Calculus 2 notes*

Attila Máté
Brooklyn College of the City University of New York

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Contents

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1 Inverse trigonometric functions

1.1 The inverse of a function

Let $f : A \to B$ be a function from the set A into the set B. f is called *one-to-one*, or 1-1, if for every x_1 and x_2 in the set A, if $x_1 \neq x_2$ then we have $f(x_1) \neq f(x_2)$. This means that if we know $y = f(x) \in B$, then we can determine $x \in A$ uniquely. f is called *onto* B if for every $y \in B$ there is an $x \in A$ such that $f(x) = y$.

If these conditions are satisfied, then we can define the *inverse* of f. That is, if $f : A \to B$ is one-to-one and onto B, then its inverse $g : B \to A$ is defined as the function such that for $y \in B$ $g(y) = x$ if and only if $f(x) = y$. The inverse of f is often written as f^{-1}, though one should use this notation with some care so as to avoid occasional ambiguities.

Note that here $\mathrm{dom}(f) = A$ and $\mathrm{ra}(f) = B$, and $\mathrm{dom}(f^{-1}) = \mathrm{dom}(g) = B$ and $\mathrm{ra}(f^{-1}) = \mathrm{ra}(g) = A$. In general, for any one-to-one function f and its inverse f^{-1} it is always true that

$$\mathrm{dom}(f^{-1}) = \mathrm{ra}(f) \qquad \text{and} \qquad \mathrm{ra}(f^{-1}) = \mathrm{dom}(f). \tag{1.1}$$

Further, note that writing $y = f(x)$, we have $g(y) = x)$, $g(f(x)) = g(y) = x$ for any $x \in A$, and $f(g(y)) = f(x) = y$ for any $y \in B$. Thus, for any set S. writing id_S for the identity function on S, that is the function $\mathrm{id} : S \to S$ such that $\mathrm{id}(s) = s$ for all $s \in S$. this means that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. In general, for any one-to-one function f and its inverse g we have

$$g \circ f = \mathrm{id}_{\mathrm{dom}\, f} \qquad \text{and} \qquad f \circ g = \mathrm{id}_{\mathrm{ra}\, f}\,. \tag{1.2}$$

Conversely, these equations imply that f and g are inverses to each other; we leave it to the reader to show this.

In general, one cannot talk about how the graph of a function relates to the graph of its inverse for the simple reason that the sets A and B can be very abstract sets so the function f is not amenable to graphing. In the special case when A and B are intervals on the real line and the function f is nice, the function f can be graphed as a curve. The fact that this curve represents a function is equivalent to saying that every vertical line in the Cartesian coordinate system intersects the curve at most once, corresponding to the fact that a function only assumes one value at a place, that is, given $x \in \operatorname{dom}(f)$, $f(x)$ is a unique value. The fact that f is one-to-one can be described by saying that each horizontal line intersects the graph of f at most once. The graph of f itself is the set of points

$$\{(x, y) : y = f(x)\},$$

while if g is the inverse of f, the graph of g is

$$\{(x, y) : y = g(x)\}.$$

As the equation $y = g(x)$ is equivalent to the equation $x = f(y)$, this set can be described as

$$\{(x, y) : x = f(y)\} = \{(y, x) : y = f(x)\}.$$

That is, the coordinates of the points of the graph of the inverse can be obtained by reversing the coordinates of the points of the graph of the function. Visually, this means that the graph of the inverse can be obtained by reflecting the graph of the function about the line $y = x$ (indeed, the points (a, b) and (b, a) are reflections of each other about the line $y = x$).

One often talks aboout the inverses of the trigonometric functions sin, cos, tan, sec, csc, though none these functions are not one-to-one, so in a strict sense they do not have inverses.

1.2 The inverse of sine

In order to define the inverse of sine, we introduce a restriction of $\sin x$:[1.1]

$$\operatorname{Sin} x \overset{def}{=} \begin{cases} \sin x & \text{if } -\pi/2 \le x \le \pi/2, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

One might call the function Sin the *principal* (i.e., main) part of sin. The “inverse” of sin will be defined as the inverse of Sin:

$$y = \arcsin x \quad \text{means that} \quad x = \operatorname{Sin} y.$$

In other words, we say that $y = \arcsin x$ if and only if $x = \sin y$ and $-\pi/2 \le y \le \pi/2$, and we call arcsin the inverse of sin.[1.2] According to equation (1.1), we have

$$\operatorname{dom}(\arcsin) = \operatorname{ra}(\operatorname{Sin}) = \operatorname{ra}(\sin) = [-1, 1]$$

and

$$\operatorname{ra}(\arcsin) = \operatorname{dom}(\operatorname{Sin}) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Further, we have

$$\operatorname{Sin}(\arcsin x) = x \qquad x \in \operatorname{ra}(\operatorname{Sin}) = [-1, 1]$$

according to the second equation in (1.2); that is,

$$\sin(\arcsin x) = x \qquad (x \in [-1, 1]), \tag{1.3}$$

since $\operatorname{Sin} x = \sin x$ whenever $\operatorname{Sin} x$ is defined.

