riemann sums $S_1(l)$ and $S_2(l)$ associated with the partition $P^{(l)}$ such that $S_1(l) \le S_1(l+1)$ and $S_2(l) \ge S_2(l+1)$. The Riemann sums $S_1 = S_1(0)$ and $S_2 = S_2(0)$ associated with the partition $P = P(0)$ will then satisfy $S_1 \le S \le S_2$. This establishes the conclusion of the lemma in its full generality.

Another technical result that allows us to navigate between partitions is the following.

Second Refinement Lemma. *Let f be a bounded real-valued function on $[a,b]$, and let M be a bound for $|f|$; that is, assume we have $|f(x)| \le M$ for every $x \in [a,b]$. Let $\delta > 0$, let P be a partition of width $< \delta$ of $[a,b]$, and let S be a Riemann sum for f associated with the partition P. Let $m \ge 0$ be an integer, and assume that the refinement P' of the partition P results by the addition of m partition points to P. Then there is a Riemann sum S' for f associated with P' such that*

$$|S' - S| < 2mM\delta.$$

Proof. It is sufficient to prove the result in case of $m = 1$. Indeed, once the result is established for $m = 1$, the general result can be obtained by the going from the partition P to P' by the successive additions one point at a time, similarly to the way this was done in the proof of the preceding lemma.

As before, write

$$P : a = x_0 < x_1 < \ldots < x_N = b,$$

and assume that in addition to these points, P' contains an additional partition point c in the interval (x_{k-1}, x_k) for some k with $1 \le k \le N$. Let $\xi_i \in [x_{i-1}, x_i]$ for $1 \le i \le N$ be the tags giving rise to the Riemann sum S associated with the partition P. Determine the Riemann sum associated with the partition P' by using the same tags for $i \ne k$, and pick $\eta_1 \in [x_{i-1}, c]$ and $\eta_2 \in [c, x_i]$ arbitrarily as tags associated with the two new intervals bordering on c. Then, as before, when forming the difference $S' - S$, all terms in S corresponding to the intervals $[x_{i-1}, x_i]$ with $i \ne k$ will cancel:

$$\begin{aligned} S' - S &= f(\eta_1)(c - x_{k-1}) + f(\eta_2)(x_k - c) - f(\xi_k)(x_k - x_{k-1}) \\ &= (f(\eta_1) - f(\xi_k))(c - x_{k-1}) + (f(\eta_2) - f(\xi_k))(x_k - c). \end{aligned}$$

As $|f(x)| \le M$ for all $x \in [a,b]$ and $x_i - x_{i-1} < \delta$ by out assumptions, it follows that

$$|S' - S| \le 2M \cdot (x_k - c) + 2M \cdot (c - x_{k-1}) = 2M \cdot (x_k - x_{k-1}) < 2M\delta.$$

This proves the result for $m = 1$, and thus, as we remarked above, the result for any integer $m \ge 0$ also follows. The proof is complete.

Next we can turn to the proof of the missing part of the Modified First Criterion for Riemann Integrability.

Proof of Sufficiency of the Modified First Criterion of Riemann Integrability. Assume that the described criterion holds for the function f on the interval $[a,b]$. Then f is bounded on $[a,b]$ by the Boundedness Lemma. Let M be such that $|f(x)| \le M$ for all $x \in [a,b]$. Let $\epsilon > 0$ be arbitrary, and let P_0 be a partition such that for any two Riemann sums S_1 and S_2 for f associated with the partition P_0 we have $|S_1 - S_2| < \epsilon/2$. Let $m + 2$ be the number of partition points in P_0 ($m \ge 0$). Let $\delta = \frac{\epsilon}{8mM+1}$. Let P be a partition of width $< \delta$ and let S be a Riemann sum for f associated with the partition P. Let P' be the partition $P \cup P_0$; then P' is obtained from P by adding at most m partition points (since the elements a and b of P_0 also belong to P; there may be other elements of P_0 that also belong to P). Thus, by the Second Refinement Lemma above, there is a Riemann sum for f associated with the partition P' such that

$$|S' - S| \le 2mM\delta < \frac{\epsilon}{4},$$

where the second inequality holds in virtue of the choice of δ. By the First Refinement Lemma, there are Riemann sums S_1 and S_2 for f associated with the partition P_0 such that $S_1 \le S' \le S_2$. Thus

$$S_1 - \frac{\epsilon}{4} \le S' - \frac{\epsilon}{4} < S < S' + \frac{\epsilon}{4} \le S_2 + \frac{\epsilon}{4},$$

i.e.,

$$S_1 - \frac{\epsilon}{4} < S < S_2 + \frac{\epsilon}{4}.$$

Similarly, if R is another Riemann for f sum associated with a partition of width $< \delta$, then there are Riemann sums R_1 and R_2 for f associated with the partition P_0 such that

$$R_1 - \frac{\epsilon}{4} < R < R_2 + \frac{\epsilon}{4}.$$

To show from these inequalities that $|S-R| < \epsilon$, first assume that $S \le R$. By the assumption on P_0, we have $|R_2 - S_1| < \epsilon/2$ for the Riemann sums R_2 and P_1 associated with the partition P_0. Thus

$$S_1 - \frac{\epsilon}{4} < S \le R < R_2 + \frac{\epsilon}{4} < S_1 + \frac{\epsilon}{2} + \frac{\epsilon}{4} = S_1 + \frac{3\epsilon}{4}.$$

Thus $R - S < \epsilon$ follows in this case. Similarly $S - R < \epsilon$ follows in case $R \le S$. Thus $|R - S| < \epsilon$ holds for any two Riemann sums R and S for f associated with partitions of width $< \delta$ of $[a, b]$. Thus f is Riemann integrable on $[a, b]$ according to the First Criterion for Riemann Integrability. The proof is complete.

26. The Riemann integral: an example

Example. Let a, b, α, and β be real numbers with $a \le \alpha < \beta \le b$, and define the function f on $[a, b]$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in (\alpha, \beta), \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\int_a^b f = \beta - \alpha.$$

Proof. In order to make it easier to follow the arguments below, we restate the definition of f as

$$f(x) = \begin{cases} 1 & \text{if } \alpha < x < \beta, \\ 0 & \text{if } a \le x \le \alpha \text{ or } \beta \le x \le b. \end{cases}$$

Let $\epsilon > 0$ be arbitrary, and let

$$P : a = x_0 < x_1 < x_2 < \ldots < x_N = b$$

be a partition of $[a, b]$ with width $< \delta$, where the number $\delta > 0$ will be determined later. Further, let $\xi_i \in [x_{i-1}, x_i]$ for i with $1 \le i \le N$ be tags for this partition. Let the integers p and q with $1 \le p, q \le N$ be determined by the inequalities

$$x_{p-1} \le \alpha < x_p \quad \text{and} \quad x_{q-1} < \beta \le x_q. \tag{35}$$

Then $f(\xi_i) = 0$ if $i < p$ or $i > q$, and $f(\xi_i) = 1$ if $p < i < q$.[58] Thus, for the corresponding Riemann sum

$$S = \sum_{i=1}^{N} f(\xi_i)(x_i - x_{i-1})$$

[58] To see this, it is important to remember that $f(\alpha) = f(\beta) = 0$ by the definition of f. For $i = p$ or $i = q$ we may have $f(\xi_i) = 0$ or $f(\xi_i) = 1$.

we have

$$\sum_{i:\ p<i<q}(x_i - x_{i-1}) \le S \le \sum_{i:\ p\le i\le q}(x_i - x_{i-1}). \tag{36}$$

Since we have $p \le q$ by the definitions of p and q, the sum on the right-hand side telescopes:[59]

$$\begin{aligned}\sum_{i:\ p\le i\le q}(x_i - x_{i-1}) &= \sum_{i=p}^{q}(x_i - x_{i-1}) = (x_p - x_{p-1}) + (x_{p+1} - x_p)\\ &+ (x_{p+2} - x_{p+1}) + \cdots + (x_q - x_{q-1}) = x_q - x_{p-1}.\end{aligned} \tag{37}$$

Similarly, for the sum on the left-hand side we have

$$\sum_{i:\ p<i<q}(x_i - x_{i-1}) = \sum_{i=p+1}^{q-1}(x_i - x_{i-1}) \ge x_{q-1} - x_p. \tag{38}$$

One expects equality on the right-hand side here, but equality does not always hold. One has equality when $p+1 \le q-1$ (i.e., when $q \ge p+2$), in which case the sum on the left-hand side is not an empty sum, and therefore it telescopes. One also has equality in case $q = p+1$, when the sum on the left is the empty sum, and so it is zero, but the right-hand side is also zero in this case. However, in case $q = p$, the sum on the left-hand side is the empty sum, so it is zero, while the right-hand side is $x_{q-1} - x_p = x_{q-1} - x_q < 0$, so the inequality in this case still holds. Of course, $q < p$ is not possible according to (35), as $\alpha < \beta$. So the inequality in (38) certainly holds.

