The Bolzano–Weierstrass theorem*

Attila Máté
Brooklyn College of the City University of New York

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1 The Bolzano–Weierstrass theorem

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of natural numbers, and let $\mathbb{R}$ be the set of reals.

Definition 1.1. A sequence of natural numbers is a function $\pi : \mathbb{N} \to \mathbb{N}$. π is called increasing if $\pi(n+1) > \pi(n)$ for all $n \in \mathbb{N}$. A sequence of reals is a function $f : \mathbb{N} \to \mathbb{R}$, a subsequence g of f is a composition $f \circ \pi$ where $\pi : \mathbb{N} \to \mathbb{N}$ is increasing.

In common parlance, by a sequence of reals one means an infinite list of real numbers a_1, a_2, a_3, a_4, a_5, a_6, a_7 a_8, a_9, a_{10}, a_{11}, $\ldots$, and by a subsequence one means an infinite list containing only some of these numbers, arranged in the same order; for example a_2, a_3, a_5, a_7, a_8, a_{10}, a_{11}, $\ldots$ is a subsequence. In the sense of the above definition, one can take $f(n) = a_n$ and $\pi(1) = 2$, $\pi(2) = 3$, $\pi(3) = 5$, $\pi(4) = 7$, $\pi(5) = 8$, $\pi(6) = 10$, $\pi(7) = 11$, $\ldots$. Then, for the above subsequence, one has $g = f \circ \pi$.

Lemma 1.1. *If π is an increasing sequence of integers then $\pi(n) \geq n$ for all $n \in \mathbb{N}$.*

Proof. We use induction of n. For $n = 1$, we clearly have $\pi(n) \geq n$ since $\pi(n) \in \mathbb{N}$. If $\pi(n) \geq n$ for some n, then $\pi(n+1) > \pi(n) \geq n$, so $\pi(n+1) \geq n+1$. $\square$

If $f : \mathbb{N} \to \mathbb{R}$ is a convergent sequence of reals, write

$$\lim f \overset{def}{=} \lim_{n\to\infty} f(n).$$

We have

Lemma 1.2. *If f is a convergent sequence of reals and g is a subsequence, them $\lim g = \lim f$.*

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Proof. Write $s = \lim f$. Then, for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $|f(n) - s| < \epsilon$ for every $n > N$. Since we then have $\pi(n) \geq n > N$ by Lemma 1.1, we also have $|g(n) - s| = |f(\pi(n)) - s| < \epsilon$ for every $n > N$. Hence, indeed, $\lim g = s$. $\square$

As an example, if $0 < a < 1$, then the sequence f defined as $f(n) = a^n$ for $n \in \mathbb{N}$ is convergent, since it is a bounded decreasing sequence. The sequence g defined as $g(n) = a^{n+1}$ $(n \in \mathbb{N})$ is a subsequence of f; indeed, $g = f \circ \pi$ where $\pi(n) = n + 1$ for all $n \in \mathbb{N}$. Hence, $\lim g = \lim f$, that is,

$$\lim_{n\to\infty} a^{n+1} = \lim_{n\to\infty} a^n.$$

This equation was the key in showing that $\lim_{n\to\infty} a^n = 0$.

Theorem 1.1. *If $\{a_n\}_{n=1}^{\infty}$ is a sequence of reals, then it has a monotone subsequence.*

Proof. Let

$$M = \{n \in \mathbb{N} : \text{ for every } n' > n \text{ we have } a_{n'} > a_n\}.$$

If M is infinite, then the sequence $\langle a_n : n \in M\rangle$ is strictly increasing. If M is finite, let $n_1 > \max M$. Let $n_2 > n_1$ be such that $a_{n_2} \leq a_{n_1}$; there is such an n_2, since $n_1 \notin M$. Let $n_3 > n_2$ be such that $a_{n_3} \leq a_{n_2}$; there is such an n_3, since $n_2 \notin M$. Continuing this way, if n_k has been selected for $k \geq 1$, let $n_{k+1} > n_k$ be such that $a_{n_{k+1}} \leq a_{n_k}$; there is such an n_{k+1}, since $n_k \notin M$. In this way we found a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$ that is decreasing in the wider sense. $\square$

The following result is known as the Bolzano–Weierstrass theorem (the links are clickable, and lead to the Wikipedia pages of the authors of the result).

Corollary 1.1. *Every bounded sequence of reals has a convergent subsequence.*

Proof. By the preceding theorem, take a monotone subsequence of the given sequence. Since a bounded monotone sequence is convergent, this subsequence is convergent. $\square$

Definition 1.2. If $\{a_n\}_{n=1}^{\infty}$ is a sequence of reals then

$$\liminf_{n\to\infty} a_n \stackrel{def}{=} \inf\{x : x \geq a_n \quad \text{for infinitely many values of} \quad n\} \tag{1}$$

For a set S of reals, if S is empty we put $\inf S = +\infty$, and if S is unbounded from below we put $\inf S = -\infty$. If S is nonempty and bounded from below then it is knows that $\inf S \in \mathbb{R}$. In case the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded, then the set

$$S = \{x : x \geq a_n \quad \text{for infinitely many values of} \quad n\} \tag{2}$$

is nonempty and bounded from below, and so in this case $\liminf_{n\to\infty} = \inf S \in \mathbb{R}$.

The next theorem supplies another proof of the Bolzano–Weierstrass theorem.

Theorem 1.2. *If $\{a_n\}_{n=1}^{\infty}$ is a bounded sequence of reals then if has a subsequence that converges to $\liminf_{n\to\infty} a_n$.*

This theorem was stated toward the end of the class; I tried to rush through the proof, but I made some mistakes. What follows is a corrected proof.

The proof of his theorem relies on the next lemma:

Lemma 1.3. *Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence of reals, and let $s = \liminf_{n\to\infty} a_n$. Let $\epsilon > 0$ be arbitrary. Then the interval $(s - \epsilon, s + \epsilon)$ contains a_n for infinitely many values of n.*

Proof. Let S be the set defined by formula (2). Then S is nonempty, bounded, and $s = \inf S$. We have $s - \epsilon < \inf S$, so $s - \epsilon \notin S$. Hence, there are only finitely many values of n for which $a_n \leq s - \epsilon$. On the other hand, $s + \epsilon > \inf S$, so $s + \epsilon$ is not a lower bound of S. Hence there is an $x \in S$ for which $x < s + \epsilon$. Therefore, we have $x \geq a_n$ for infinitely many values of n. Thus, *a fortiori*,[1.1] we have $s + \epsilon > a_n$ for infinitely many values of n. As we saw just before, for only finitely many of these values of n do we also have $s - \epsilon \geq a_n$. So, indeed, we have $a_n \in (s - \epsilon, s + \epsilon)$ for infinitely many values of n. $\square$

Proof of Theorem 1.2. Write $s = \liminf_{n\to\infty} a_n$. Using the lemma just proved, let n_1 be such that $a_{n_1} \in (s - 1, s + 1)$. Let $n_2 > n_1$ be such that $a_{n_2} \in (s - 1/2, s + 1/2)$. If for $k > 1$ we have selected n_{k-1}, let $n_k > n_{k-1}$ be such that $a_{n_k} \in (s - 1/k, s + 1/k)$. Then, clearly, the subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$ converges to s. $\square$

There is more about the Bolzano–Weierstrass theorem on pp. 23–24 (pdf pp. 29–39) in my notes (a clickable link) Supplementary Notes on Introduction to Analysis by Maxwell Rosenlicht, using various other techniques for the proof.

[1.1] For even stronger or greater reason (Latin)