[1.1]The symbol Sin is only introduced for the present discussion; it is not commonly used otherwise.

[1.2]The book [9] often uses $\sin^{-1} x$ instead of $\arcsin x$. We would strongly discourage the use of the former, since one could also interpret $\sin^{-1} x$ as $(\sin x)^{-1} = 1/\sin x$ in analogy with $\sin^2 x$ meaning $(\sin x)^2$.

1.2.1 The derivative of $\arcsin x$

If we have

$$y = \arcsin x$$

then we also have

$$x = \sin y,$$

though the two equations are not equivalent, since the first equation implies that

$$-\pi/2 \le y \le \pi/2, \tag{1.4}$$

whereas the second equation can hold also without this inequality being true. Differentiating the second equation with respect to x, we obtain

$$1 = y' \cos y,$$

whence we have

$$y' = \frac{1}{\cos y} = \frac{1}{\pm\sqrt{1 - \sin^2 y}}.$$

As for the $\pm$ sign, it should be $+$ since $\cos y \ge 0$ for y satisfying inequality (1.4). Thus,

$$y' = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}},$$

where the second equation follows according to (1.3). That is,

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1 - x^2}}.$$

As a side benefit, we migh want to note that we also obtained the equation

$$\cos(\arcsin x) = \sqrt{1 - x^2} \qquad (-1 \le x \le 1),$$

where the inequality ensures that $\arcsin x$ is defined. This equation will be of use later. By reversing the equation we obtained for the derivative of $\arcsin x$, we obtain the integral formula

$$\int \frac{dx}{\sqrt{1 - x^2}} = \arcsin x + C, \tag{1.5}$$

a formula that we want to add to our list of basic integrals.

1.3 The inverse of cosine

Proceeding similarly to defining the inverse of sine, we define the principal part of cosine as

$$\operatorname{Cos} x \overset{def}{=} \begin{cases} \cos x & \text{if } 0 \le x \le \pi, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

We define the "inverse" of cos as the inverse of Cos:

$$y = \arccos x \quad \text{means that} \quad x = \operatorname{Cos} y.$$

In other words, we say that $y = \arccos x$ if and only if $x = \cos y$ and $0 \le y \le \pi$. According to equation (1.1), we have

$$\operatorname{dom}(\arccos) = \operatorname{ra}(\operatorname{Cos}) = \operatorname{ra}(\cos) = [-1, 1]$$

and

$$\operatorname{ra}(\arccos) = \operatorname{dom}(\operatorname{Cos}) = [0, \pi]\,.$$

Further, we have

$$\operatorname{Cos}(\arccos x) = x \qquad x \in \operatorname{ra}(\operatorname{Cos}) = [-1, 1]$$

according to the second equation in (1.2); that is,

$$\cos(\arccos x) = x \qquad (x \in [-1, 1]), \tag{1.6}$$

since $\operatorname{Cos} x = \cos x$ whenever $\operatorname{Cos} x$ is defined.

1.3.1 The derivative of $\arccos x$

If we have

$$y = \arccos x$$

then we also have

$$x = \cos y,$$

though the two equations are not equivalent, since the first equation implies that

$$0 \le y \le \pi, \tag{1.7}$$

whereas the second equation can hold also without this inequality being true. Differentiating the second equation with respect to x, we obtain

$$1 = y'(-\sin y).$$

From here we obtain the equation

$$y' = -\frac{1}{\sin y} = -\frac{1}{\pm\sqrt{1 - \cos^2 y}}.$$

As for the $\pm$ sign, it should be $+$ since $\sin y \ge 0$ for y satisfying inequality (1.7). Hence,

$$y' = -\frac{1}{\sqrt{1 - \cos^2 y}} = -\frac{1}{\sqrt{1 - x^2}},$$

where the second equation follows according to (1.6). That is,

$$\frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1 - x^2}}.$$

As a side benefit, we migh want to note that we also obtained the equation

$$\sin(\arccos x) = \sqrt{1 - x^2} \qquad (-1 \le x \le 1),$$

where the inequality ensures that $\arccos x$ is defined. This equation will be of use later. There is no reason to reverse the equation obtaine for the derivativ of $\arccos x$, since the integration formula so obtained would be essentially identical to formula (1.5).