Now, taking (37) and (38) into account, (36) becomes

$$x_{q-1} - x_p \le S \le x_q - x_{p-1};$$

thus

$$(x_{q-1} - \beta) - (x_p - \alpha) \le S - (\beta - \alpha) \le (x_q - \beta) - (x_{p-1} - \alpha).$$

We have $|x_{q-1} - \beta| < \delta$, $|x_p - \alpha| < \delta$, $|x_q - \beta| < \delta$, and $|x_{p-1} - \alpha| < \delta$ according to (35) (and the assumption that the width of P is $< \delta$). Thus, the last displayed inequality implies that

$$-2\delta < S - (\beta - \alpha) < 2\delta,$$

i.e., that $|S - (\beta - \alpha)| < 2\delta$. Hence we will have $|S - (\beta - \alpha)| < \epsilon$ in case $\delta \le \epsilon/2$. As $\epsilon > 0$ was arbitrary, this shows that

$$\int_a^b f = \beta - \alpha.$$

The proof is complete.

[59] I.e., all its terms except for the extreme ones will cancel. The significance of the inequality $p \le q$ is that it guarantees that the sum in question is not the empty sum, so we indeed do have cancelations.

27. Integrability of monotonic and of continuous functions

In this section we establish basic results about the integrability of certain functions. We start with the

Second Criterion for Riemann Integrability. *A real valued function f on $[a,b]$ is Riemann integrable if and only if for every $\epsilon > 0$ there are step functions f_1 and f_2 on $[a,b]$ such that $\int_a^b (f_2 - f_1) < \epsilon$ and $f_1(x) \le f(x) \le f_2(x)$ for every $x \in [a,b]$*

As for the sufficiency of this criterion, there is no need to assume that the functions f_1 and f_2 are step functions; it is enough to assume that they are Riemann integrable:

Lemma. *Let f be a real-valued function on the interval $[a,b]$. Assume that for every $\epsilon > 0$ there are Riemann-integrable functions f_1 and f_2 on $[a,b]$ such that $f_1(x) \le f(x) \le f_2(x)$ and $\int_a^b (f_2 - f_1) < \epsilon$. Then f is Riemann integrable on $[a,b]$.*

First Proof. Let $\epsilon > 0$ be arbitrary. Then, according to the assumptions, there are Riemann integrable functions f_1 and f_2 such that $f_1(x) \le f(x) \le f_2(x)$ for all $x \in [a,b]$ and $\int_a^b (f_2 - f_1) < \epsilon/3$. Let $\delta_1 > 0$ be such that for any partition P of width $< \delta_1$, every Riemann sum S_1 for f_1 associated with the partition P is such that $|\int_a^b f_1 - S_1| < \epsilon/3$. Similarly, Let $\delta_2 > 0$ be such that for any partition P of width $< \delta_2$, every Riemann sum S_2 for f_2 associated with the partition P is such that $|\int_a^b f_2 - S_2| < \epsilon/3$.

Now, let $\delta = \min\{\delta_1, \delta_2\}$, and let

$$P : a = x_0 < x_1 < \ldots < x_N$$

be an arbitrary partition of width $< \delta$, and for each i with $1 \le i \le N$ let $\xi_i \in [x_{i-1}, x_i]$ be an arbitrary tag. Let S, S_1, and S_2 be the corresponding Riemann sums for f, f_1, and f_2, respectively, that is,

$$S = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}), \qquad S_1 = \sum_{i=1}^{n} f_1(\xi_i)(x_i - x_{i-1}),$$

$$\text{and} \quad S_2 = \sum_{i=1}^{n} f_2(\xi_i)(x_i - x_{i-1}).$$

Then we have

$$\int_a^b f_1 - \frac{\epsilon}{3} < S_1 \le S \le S_2 < \int_a^b f_2 + \frac{\epsilon}{3}.$$

Thus, for any Riemann sum S for f associated with a partition of width $< \delta$ of the interval $[a,b]$ we have

$$S \in \left(\int_a^b f_1 - \frac{\epsilon}{3}, \int_a^b f_2 + \frac{\epsilon}{3} \right);$$

the length of this interval is

$$\int_a^b f_2 - \int_a^b f_1 + \frac{2\epsilon}{3} \le \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon.$$

Now, any two Riemann sums S and S' for f associated with partitions of width $< \delta$ of the interval $[a,b]$ belong to this interval. Thus $|S' - S| < \epsilon$. Since $\epsilon > 0$ was arbitrary, the Riemann integrability of f follows from the First Criterion for Riemann Integrability. The proof is complete.

Note. The statement just proved differs from the sufficiency of the Second Criterion for Riemann Integrability merely in that instead of assuming that f_1 and f_2 are step functions it is assumed only that they are Riemann integrable. This makes no practical difference in the proof, and the above proof is virtually identical to the way sufficiency of the Second Criterion for Riemann Integrability can be derived from the the First Criterion for Riemann Integrability. A proof of the above statement based on the Second Criterion of Riemann integrability is much shorter, but perhaps it gives less insight:

Second Proof. Let $\epsilon > 0$ be arbitrary, and let f_1 and f_2 be Riemann integrable functions on $[a,b]$ such that $f_1(x) \le f(x) \le f_2(x)$ for all $x \in [a,b]$ and $\int_a^b (f_2 - f_1) < \epsilon/3$; such f_1 and f_2 exist according to the assumptions.

As f_1 is Riemann integrable on $[a,b]$, by the Second Criterion for Riemann Integrability there are step functions ϕ_1 and ϕ_2 on $[a,b]$ such that $\phi_1(x) \le f_1(x) \le \phi_2(x)$ for all $x \in [a,b]$ and $\int_a^b (\phi_2 - \phi_1) < \epsilon/3$. Similarly, as f_2 is Riemann integrable on $[a,b]$ by the Second Criterion for Riemann Integrability there are step functions ψ_1 and ψ_2 on $[a,b]$ such that $\psi_1(x) \le f_2(x) \le \psi_2(x)$ for all $x \in [a,b]$ and $\int_a^b (\psi_2 - \psi_1) < \epsilon/3$.

Then ϕ_1 and ψ_2 are step functions on $[a,b]$ such that $\phi_1(x) \le f(x) \le \psi_2(x)$ for all $x \in [a,b]$, and

$$\int_a^b (\psi_2 - \phi_1) \le \int_a^b (\psi_2 - \psi_1) + \int_a^b (\psi_1 - f_2) + \int_a^b (f_2 - f_1) \\ + \int_a^b (f_1 - \phi_2) + \int_a^b (\phi_2 - \phi_1) < \frac{\epsilon}{3} + 0 + \frac{\epsilon}{3} + 0 + \frac{\epsilon}{3} = \epsilon.$$

Thus f is Riemann integrable according to the Second Criterion for Riemann Integrability.

Lemma. *Assume that f is an increasing function on the interval $[a,b]$. Then f is Riemann integrable on $[a,b]$.*

Proof. Let N be a positive integer, and consider the partition $P_N : a = x_0 < x_1 < \ldots < x_N = b$ obtained by dividing the interval $[a,b]$ into N equal parts; that is, put $x_i = a + (b-a)i/N$ for i with $1 \le i \le N$. Consider the sums

$$L_N = \sum_{i=1}^N f(x_{i-1})(x_i - x_{i-1}) \quad \text{and} \quad U_N = \sum_{i=1}^N f(x_i)(x_i - x_{i-1}).$$

As $x_{i-1}, x_i \in [x_{i-1}, x_i]$, these are Riemann sums for the partition P_N. In fact, as f is increasing, we have $f(x_{i-1}) \le f(x) \le f(x_i)$ for every $x \in [x_{i-1}, x_i]$; thus, L_N is the smallest and U_N is the largest Riemann sum for f associated with the partition P_N. Now, noting that $x_i - x_{i-1} = \frac{b-a}{N}$, we have

$$U_N - L_N = \sum_{i=1}^N (f(x_i) - f(x_{i-1}))(x_i - x_{i-1}) = \frac{b-a}{N} \sum_{i=1}^N (f(x_i) - f(x_{i-1})) \\ = \frac{b-a}{N}(f(x_N) - f(x_0)) = \frac{b-a}{N}(f(b) - f(a)).$$

From here one can conclude that f is integrable in at least two different ways. First, defining the step functions f_1 and f_2 on $[a,b]$ by putting $f_1(a) = f_2(a) = f(a)$ and

$$f_1(x) = f(x_{i-1}) \text{ and } f_2(x) = f(x_i) \quad \text{for} \quad x \in (x_{i-1}, x_i] \quad (1 \le i \le N),$$

we have $f_1(x) \le f(x) \le f_2(x)$ for all $x \in [a,b]$ and

$$\int_a^b f_1 = L_N \quad \text{and} \quad \int_a^b f_2 = U_N.$$

Thus, if we start with an arbitrary $\epsilon > 0$ and pick N such that $N > \frac{b-a}{\epsilon}(f(b) - f(a))$, we have $\int_a^b (f_2 - f_1) = U_N - L_N < \epsilon$. Thus, the integrability of f follows from the Second Criterion for Riemann Integrability.

More directly, the integrability of f follows from the Modified First Criterion for Riemann Integrability, since for any Riemann sum S for f associated with the partition P_N we have $L_N \le S \le U_N$. Thus, if S_1 and S_2 are two Riemann sums associated with the partition P_N, we have $|S_2 - S_1| \le U_N - L_N < \epsilon$.

The following theorem says that every continuous function is integrable.

Theorem. *Let a, b be real numbers, $a < b$, and assume the function $f : [a, b] \to \mathbb{R}$ is continuous. Then f is Riemann integrable on $[a, b]$.*

Proof. As f is continuous on the closed interval $[a, b]$, it follows that f is uniformly continuous on $[a, b]$.[60] Let $\epsilon > 0$ be arbitrary, and let $\delta > 0$ such that for any $x, y \in [a, b]$ we have

$$|f(x) - f(y)| < \frac{\epsilon}{b-a} \quad \text{whenever} \quad |x - y| < \delta. \tag{39}$$

Let

$$P : a = x_0 < x_1 < \ldots < x_N = b$$

be a partition of $[a, b]$ of width $< \delta$. Let

$$m_i = \min\{f(x) : x \in [x_{i-1}, x_i]\} \qquad \text{and} \qquad M_i = \max\{f(x) : x \in [x_{i-1}, x_i]\}$$

for i with $1 \le i \le N$.[61] Observe that

$$M_i - m_i < \frac{\epsilon}{b-a} \qquad \text{for each } i \text{ with } 1 \le i \le N. \tag{40}$$

This is because, given i, $m_i = f(\xi)$ and $M_i = f(\eta)$ for some $\xi, \eta \in [x_{i-1}, x_i]$. Then $|\eta - \xi| \le x_i - x_{i-1} < \delta$, and so $M_i - m_i = f(\eta) - f(\xi) < \epsilon/(b - a)$ according to (39), as claimed.

For $x \in [a, b]$, let

$$f_1(x) = \begin{cases} m_i & \text{if } x \in (x_{i-1}, x_i) \text{ for some } i \text{ with } 1 \le i \le N, \\ f(x) & \text{if } x = x_i \text{ for some } i \text{ with } 0 \le i \le N \end{cases}$$

(note that the ranges of i are different in the two cases), and

$$f_2(x) = \begin{cases} M_i & \text{if } x \in (x_{i-1}, x_i) \text{ for some } i \text{ with } 1 \le i \le N, \\ f(x) & \text{if } x = x_i \text{ for some } i \text{ with } 0 \le i \le N. \end{cases}$$

Then $f_1(x)$ and $f_2(x)$ are step functions such that $f_1(x) \le f(x) \le f_2(x)$ for all $x \in [a, b]$. Furthermore,

$$\begin{aligned} \int_a^b (f_2 - f_1) &= \int_a^b f_2 - \int_a^b f_1 \\ &= \sum_{i=1}^N M_i(x_i - x_{i-1}) - \sum_{i=1}^N m_i(x_i - x_{i-1}) = \sum_{i=1}^N (M_i - m_i)(x_i - x_{i-1}) \\ &< \sum_{i=1}^N \frac{\epsilon}{b-a}(x_i - x_{i-1}) = \frac{\epsilon}{b-a} \sum_{i=1}^N (x_i - x_{i-1}) = \frac{\epsilon}{b-a}(b - a) = \epsilon; \end{aligned}$$

[60] A closed interval on the real line is compact, and a continuous function on a compact space is uniformly continuous.

[61] The definitions of m_i and M_i are meaningful, since a continuous function on a closed interval has both a minimum and a maximum, according to the Maximum Value Theorem.

The inequality here follows from (40), and the last but one equality here follows since the sum on the left of this equality telescopes to $x_N - x_0 = b - a$. Since $\epsilon > 0$ was arbitrary, the step functions f_1 and f_2 witness the integrability of f, according to the Second Criterion of Riemann Integrability. The proof is complete.

28. The Newton–Leibniz formula

The following is a version of the Newton–Leibniz formula.

Theorem. *Assume F is continuous and f is Riemann-integrable on the interval $[a, b]$. Assume, further, that $f(x) = F'(x)$ for every $x \in (a, b)$. Then*

$$\int_a^b f = F(b) - F(a).$$

In the usual form of the Newton–Leibniz formula the continuity of f is assumed. That is necessary if the result is derived from the Fundamental Theorem of Calculus, saying that, under the additional assumption that f is continuous on $[a, b]$ we have

$$\frac{d}{dx} \int_a^x f(t)\,dt = f(x)$$

for all $x \in (a, b)$ (the formula is also true at $x = a$ if we take differentiation from the left, and at $x = b$ if we take differentiation from the right). Here we derive the formula from the Mean-Value Theorem for differentiation.

Proof. Let $\epsilon > 0$, and let $\delta > 0$ such that for any partition

$$P : a = x_0 < x_1 < \ldots < x_N = b$$

with norm $\|P\| = \max\{x_i - x_{i-1} : 1 \le i \le N\}$ we have $\left|\int_a^b f - S\right| < \epsilon$ for any Riemann sum for f associated with the partition P.

Fix a partition P as above with norm less that δ. For each i with $1 \le i \le N$ pick ξ_i in the interval (x_{i-1}, x_i) such that $F(x_i) - F(x_{i-1}) = F'(\xi_i)(x_i - x_{i-1})$; there is such a ξ according to the Mean-Value Theorem for differentiation. Noting that $F'(\xi_i) = f(\xi_i)$ by our assumptions, we have

$$\begin{aligned} S &= \sum_{i=1}^N f(\xi_i)(x_i - x_{i-1}) = \sum_{i=1}^N (F(x_i) - F(x_{i-1})) \\ &= F(x_N) - F(x_0) = F(b) - F(a) \end{aligned}$$

for the corresponding Riemann sum S. A graphic description for the cancelation in the second sum used to obtain the second equality is to say that the second sum *telescopes* (the image here is a telescope consisting of pipes of increasing diameters, so that these pipes can be pushed into one another so as to collapse the telescope); such a sum is called a *telescoping sum.*

By the choice of δ, we have $\left|\int_a^b f - S\right| < \epsilon$, i.e., $\left|\int_a^b f - (F(b) - F(a))\right| < \epsilon$. Since the above argument can be carried out for any $\epsilon > 0$, the latter inequality is valid for every $\epsilon > 0$; thus we must have $\int_a^b f = F(b) - F(a)$. The proof is complete.

The above theorem is applicable with $[a,b]=[-1,1]$,

$$F(x)=\begin{cases} x^2\sin\frac{1}{x} & \text{if } x\neq 0,\\ 0 & \text{if } x=0,\end{cases}$$

and

$$f(x)=F'(x)=\begin{cases} 2x\sin\frac{1}{x}-\cos\frac{1}{x} & \text{if } x\neq 0,\\ 0 & \text{if } x=0.\end{cases}$$

$f(x)$ is not continuous at $x=0$. It is not hard to show, however, that f is Riemann-integrable on $[-1,1]$.

Indeed, let ϵ be arbitrary with $0<\epsilon<1$, and determine the step functions f_1 and f_2 on $[-1,-\epsilon/16]$ such that $f_1(x)\le f(x)\le f_2(x)$ as follows:

$$\int_{-1}^{\epsilon/16}(f_2(x)-f_1(x))\,dx<\frac{\epsilon}{4};$$

this is possible, since f is continuous, and therefore integrable, on the interval $[-1,-\epsilon/16]$.[62] Similarly, determine f_1 and f_2 on the interval $[\epsilon/16,1]$ such that

$$\int_{\epsilon/16}^{1}(f_2(x)-f_1(x))\,dx<\frac{\epsilon}{4}.$$

Finally, put

$$f_1(x)=-2 \text{ and } f_2(x)=2 \quad \text{for} \quad x\in\left[-\frac{\epsilon}{16},\frac{\epsilon}{16}\right].$$

It is easy to see that

$$\int_{-1}^{1}(f_2(x)-f_1(x))\,dx<\frac{\epsilon}{4}+\frac{\epsilon}{2}+\frac{\epsilon}{4}=\epsilon.$$

Since $\epsilon>0$ was arbitrary, it follows that f is integrable on $[1,-1]$.