It is, however, worth pointing out that our results show that[1.3]

$$\frac{d}{dx}(\arcsin x + \arccos x) = 0 \qquad (-1 < x < 1).$$

This implies that $\arcsin x + \arccos x$ is constant on the interval $[-1, 1]$.[1.4] For $x = 0$ we have $\arcsin x = 0$ and $\arccos x = \pi/2$; so the value of this function is $\pi/2$ for all $x \in [-1, 1]$. That is,

$$\arcsin x + \arccos x = \frac{\pi}{2} \qquad (-1 \le x \le 1). \tag{1.8}$$

This equation can be established directly from the equation

$$\cos t = \sin\left(\frac{\pi}{2} - t\right). \tag{1.9}$$

1.4 The inverse of tangent

Proceeding similarly as before, we define the principal part of tangent as

$$\operatorname{Tan} x \overset{def}{=} \begin{cases} \tan x & \text{if } -\pi/2 < x < \pi/2, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Note that $x = \pm\pi/2$ is not allowed here since $\tan x$ is not defined at those places. We define the "inverse" of tan as the inverse of Tan:

$$y = \arctan x \quad \text{means that} \quad x = \operatorname{Tan} y.$$

In other words, we say that $y = \arctan x$ if and only if $x = \tan y$ and $-\pi/2 < y < \pi/2$. According to equation (1.1), we have

$$\operatorname{dom}(\arctan) = \operatorname{ra}(\operatorname{Tan}) = \operatorname{ra}(\tan) = (-\infty, \infty) = \mathbb{R}$$

and

$$\operatorname{ra}(\arctan) = \operatorname{dom}(\operatorname{Tan}) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Further, we have

$$\operatorname{Tan}(\arctan x) = x \qquad x \in \operatorname{ra}(\operatorname{Tan}) = \mathbb{R}$$

according to the second equation in (1.2); that is,

$$\tan(\arctan x) = x \qquad \text{for all} \quad x \in \mathbb{R}, \tag{1.10}$$

since $\operatorname{Tan} x = \tan x$ whenever $\operatorname{Tan} x$ is defined.

[1.3] While $\arcsin x$ and $\arccos x$ are also defined when $x = 1$ or $x = -1$, their derivatives are not, so we take $-1 < x < 1$ next

[1.4] On the closed interval $[-1, 1]$. Even though the derivative of $\arcsin x + \arccos x$ is not defined at the endpoints of this interval, the function itself is continuous on the closed interval; hence it is constant on the closed interval.

1.4.1 The derivative of $\arctan x$

If we have

$$y = \arctan x$$

then we also have

$$x = \tan y,$$

though the two equations are not equivalent, since the first equation implies that

$$-\frac{\pi}{2} < y < \frac{\pi}{2}, \tag{1.11}$$

whereas the second equation can hold also without this inequality being true. Differentiating the second equation with respect to x, we obtain

$$1 = y'(\sec^2 y).$$

From here we obtain the equation

$$y' = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2};$$

the last equation follows according to (1.10). That is,

$$\frac{d}{dx} \arctan x = \frac{1}{1 + x^2}.$$

By reversing this equation, we obtain the integral formula

$$\int \frac{dx}{1 + x^2} = \arctan x + C, \tag{1.12}$$

a formula that we want to add to our list of basic integrals. Finally, we already used the equation that with $y = \arctan x$ we have

$$\sec(\arctan x) = \sec y = \pm\sqrt{1 + \tan^2 y} = \pm\sqrt{1 + x^2},$$

but we did not need to establish which of the $\pm$ sign applies. However, we have $-\pi/2 < y < \pi/2$ according to the definition of $\arctan x$, and for these values of y we have $\sec y > 0$ (in fact, we have $\sec y \geq 1$. Hence, we have

$$\sec(\arctan x) = \sqrt{1 + x^2},$$

an equation of importance in our discussion below.

The inverse of cot, arccot will not be defined here, but if one wants to define it, the definition should satisfy the analog of equation (1.8), that is, we should have

$$\arctan x + \operatorname{arccot} x = \frac{\pi}{2} \qquad \text{for all} \quad x \in \mathbb{R}.$$

Note for this that, in analogy with equation (1.9), we have

$$\cot t = \tan\left(\frac{\pi}{2} - t\right).$$

1.5 The inverse of secant

Defining the inverse of secant has important uses in techniques of integration. In our present approach, we define the principal part of secant as

$$\operatorname{Sec} x \stackrel{def}{=} \begin{cases} \sec x & \text{if } 0 \le x < \pi/2 \text{ or } \pi \le x < 3\pi/2 \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Note that $x = \pi/2$ or $x = 3\pi/2$ is not allowed here since $\sec x$ is not defined at those places. We define the "inverse" of sec as the inverse of Sec:

$$y = \operatorname{arcsec} x \quad \text{means that} \quad x = \operatorname{Sec} y.$$

In other words, we say that $y = \operatorname{arcsec} x$ if and only if $x = \sec y$ and $0 \le y < \pi/2$ or $\pi \le y < 3\pi/2$. While this is the definition we will follow in the present course since it helps simplify certain integration techniques, this is not the only way arcsec is defined in the literature. In fact, the definition based on the principal part

$$\operatorname{Sec}_1 x \stackrel{def}{=} \begin{cases} \sec x & \text{if } 0 \le x < \pi/2 \text{ or } \pi/2 < x \le \pi \\ \text{undefined} & \text{otherwise.} \end{cases}$$

is more common, and has other advantages. In fact, one should consider the definition based on $\operatorname{Sec}_1 x$ to be the standard definition of arcsec, since that definition is used by all major computer algebra software such as Maxima, Maple, and Mathematica, and it is also given by the NIST (National Institute of Standards and Technology). See [6] for more details; these notes also discuss the minor modifications that need to be made so as to apply the integration techniques below with the definition of arcsec based on the principal part $\operatorname{Sec}_1 x$.

According to equation (1.1), we have

$$\operatorname{dom}(\operatorname{arcsec}) = \operatorname{ra}(\operatorname{Sec}) = \operatorname{ra}(\sec) = (-\infty, -1] \cup [1, \infty)$$

and

$$\operatorname{ra}(\operatorname{arcsec}) = \operatorname{dom}(\operatorname{Sec}) = \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right).$$

Further, we have

$$\operatorname{Sec}(\operatorname{arcsec} x) = x \qquad x \in \operatorname{ra}(\operatorname{Sec}) = \operatorname{ra}(\sec) = (-\infty, -1] \cup [1, \infty)$$

according to the second equation in (1.2); that is,

$$\sec(\operatorname{arcsec} x) = x \qquad \text{if} \quad x \le -1 \quad \text{or} \quad x \ge 1. \tag{1.13}$$

since $\operatorname{Sec} x = \sec x$ whenever $\operatorname{Sec} x$ is defined.

Problem 1.1. Find $\operatorname{arcsec}(-2)$.

Solution. Writing $y = \operatorname{arcsec}(-2)$, we have $\sec y = -2$ and $0 \le y < \pi/2$ or $\pi \le y < 3\pi/2$. Given that $\sec y = -2$, we have $\cos y = 1/\sec y = -1/2$. The only values of y with $0 \le y < 2\pi$ for which $\cos y = -1/2$ are $y = 2\pi/3$ or $y = 2\pi - 2\pi/3 = 4\pi/3$. We have $\pi \le 4\pi/3 < 3\pi/2$, so $y = 4\pi/3$ satisfies all the requirements. That is,

$$\operatorname{arcsec}(-2) = \frac{4\pi}{3}.$$

1.5.1 The derivative of $\operatorname{arcsec} x$

If we have

$$y = \operatorname{arcsec} x$$

then we also have

$$x = \sec y,$$

though the two equations are not equivalent, since the first equation implies that

$$0 \le y < \frac{\pi}{2} \qquad \text{or} \quad \pi \le y < \frac{3\pi}{2}, \tag{1.14}$$

whereas the second equation can hold also without this inequality being true. Differentiating the second equation with respect to x, we obtain

$$1 = y'(\sec y \tan y).$$

From here we obtain the equation

$$y' = \frac{1}{\sec y \tan y}.$$

Here $\sec y = x$ according to (1.13). Further,

$$\tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2 - 1}.$$

However, for y satisfying (1.14) we have $\tan y \ge 0$, and so, given that $y = \operatorname{arcsec} x$, we can conclude that

$$\tan \operatorname{arcsec} x = \sqrt{x^2 - 1};$$

this equation will have important uses in what follows. As for the deriative of $\operatorname{arcsec} x$, we can now conclude that

$$\frac{d}{dx} \operatorname{arcsec} x = \frac{1}{x\sqrt{x^2 - 1}}.$$

The inverse of csc, arccsc will not be defined here, but if one wants to define it, the definition should satisfy the analog of equation (1.8), that is, we should have

$$\operatorname{arcsec} x + \operatorname{arccsc} x = \frac{\pi}{2} \qquad \text{if} \quad x \le -1 \quad \text{or} \quad x \ge 1.$$