Thus, by the above Theorem we have

$$\int_{-1}^{1}f=F(1)-F(-1)=\sin 1-\sin(-1)\approx .84147-(-.04147)=1.68294.$$

29. Integration by parts

Given a real-valued function f, write

$$f_+(x)=\begin{cases} f(x) & \text{if } f(x)\ge 0,\\ 0 & \text{if } f(x)<0,\end{cases} \quad \text{and} \quad f_-(x)=\begin{cases} 0 & \text{if } f(x)\ge 0,\\ -f(x) & \text{if } f(x)<0.\end{cases}$$

f_+ is called the *positive part*, and f_-, the *negative part*, of f. We have

[62] see the integrability criterion given in the Proposition on p. 120 in Rosenlicht.

Lemma. *Let f be a real-valued function on the interval $[a,b]$, and assume f is Riemann integrable. Then the functions f_+, f_-, $|f|$, and f^2 are also Riemann integrable.*

Proof. We will use the Second Criterion of Riemann Integrability. Let $\epsilon > 0$ be arbitrary, and let g and h be step functions on $[a,b]$ such that $g(x) \le f(x) \le h(x)$ for all $x \in [a,b]$ and $\int_a^b (h-g) < \epsilon$.

Then g_+ and h_+ are step functions such that $g_+(x) \le f_+(x) \le h_+(x)$ and $h_+(x) - g_+(x) \le h(x) - g(x)$ for all $x \in [a,b]$. The latter inequality implies that $\int_a^b (h_+ - g_+) \le \int_a^b (h-g) < \epsilon$. Thus f_+ is integrable.

The integrability of f_- can be shown in a similar way, or, more simply, by observing that $f_-(x) = (-f)_+(x)$ for all $x \in [a,b]$. As we have $|f| = f_+ + f_-$, the integrability of $|f|$ also follows.

To show the integrability of f^2, we may now assume that $f \ge 0$ (since $f^2 = |f|^2$). As f is integrable, it is bounded; write M for an upper bound of f. Let $\epsilon > 0$ be arbitrary, and let g and h be step functions such that $g(x) \le f(x) \le h(x)$ for all $x \in [a,b]$ and $\int_a^b (h-g) < \epsilon/(2M)$. We may also assume that $g(x) \ge 0$ (since we may replace g with g_+). Furthermore, we may assume that $h(x) \le M$ for all $x \in [a,b]$ (since we may replace $h(x)$ with $\min\{h(x), M\}$). Then $(g(x))^2 \le (f(x))^2 \le (h(x))^2$ for all $x \in [a,b]$. Furthermore,

$$\begin{aligned}\int_a^b (h(x)^2 - (g(x)^2)\,dx &= \int_a^b (h(x)+g(x))(h(x)-g(x))\,dx \\ &\le \int_a^b 2M(h(x)-g(x))\,dx \le 2M \cdot \frac{\epsilon}{2M} = \epsilon.\end{aligned}$$

Thus the integrability of f^2 also follows.

Corollary. *Let f and g be Riemann-integrable functions on $[a,b]$. Then fg is also Riemann integrable.*

Proof. We have

$$fg = \frac{(f+g)^2 - (f-g)^2}{4},$$

and so the integrability of fg follows from the above Lemma.

Given a function f on the interval $[a,b]$, we will say that f is differentiable on $[a,b]$ if f is differentiable for every $x \in (a,b)$, and f is differentiable from the right at a, and it is differentiable from the left at b. When there is no danger of misunderstanding, we will write $f'(a)$ for the derivative of f at a from the right, and $f'(b)$ for its derivative at b from the left.[63] If f is differentiable on $[a,b]$ then f is also continuous on $[a,b]$, since differentiability implies continuity.[64]

Integration by Parts Theorem. *Let f and g be real-valued functions that are differentiable on $[a,b]$. Assume that f' and g' are Riemann-integrable on $[a,b]$. Then $f'g$ and fg' are also Riemann-integrable on $[a,b]$, and*

$$\int_a^b f'g = f(b)g(b) - f(a)g(a) - \int_a^b fg'.$$

In the usual formulation if this Theorem, one assumes that f' and g' are continuous, even though the weaker assumption of their integrability also suffices.

[63] When there is a need to protect against misunderstanding, one can write $f'_+(x)$ for the derivative of f at x from the right, and $f'_-(x)$ of for its derivative at x from the left.

[64] One-sided differentiability implies one-sided continuity, as shown by an argument similar to the one showing that differentiability implies continuity.

Proof. As f and g are differentiable, they are also continuous, hence integrable, since continuous functions are integrable. The integrability of $f'g$ and fg' then follows from the Corollary. As $(fg)' = f'g + fg'$ is then also integrable, we can use the Newton–Leibniz formula:

$$\int_a^b (f'g + fg') = f(b)g(b) - f(a)g(a).$$

This equation is equivalent to the one claimed in the Theorem, completing the proof.

30. Change of variable

Given a function g on the interval $[a, b]$, we will say that g is differentiable on $[a, b]$ if g is differentiable for every $x \in (a, b)$, and g is differentiable from the right at a, and it is differentiable from the left at b. When there is no danger of misunderstanding, we will write $g'(a)$ for the derivative of g at a from the right, and $g'(b)$ for its derivative at b from the left.[65] Assuming that g is differentiable on $[a, b]$, it is easy to see that the set

$$\{g(x) \colon x \in [a, b]\}.$$

is a closed interval. Indeed, since differentiability implies continuity, the function g is continuous on $[a, b]$.[66] Hence the above set is connected and compact, it being a continuous image of a connected compact set, and as such it must be a closed interval.

In order to establish our main result, we will need the following

Lemma. *Let f be a function on the interval $[a, b]$, let $\epsilon > 0$, let $\epsilon > 0$, and let*

$$P \colon x_0 < x_1 < \ldots < x_N$$

be a partition of $[a, b]$ such that for any two Riemann sums S_1 and S_2 associated with this partition we have $|S_1 - S_2| < \epsilon$. Let $\xi, \eta_i \in [x_{i-1}, x_i]$ be a pair of tags for the interval $[x_{i-1}, x_i]$ for each i with $1 \le i \le N$. Then

$$\sum_{i=1}^{N} |f(\xi_i) - f(\eta_i)|(x_i - x_{i-1}) < \epsilon.$$

Proof. For each i with $1 \le i \le N$ put

$$u_i = \begin{cases} \xi_i & \text{if } f(\xi_i) \ge f(\eta_i), \\ \eta_i & \text{if } f(\xi_i) < f(\eta_i), \end{cases} \quad \text{and} \quad v_i = \begin{cases} \xi_i & \text{if } f(\xi_i) < f(\eta_i), \\ \eta_i & \text{if } f(\xi_i) \ge f(\eta_i). \end{cases}$$

We have

$$\sum_{i=1}^{N} |f(\xi_i) - f(\eta_i)|(x_i - x_{i-1}) = \sum_{i=1}^{N} f(u_i)(x_i - x_{i-1}) - \sum_{i=1}^{N} f(v_i)(x_i - x_{i-1}) < \epsilon;$$

the inequality holds because on the left-hand side of this inequality we have the difference between two Riemann sums associated with the partition P.

The following is a version of the Change of Variable Theorem for the Riemann Integral.

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[65] When there is a need to protect against misunderstanding, one can write $g'_+(x)$ for the derivative of g at x from the right, and $g'_-(x)$ of for its derivative at x from the left.

[66] One-sided differentiability implies one-sided continuity, as shown by an argument similar to the one showing that differentiability implies continuity.

Change of Variable Theorem. *Let $[a,b]$ be a closed interval in $\mathbb{R}$, and let g be a function that is differentiable on $[a,b]$. Let f be a bounded function defined on the interval*

$$[c,d] \stackrel{def}{=} \{g(x)\colon x \in [a,b]\}.$$

Assume, further, that the integrals

$$\int_a^b f(g(x))g'(x)\,dx, \quad \int_a^b g'(x)\,dx, \quad \text{and} \quad \int_{g(a)}^{g(b)} f(t)\,dt$$

exist. Then the first and the last of these integrals are equal, that is,

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(t)\,dt.$$

In the usual statement of this result, the continuity of f on $[c,d]$ and that of g' on $[a,b]$ is assumed; under these assumptions, the existence of the three integrals in question is assured.

Proof. Assume $g(b) > g(a)$ for the sake of simplicity. At the end, we will comment on what modifications are needed in case $g(b) < g(a)$ or $g(b) = g(a)$. For now, this assumption allows us to talk about the interval $[g(a), g(b)]$.

Let M be a bound for $|f|$ on $[c,d]$. Let $\epsilon > 0$ be arbitrary. Let $\delta_1 > 0$ be such that for any Riemann sum S for the function $(f \circ g)\cdot g'$ associated with a partition of width $< \delta_1$ of the interval $[a,b]$ we have $\left|S - \int_a^b (f \circ g)\cdot g'\right| < \epsilon/3$. Let $\delta_2 > 0$ be such that for any Riemann sum S for the function g' associated with a partition of width $< \delta_2$ of the interval $[a,b]$ we have $\left|S - \int_a^b g'\right| < \epsilon/(6M)$; then, for any two Riemann sums S_1 and S_2 for the function g' associated with a partition of width $< \delta_2$ of the interval $[a,b]$ we have $|S_1 - S_2| < \epsilon/(3M)$. Similarly, let Let $\delta_3 > 0$ be such that for any Riemann sum S for the function f associated with a partition of width $< \delta_3$ of the interval $[g(a), g(b)]$ we have $\left|S - \int_{g(a)}^{g(b)} f\right| < \epsilon/3$. As g is differentiable on $[a,b]$, it is continuous there, and so it is also uniformly continuous. Let δ with $0 < \delta \le \min\{\delta_1, \delta_2\}$ be such that we have $|g(x) - g(y)| < \delta_3$ for any $x, y \in [a,b]$ with $|x - y| < \delta$.

Let $P_0\colon a = s_1 < s_2 < \ldots < s_{N_0} = b$ be a partition of width $< \delta$ of the interval $[a,b]$. We may assume that there are i and i' with $1 \le i, i' \le N_0$ such that $g(s_i) = c$ and $g(s_{i'}) = d$; if this not already the case, then we may add two additional partition points to ensure this.

We will add further partition points to P_0 as follows: Let i with $1 \le i \le N_0$ be arbitrary. If $g(s_{i-1}) < g(s_i)$ and there is a k with $0 \le k \le N_0$ such that $g(s_{i-1}) < g(s_k) < g(s_i)$ then add the least number s with $s_{i-1} < s < s_i$ such that $g(s) = g(s_k)$ as a new partition point; that there is such an s can be seen by the Intermediate-Value Theorem, as g is continuous. That there is a least such s can also be easily seen. Indeed, we can take

$$s = \inf\{x\colon s_{i-1} < x < s_i \ \ \&\ \ g(x) = g(s_k)\}.$$

Similarly, if $g(s_{i-1}) > g(s_i)$ and there is a k with $0 \le k \le N_0$ such that $g(s_{i-1}) > g(s_k) > g(s_i)$ then add the largest number s with $s_{i-1} < s < s_i$ such that $g(s) = g(s_k)$ as a new partition point; that there is such an s can be seen by the Intermediate-Value Theorem, as g is continuous. That there is a largest such s can also be easily seen. Indeed, we can take

$$s = \sup\{x\colon s_{i-1} < x < s_i \ \ \&\ \ g(x) = g(s_k)\}.$$

Let

$$P\colon a = x_0 < x_1 < \ldots < x_N = b$$

be the partition so obtained. An important consequence of the construction of the partition P is that, for each i with $1 \le i \le N$, $g(x_{i-1})$ and $g(x_i)$ are adjacent elements of the set

$$\{g(x_k) \colon 0 \le k \le N\};$$

i.e., there is no k with $0 \le k \le N$ such that $g(x_k)$ lies strictly between $g(x_{i-1})$ and $g(x_i)$.[67] For each interval $[x_{i-1}, x_i]$ $(1 \le \xi \le N)$ let the tag ξ_i be defined as

$$\xi_i = \begin{cases} x_{i-1} & \text{if } g(x_{i-1}) > g(x_i), \\ x_i & \text{if } g(x_{i-1}) \le g(x_i). \end{cases}$$

This will ensure that

$$g(\xi_i) = \max\{g(x_{i-1}), g(x_i)\}. \tag{1}$$

Let S be the Riemann sum

$$S = \sum_{i=1}^{N} f(g(\xi_i))g'(\xi_i)(x_i - x_{i-1}).$$

Then we have

$$\left| \int_a^b f(g(x))g'(x)\,dx - S \right| < \frac{\epsilon}{3} \tag{2}$$

by our assumptions. Furthermore, writing[68]

$$S_1 = \sum_{i=1}^{N} f(g(\xi_i))(g(x_i) - g(x_{i-1})),$$

we claim that

$$|S - S_1| < \frac{\epsilon}{3}. \tag{3}$$

Indeed, for each i with $1 \le i \le N$ let $u_i \in [x_{i-1}, x_i]$ be such that

$$g'(u_i)(x_i - x_{i-1}) = g(x_i) - g(x_{i-1}); \tag{4}$$

such a u_i exists in view of the Mean-Value Theorem for Differentiation. Using this equation, by our assumptions (and the Lemma above), we have

$$\begin{aligned} |S - S_1| &= \left| \sum_{i=1}^{N} f(g(\xi_i))g'(\xi_i)(x_i - x_{i-1}) - \sum_{i=1}^{N} f(g(\xi_i))(g(x_i) - g(x_{i-1})) \right| \\ &= \left| \sum_{i=1}^{N} f(g(\xi_i))g'(\xi_i)(x_i - x_{i-1}) - \sum_{i=1}^{N} f(g(\xi_i))g'(u_i)(x_i - x_{i-1}) \right| \\ &\le \sum_{i=1}^{N} |f(g(\xi_i))| \, |g'(\xi_i) - g'(u_i)| \, (x_i - x_{i-1}) \\ &\le M \sum_{i=1}^{N} |g'(\xi_i) - g'(u_i)| \, (x_i - x_{i-1}) < M \cdot \frac{\epsilon}{3M} = \frac{\epsilon}{3}. \end{aligned}$$

[67] One may also observe, for whatever it is worth, that this set is the same as

$$\{g(s_k) \colon 0 \le k \le N_0\}.$$

[68] This sum is a Riemann-Stieltjes sum for the Stieltjes integral

$$\int_a^b f(g(x))\,dg(x);$$

however, we do not really need to get involved with Stieltjes integrals for our present purposes.

Indeed, the second equality follows by (4), the second inequality holds since M is an upper bound for f on $[c,d]$, and the third inequality is valid in view of the assumptions on the Riemann sums for g' and by virtue of the Lemma above.

Now, let

$$Q\colon g(a) = t_0 < t_1 < \ldots < t_K = g(b)$$

be the partition whose partition points are exactly the elements of the set

$$\{g(x_i) : 0 \le i \le N \ \ \& \ \ g(a) \le g(x_i) \le g(b)\}.$$

We then have

$$S_1 = \sum_{i=1}^{K} f(t_i)(t_i - t_{i-1}). \tag{5}$$

This can be seen as follows. For each interval $[x_{i-1}, x_i]$ such that $g(x_{i-1}) > g(x_i)$, where $1 \le i \le N$, we have a matching interval $[x_{i'-1}, x_{i'}]$ for some i' with $1 \le i' \le N$ such that $g(x_{i'-1}) = g(x_i)$ and $g(x_{i'}) = g(x_{i-1})$; to ensure that this is the case was the reason that we inserted additional points into the starting partition P_0 of $[a,b]$. According to the definition of the ξ_i's, we then have $\xi_i = x_{i-1}$ and $\xi_{i'} = x_{i'}$. Therefore the terms

$$f(g(\xi_i))(g(x_i) - g(x_{i-1})) \quad \text{and} \quad (g(\xi_{i'}))(g(x_{i'}) - g(x_{i'-1}))$$

cancel each other. This will cause the the cancelation of all terms in S_1 for which $g(x_i)$ or $g(x_{i-1})$ are outside the interval $[g(a), g(b)]$. Furthermore, for each i with $1 \le i \le K$ there will be only one j with $1 \le j \le N$ such that $t_{i-1} = g(x_{j-1})$, $t_i = g(x_j)$ and for which the term

$$f(g(\xi_j))(g(x_i) - g(x_{j-1}))$$

in S_1 survives; in this case $\xi_j = t_i$ (cf. (1)); that is, if there is more than one j satisfying these requirements, the corresponding terms for all but one of them will be canceled. We then have

$$f(g(\xi_j))(g(x_j) - g(x_{j-1})) = f(t_i)(t_i - t_{i-1}).$$

Thus, (5) follows. Moreover, we have

$$t_i - t_{i-1} = g(x_j) - g(x_{j-1}) < \delta_3$$

since $|x_j - x_{j-1}| < \delta$ (recall the definition of δ to see this). Therefore, the width of the partition Q is $< \delta$. Hence, by (5) we have

$$\left| S_1 - \int_{g(a)}^{g(b)} f \right| < \frac{\epsilon}{3}$$

by the choice of δ_3. This inequality, together with (2) and (3) implies

$$\left| \int_a^b f(g(x))g'(x)\,dx - \int_a^b f(t)\,dt \right| < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, this establishes the equality claimed in the theorem.

As for the assumption $g(a) < g(b)$, no serious use of this assumption was made in the proof. In particular, in case $g(a) = g(b)$, when deriving equation (5), all terms occurring in the definition of S_1 cancel, yielding $S_1 = 0$ instead of (5), as expected; the only reason for not including the possibility $g(a) = g(b)$ above was to ensure that we can talk about the interval $[g(a), g(b)]$. The case $g(b) > g(a)$ can be discussed in a similar way, or else, we can eliminate this case by using the change of variable $t = -\tau$ and working with the function $\tilde{g}$ defined as $\tilde{g}(t) = g(-t)$ instead of g.[69]

[69]The change of variable $t = -\tau$ does not rely on the Change of Variable Theorem. Instead, it relies on the fact that $\int_a^b f = -\int_b^a f$, true by the definition of $\int_a^b f$ in case $a > b$.

31. Compactness product spaces

The *Cartesian product* of two sets S_1 and S_2 is defined as the set of pairs

$$S_1 \times S_2 \overset{def}{=} \{(p_1, p_2) : p_1 \in S_1 \quad \text{and} \quad p_2 \in S_2\}.$$

If S_1 and S_2 are underlying sets of metric spaces, then a distance function can be introduced on their Cartesian product:[70]

Definition. Let (S_1, d_1) and (S_2, d_2) be metric spaces. The *Cartesian product*, or, shortly, the product (S, ρ) of these spaces is defined by putting $S = S_1 \times S_2$ and

$$d((p_1, p_2), (q_1, q_2)) \overset{def}{=} \sqrt{(d_1(p_1, q_1))^2 + (d_2(p_2, q_2))^2}.$$

For example, if E^1 denotes the metric space $(\mathbb{R}, d)$ with the usual distance $d(x, y) = |y - x|$, then the Euclidean E^2 with the usual distance can be written as $E^2 = E^1 \times E^1$. For this reason, one often writes $\mathbb{R}^2$ for the Euclidean plane.[71]

To show that the above definition is sound, we need to show that the distance d defined above satisfies the triangle inequalities, i.e., that for $p_1, q_1, r_1 \in S_1$ and $p_2, q_2, r_2 \in S_2$ we have

$$d((p_1, p_2), (r_1, r_2)) \le d((p_1, p_2), (q_1, q_2)) + d((q_1, q_2), (r_1, r_2)). \tag{41}$$

Writing $A_1 = d_1(p_1, q_1)$, $A_2 = d_2(p_2, q_2)$, $B_1 = d_1(q_1, r_1)$, and $B_2 = d_2(q_2, r_2)$, we have $d(p_1, r_1) \le A_1 + B_1$ and $d(p_2, r_2) \le A_2 + B_2$ and by the triangle inequality in S_1 and S_2, respectively, so we have

$$d((p_1, p_2), (r_1, r_2)) = \sqrt{(d_1(p_1, r_1))^2 + (d_2(p_2, r_2))^2} \le \sqrt{(A_1 + B_1)^2 + (A_2 + B_2)^2}.$$

Note that $d((p_1, p_2), (q_1, q_2)) = \sqrt{A_1^2 + A_2^2}$ and $d((q_1, r_2), (r_1, r_2)) = \sqrt{B_1^2 + B_2^2}$. Thus (41) will follow if we establish

$$\sqrt{(A_1 + B_1)^2 + (A_2 + B_2)^2} \le \sqrt{A_1^2 + A_2^2} + \sqrt{B_1^2 + B_2^2} \tag{42}$$

for all nonnegative reals A_1, A_2, B_1, and B_2. Squaring both sides and making obvious cancelations, we can see that this inequality is equivalent to the following:

$$A_1 B_1 + A_2 B_2 \le \sqrt{A_1^2 + A_2^2} \cdot \sqrt{A_1^2 + A_2^2}.$$

Again, squaring both sides (since the numbers A_1, A_2, B_1, and B_2 are nonnegative, both sides are nonnegative), and making cancelations, this is seen to be equivalent to

$$2A_1 B_1 A_2 B_2 \le A_1^2 B_2^2 + A_2^2 B_1^2.$$

This latter inequality is certainly true, since it can also be written as

$$0 \le (A_1 B_2 - A_2 B_1)^2.$$

Thus the equivalent inequality (42) follows. Hence (41) holds *a fortiori*.[72]

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[70] The habit of using the letter E for metric spaces seems to originate with the French word *espace* for space; for example, Rosenlicht usually uses the letter E to denote metric space. Here we broke the habit.

[71] Note the notational confusion that one often uses the same symbol, such as e.g. E^2, to denote a metric space as well as its underlining set. Thus, $\mathbb{R}^2$ is used both to denote the set of pairs of reals and the Euclidean plane with the usual distance function. Most of the time, such notational confusion is harmless – at other times, it can be avoided by using different fonts, such as, e.g., $\mathcal{E}^2$ for the space and E^2 for the underlying set.

[72] The Latin phrase *a fortiori*, meaning "even more so," is frequently used in the mathematical literature.

A base of a metric space. Many properties of a metric space can be derived solely from knowing what the open sets are in the space, and the distance function itself is not used directly (i.e., beyond using it to determine the open sets). In fact, in the mathematical discipline of *topology*, one dispenses with the distance function entirely and studies the properties of the space describable in terms of open the open sets.[73] It is often useful to describe the open sets with the aid of the concept of a *base*:

Definition. Let (S,d) be a metric space. A set $\mathcal{B}$ of open subsets of S is called a *base* of (S,d) if for every point $p \in S$ and every open set $U \subset S$ with $p \in U$ there is a $B \in B$ such that $p \in B \subset U$.

It is clear that B of S can be used to describe the open sets of S. As can be easily shown, a set $U \subset S$ is open if for every $p \in U$ there is a B in $\mathcal{B}$ such that $p \in B \subset U$. Note that the point here is somewhat subtle and requires careful reading, since the last sentence is almost the same as the sentence defining the concept of a base.

An example for base in a metric space is the set of open balls of that space. For the product of two metric spaces, a useful base is the set of *open rectangles*, which we will proceed to describe. Let (S_1, d_1) and $(S_2, d_2$ be metric spaces, and let (S,d) be their Cartesian products. Denoting by $\mathcal{O}_1$ and $\mathcal{O}_2$ the set of open subsets of S_1 and S_2, respectively, the

$$\{U_1 \times U_2 : U_1 \in \mathcal{O}_1 \quad \text{and} \quad U_2 \in \mathcal{O}_2\}$$

is a base of (S,d). Indeed, if $U \subset S$ is an open set and $(p_1,p_2) \in U$, then there is an $\epsilon > 0$ such that the open ball $\{(q_1,q_2) : (q_1,q_2) \in S \text{ and } d((p_1,p_2),(q_1,q_2)) < \epsilon\}$ of center (p_1,p_2) and radius ϵ is a subset of U. Then, writing $U_1 = \{q_1 : q_1 \in S_1 \text{ and } d(p_1,q_1) < \epsilon/\sqrt{2}\}$ and $U_2 = \{q_2 : q_2 \in S_2 \text{ and } d(p_2,q_2) < \epsilon/\sqrt{2}\}$, these sets are open balls of radius $\epsilon/\sqrt{2}$ in S_1 and S_2, respectively. We clearly have $U_1 \times U_2 \subset U$. We will call elements of the base just described *open rectangles*.[74]

A simple but important application of bases is the following.

Lemma. *Let (S,d) be a metric space and let $\mathcal{B}$ be a base of S. Let C be a subset of S. Assume that for any subset $\mathcal{B}'$ of $\mathcal{B}$ such that $C \subset \bigcup \mathcal{B}'$ there is a finite set $\mathcal{B}'' \subset \mathcal{B}'$ such that $C \subset \bigcup B''$. Then C is compact.*

The statement of the Lemma is almost identical to the definition of a compact set, except that instead of open covers only covers by members of a base are considered.

Proof. For each open set U of S and for every point $p \in U$ let $B(p,U) \in \mathcal{B}$ such that

$$p \in B(p,U) \in U.$$

Let $U_\iota : \iota \in I$ be an arbitrary collection of open sets[75] I may not be a countable set such that $C \subset \bigcup_{\iota \in I} U_\iota$. Then the set

$$\mathcal{B}' = \{B(p,U_\iota) : \iota \in I \quad \text{and} \quad p \in U_\iota\}$$

covers C; indeed, it is clear from the definition of a base that

$$\bigcup B' = \bigcup_{\iota \in I} U_\iota.$$

[73] The above meaning of *topology* is often described as *point set topology*. Another meaning of the word is *differential topology*; this is a sub-discipline of *differential geometry*.

[74] Other authors use the term open rectangle to describe the Cartesian products of open balls; it is easy to see that these sets also form a base of the product space.

[75] The symbol ι is the lower case Greek letter *iota*. It, rather than the letter i, is used to suggest that the set I may possibly be too large to be put into a one-to-one correspondence with the integers – i is often, but not always, reserved to denote integers.

Thus, by our assumption, a finite set $\mathcal{B}'' \subset \mathcal{B}'$ such that $C \subset \bigcup \mathcal{B}''$. The set $\mathcal{B}'$ can be written in the form

$$\mathcal{B}'' = \{B(p, U_\iota) : \iota \in I' \quad \text{and} \quad p \in X_\iota\}$$

where I' is a finite subsets of I and, for each $\iota \in I'$, X_ι is an appropriate finite subset of U_ι. Then we have

$$C \subset \bigcup_{\iota \in I'} U_\iota.$$

Thus the finite collection $\{U_\iota : \iota \in I'\}$ covers C. This shows that C is indeed compact, completing the proof.

Compactness of products. The following is a special case of a much more general result, called Tychonoff's Theorem. This latter asserts the compactness of the product of infinitely many spaces, but in order to discuss this result we would first need to define the product of infinitely many spaces. Here we will deal with the product of only two spaces; the result will be useful for us in discussing the interchangeability of integration and differentiation.

Theorem. *Let (S_1, d_1) and (S_2, d_2) be metric spaces and let (S, d) be their Cartesian product. Assume that S_1 and S_2 are compact. Then S is also compact.*

Proof. Let $\mathcal{B}$ be the set of open rectangles, described above, of the space (S, d), and let $\{B_\iota : \iota \in I\}$ be a subset of $\mathcal{B}$ such that $S \subset \bigcup_{\iota \in I} B_\iota$. In view of the Lemma above, we need to prove that there is a finite set $I' \subset I$ such that $S \subset \bigcup_{\iota \in I'} B_\iota$ also holds.

For each $\iota \in I$ write $B_\iota = U_\iota \times V_\iota$, where U_ι and V_ι are open subsets of S_1 and S_2, respectively. For each $p \in S_1$, write

$$I_p = \{\iota \in I : p \in U_\iota\}$$

Then $\{p\} \times S_2 \subset \bigcup_{\iota \in I_\iota} U_\iota \times V_\iota$, and so $S_2 \subset \bigcup_{\iota \in I} V_\iota$. Since S_2 is compact, there is a finite set $I'_p \subset I_p$ such that $S_2 \subset \bigcup_{\iota \in I'_p} V_\iota$. Write $O_p = \bigcap_{\iota \in I'_p} U_\iota$; the set O_p, being the intersection of finitely many open sets of the space S_1, is an open set of the space S_1.

Clearly

$$S_1 \subset \bigcup_{p \in S_1} O_p,$$

since $p \in O_p$ for each $p \in S_1$. The space S_1 being compact, this cover has a finite subcover. That is, there is a finite set $X \subset S_1$ such that

$$S_1 \subset \bigcup_{p \in X} O_p.$$

Since we have $O_p \times S_2 \subset \bigcup_{\iota \in I'_p} U_\iota \times V_\iota$ in view of the definition of O_p, it follows that

$$S_1 \times S_2 \subset \bigcup_{p \in X} \bigcup_{\iota \in I'_p} U_\iota \times V_\iota = \bigcup \{U_\iota \times V_\iota : \iota \in \bigcup_{p \in X} I'_p\}.$$

Thus, the set on the right-hand side after the union sign is a finite open cover of $S_1 \times S_2$, showing that this space is compact. The proof is complete.

As a consequence, for any reals a, b, c, d with $a \le b$ and $c \le d$, the product

$$[a, b] \times [c, d]$$

is a compact subset of E^2. Since a closed subset of a compact set is compact, it follows that any bounded closed subset of E^2 is compact. Since for any positive integer n we have[76] $E^{n+1} = E^n \times \mathbb{R}$, we obtain by induction the following.

[76] In a strictly formal sense, the equation $E^{n+1} = E^n \times \mathbb{R}$ may not be true. There is no universally agreed definition of the Cartesian product $A \times B \times C$ of the sets A, B, and C. With some of the definitions, the equation $A \times B \times C = (A \times B) \times C$ is true, with others it is not. No definition satisfies the associativity rule $(A \times B) \times C = A \times (B \times C)$. The failure of these equations, is, however, only formal; in a practical sense, these equations are as good as true. That is, there is a one-to-one mapping from $(A \times B) \times C$ onto $A \times (B \times C)$, for example.

Corollary. *Given an positive integer n, any bounded closed subset of the Euclidean space E^n is compact.*

32. Interchanging integration and differentiation

Consider a function $f\colon E^2 \to \mathbb{R}$, where E^2 is the Euclidean plane. The value of such a function at the point (x,y) of E^2 should properly be denoted as $f((x,y))$. Usually, however, one uses the notation $f(x,y)$, and one also calls f a function of two variables. For a fixed value of $x \in \mathbb{R}$, the function $f_x : \mathbb{R} \to \mathbb{R}$ is defined as $f_x(y) = f(x,y)$. Similarly, for fixed $y \in \mathbb{R}$, the function $f^y : \mathbb{R} \to \mathbb{R}$ is defined as $f^y(x) = f(x,y)$. If these functions are differentiable, their derivatives are called the partial derivatives of f: $(f_x)'(y) = \frac{\partial f}{\partial y}(x,y)$ and $(f^y)'(x) = \frac{\partial f}{\partial x}(x,y)$. If we write out the definition of the derivative, the partial derivatives can be defined as the following limits:

$$\frac{\partial f}{\partial x}(x,y) = \lim_{t\to x} \frac{f(t,y)-f(x,y)}{t-x}, \qquad \frac{\partial f}{\partial y}(x,y) = \lim_{t\to y} \frac{f(x,t)-f(x,y)}{t-y}.$$

This notation for partial derivatives is easy to object to, as the association between x and y and the arguments of f is a loose one (since one can certainly talk about $f(u,v)$), so a better notation would be $\partial_1 f$ instead of $\frac{\partial f}{\partial x}$ and $\partial_2 f$ instead of $\frac{\partial f}{\partial y}$.

The following result discusses the interchange of differentiation and integration. There are many results of this type.

Theorem. *Let a, b, c, and d be real numbers with $a < b$ and $c < d$. Let f be a continuous real-valued function on the set $[a,b]\times(c,d)$ such that $\frac{\partial f}{\partial y}$ is also continuous on this set. Then, for each fixed y with $c < y < d$ we have*

$$\frac{d}{dy}\int_a^b f(x,y)\,dx = \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx.$$

The right-hand side contains an integral of a continuous function, and we know that such an integral exists. Similarly, the integral on the left-hand side exists, it being the integral of a continuous function. The equation in the theorem asserts that this integral, that is, the function

$$F(y) = \int_a^b f(x,y)\,dx,$$

is differentiable, and its derivative $F'(y)$ equals the right-hand side. For the proof, we need a lemma about integrals involving a uniformly convergent sequence of functions.

Lemma. *Assume that the real-valued functions f_n ($1 \le n < \infty$) are continuous on the interval $[a,b]$. Assume, further, that $\lim_{n\to\infty} f_n = f$ uniformly on $[a,b]$. Then*

$$\lim_{n\to\infty}\int_a^b f_n = \int_a^b f.$$

Proof. First, f is continuous on $[a,b]$, since the continuous functions f_n converge to f uniformly on $[a,b]$. Continuous functions are integrable, so the functions f and f_n ($1 \le f < \infty$) are integrable. Let $\epsilon > 0$ be arbitrary. As the functions f_n converge to f uniformly, there is an integer $N > 0$ such that $|f(x) - f_n(x)| < \frac{\epsilon}{2(b-a)}$ for every $x \in [a, b]$ and for every integer $n > N$. That is,

$$f_n(x) - \frac{\epsilon}{2(b-a)} < f(x) < f_n(x) + \frac{\epsilon}{2(b-a)}$$

holds for every $n > N$ and for every $x \in [a, b]$. Therefore,[77]

$$\begin{aligned}\int_a^b f_n - \frac{\epsilon}{2} &= \int_a^b \left(f_n - \frac{\epsilon}{2(b-a)} \right) \le \int_a^b f \\ &\le \int_a^b \left(f_n + \frac{\epsilon}{2(b-a)} \right) = \int_a^b f_n + \frac{\epsilon}{2}.\end{aligned}$$

Thus $| \int_a^b f - \int_a^b f_n| < \frac{\epsilon}{2} < \epsilon$ for $n > N$. Hence $\int_a^b f_n$ converges to $\int_a^b f$. The proof is complete.

Note. In the proof above, we used the statement that if f and g are Riemann integrable on $[a, b]$ and $f(x) \le g(x)$ for all $x \in [a, b]$, then we have $\int_a^b f \le \int_a^b g$. Since we actually had $f(x) < g(x)$ for all $x \in [a, b]$ for the functions in question, we could have obtained the stronger conclusion $\int_a^b f < \int_a^b g$ (provided $a < b$). However, proving that the assumption $f(x) < g(x)$ for all $x \in [a, b]$ implies that $\int_a^b f < \int_a^b g$ involves some technical complications that we wanted to avoid.

Next we turn to the

Proof of the Theorem. Let c' and d' be reals with $c < c' < y < d' < d$. Then the set $[a, b] \times [c', d']$ is compact. Since the function $\frac{\partial f}{\partial y}$ is continuous on this set, it is uniformly continuous there. Let $\{t_n\}_{n=1}^{\infty}$ be an arbitrary sequence of reals different from y in the interval $[c', d']$ such that $\lim_{n\to\infty} t_n = y$. Then

$$\lim_{n\to\infty} \frac{f(x, t_n) - f(x, y)}{t_n - y} = \lim_{t\to y} \frac{f(x, t) - f(x, y)}{t - y} = \frac{\partial f}{\partial y}(x, y). \tag{43}$$

Indeed, the second equation is the definition of the partial derivative, while the first equation is the description of limits of functions in terms of limits of sequences.[78] We claim that the convergence of the sequence on the left is uniform in the interval $[a, b]$ (that is, for $x \in [a, b]$; note that y is fixed here).

To see this, let $\epsilon > 0$ be arbitrary, and let $\delta > 0$ be such that for any two points (x_1, y_1) and (x_2, y_2) in the set $[a, b] \times [c', d']$ with $d((x_1, y_1), (x_2, y_2)) < \delta$ we have

$$\left| \frac{\partial f}{\partial y}(x_1, y_1) - \frac{\partial f}{\partial y}(x_2, y_2) \right| < \epsilon$$

(here $d(.,.)$ denotes the Euclidean distance in E^2). In particular, if we choose $\delta < \min\{y - c', d' - y\}$, then for every $x \in [a, b]$ and every y_1 with $|y_1 - y| < \delta$ we have

$$\left| \frac{\partial f}{\partial y}(x, y_1) - \frac{\partial f}{\partial y}(x, y) \right| < \epsilon$$

(note that the inequality above involving y_1 guarantees that $y_1 \in [c', d']$ in view of our choice of δ). Pick $N > 0$ such that we have $|t_n - y| < \delta$ for $n > N$. Consider a fixed $n > N$

[77] See the note after the proof.

[78] See Rosenlicht [Ros], the Proposition on p. 74. That Proposition is stated in terms of continuous functions, but it can easily be restated in terms of limits of functions.

and a fixed $x \in [a, b]$ By the Mean-Value Theorem, there is a $\eta_{n,x}$ between y and t_n (i.e., $\eta_{n,x} \in (t_n, y)$ if $t_n < y$, and $\eta_{n,x} \in (y, t_n)$ if $t_n > y$) such that

$$\frac{\partial f}{\partial y}(x, \eta_{n,x}) = \frac{f(x, t_n) - f(x, y)}{t_n - y}.$$

Observing that $|\eta_{n,x} - y| < |t_n - y| < \delta$, we have

$$\left| \frac{f(x, t_n) - f(x, y)}{t_n - y} - \frac{\partial f}{\partial y}(x, y) \right| = \left| \frac{\partial f}{\partial y} f(x, \eta_{n,x}) - \frac{\partial f}{\partial y}(x, y) \right| < \epsilon.$$

This inequality shows that the convergence on the left-hand side of (43) is uniform on $[a, b]$.

Using the lemma about the integral of a uniformly convergent sequence of continuous functions, we obtain that

$$\lim_{n\to\infty} \int_a^b \frac{f(x, t_n) - f(x, y)}{t_n - y}\, dx = \int_a^b \frac{\partial f}{\partial y}(x, y)\, dx.$$

As the this is true for every sequence $\{t_n\}_{n=1}^{\infty}$ with $t_n \neq y$ and $t_n \in [c', d']$, it follows in view of the proposition describing the limits of functions in terms of limits of sequences[79] that

$$\lim_{t\to y} \int_a^b \frac{f(x, t) - f(x, y)}{t - y}\, dx = \int_a^b \frac{\partial f}{\partial y}(x, y)\, dx.$$

The left-hand side here can be written as

$$\lim_{t\to y} \frac{1}{t - y} \left(\int_a^b f(x, t)\, dx - \int_a^b f(x, y)\, dx \right) = \frac{d}{dy} \int_a^b f(x, y)\, dx.$$

Remark. The assumptions of the above theorem are unnecessarily restrictive. In fact, we have the following

THEOREM. *Let a, b, c, and d be real numbers with $a < b$ and $c < d$, and let $y \in (c, d)$. Let f be a continuous a real-valued function on the set $[a, b] \times (c, d)$ such that $\frac{\partial f}{\partial y}$ exists everywhere in this set and is continuous at every point of the set $[a, b] \times \{y\}$. Then we have*

$$\frac{d}{dy} \int_a^b f(x, y)\, dx = \int_a^b \frac{\partial f}{\partial y} f(x, y)\, dx.$$

The proof of this theorem is close to that of the above theorem, but there are some changes. First, let (E, d) and (E', d') be metric spaces, and let A and B be subsets of E with $A \subset B$. Call a function $f : B \to E'$ continuous uniformly over A within B if for every $\epsilon > 0$ there is a $\delta > 0$ such that for every $p \in A$ and $q \in B$ with $d(p, q) < \delta$ we have $d'(f(p), f(q))) < \epsilon$. Call f continuous over A within B if for every $p \in A$ and every $\epsilon > 0$ there is a $\delta > 0$ such that for every $q \in B$, if we have $d(p, q) < \delta$ then we also have $d'(f(p), f(q) < \epsilon$. We have

LEMMA. *Let (E, d) and (E', d') be metric spaces, and let A and B be subsets of E with $A \subset B$. Assume $f : B \to E'$ is continuous over A within B. If A is compact then f is continuous uniformly over A within B.*

The proof is identical to the usual proof of uniform continuity of a function that is continuous on a compact set.[80] This lemma can be used to carry out a proof of the above

[79] Cf. Rosenlicht [Ros], the Proposition on p. 74.

[80] Cf. the first proof of the Theorem on p. 80 in Rosenlicht [Ros].

theorem. In addition to obtaining a stronger result, there is an additional advantage to this approach. In this proof, only the compactness of the subset $[a, b] \times \{y\}$ of E^2 is used, and the compactness of this set is equivalent to that of the interval $[a, b]$ of $\mathbb{R}$, whereas in the proof above of the first theorem of this section the compactness of the subset $[a, b] \times [c', d']$ of E^2 was used, and the compactness of this set is significantly harder to establish than the compactness of the interval $[a, b]$ of $\mathbb{R}$.

The use in the proof of the lemma about integration of uniformly convergent sequences of functions was at the price of restating limits of functions in terms of limits of sequences; this could easily have been avoided by implicitly redoing the proof of the lemma.

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[Wiki] Wikipedia entry `http://en.wikipedia.org/wiki/Taylor%27s_theorem` as given on July 10, 2007.

LIST OF SYMBOLS

Page numbers usually refer to the place where the symbol in question was introduced. In certain cases, where important discussion concerning the symbol occurs at several places, the number of the page containing the main source of information is printed in italics.

SUBJECT INDEX

Page numbers in italics refer to the main source of information about whatever is being indexed. Often, they refer to the statement of a theorem, while the other page numbers listed refer to various applications of the same.