Note for this that, in analogy with equation (1.9), we have

$$\sec t = \csc\left(\frac{\pi}{2} - t\right).$$

1.6 Integrals leading to inverse trigonometric functions

Problem 1.2. Evaluate

$$I = \int \frac{dx}{3x^2 - 18x + 32}.$$

Solution. First note that the denominator has no real zeros; that is, the equation $3x^2 - 18x + 32 = 0$ has no real solution. One can be convinced of this by noting that the discriminant of this equation, $18^2 - 4 \cdot 3 \cdot 32 = 4(9^2 - 3 \cdot 32) = 4(81 - 96) < 0$. Hence, after an appropriate substitution the integral can be converted into the form given in equation (1.12). This can be done by first completing the square in the denominator. We have

$$I = \int \frac{dx}{3(x^2 - 6x) + 32} = \int \frac{dx}{3\big((x-3)^2 - 9\big) + 32} = \int \frac{dx}{3(x-3)^2 + 5}$$
$$= \int \frac{(1/5)\,dx}{\frac{3}{5}(x-3)^2 + 1} = \int \frac{(1/5)\,dx}{\Big(\sqrt{\frac{3}{5}}\,(x-3)\Big)^2 + 1}.$$

Making the substitution $t = \sqrt{3/5}\,(x-3)$, we have $dt = \sqrt{3/5}\,dx$, that is, $dx = \sqrt{5/3}\,dt$. Hence

$$I = \int \frac{(1/5)\sqrt{5/3}\,dt}{t^2+1} = \frac{1}{\sqrt{15}} \int \frac{dt}{t^2+1} = \frac{1}{\sqrt{15}} \arctan t + C$$
$$= \frac{1}{\sqrt{15}} \arctan\left(\sqrt{\frac{3}{5}}(x-3)\right) + C = \frac{1}{\sqrt{15}} \arctan \frac{3x-9}{\sqrt{15}} + C$$

Problem 1.3. Evaluate

$$I = \int_{3/4}^{1} \frac{dx}{\sqrt{24x - 5 - 16x^2}}.$$

Solution. First note that the polynomial under the square root does have real zeros; indeed, the equation $-16x^2 + 24x - 5 = 0$ does have real solutions. Indeed, the discriminant of this equation, $24^2 - 4 \cdot (-16)(-5) = 8^2(3^2 - 5)$, is positive. Also noting that the coefficient of x^2 is negative, after an appropriate substitution the integral can be converted into the form given by equation (1.5). This can be done by first completing the square under the square root. We have

$$I = \int_{3/4}^{1} \frac{dx}{\sqrt{-5 - 16(x^2 - 3x/2)}} = \int_{3/4}^{1} \frac{dx}{\sqrt{-5 - 16\big((x-3/4)^2 - 9/16\big)}}$$
$$= \int_{3/4}^{1} \frac{dx}{\sqrt{4 - 16(x-3/4)^2}} = \int_{3/4}^{1} \frac{dx}{2\sqrt{1 - 4(x-3/4)^2}}$$
$$= \int_{3/4}^{1} \frac{(1/2)\,dx}{\sqrt{1 - (2x - 3/2)^2}}.$$

Making the change of variables $t = 2x - 3/2$, we have $dt = 2\,dx$, i.e., $dx = (1/2)\,dt$; for $x = 3/4$ we have $t = 0$, and for $x = 1$ we have $t = 1/2$. That is,

$$I = \int_{0}^{1/2} \frac{(1/4)\,dt}{\sqrt{1-t^2}} = \frac{1}{4} \arcsin t \Big|_{t=0}^{t=1/2} = \frac{1}{4}\left(\arcsin\frac{1}{2} - \arcsin 0\right) = \frac{1}{4}\left(\frac{\pi}{6} - 0\right) = \frac{\pi}{24}.$$

1.7 Reading

[9, §1.5, pp. 62–65]. [9, §3.6, pp. 222–223].

1.8 Homework

[9, §1.5, p. 65], 69, 71, 67, 73. [9, §3.6, p. 224], 65, 69, 75, 77. See also the file `invtrig_integrals.pdf` posted online for integration problems.

2 Applications of the definite integral

2.1 The Riemann Integral

The next three definitions describe the Riemann integral.

Definition 2.1 (Partition)**.** A *partition* of the interval $[a,b]$ is a finite sequence $\langle x_0, x_1, \dots, x_n\rangle$ of points such that

$$P : a = x_0 < x_1 < x_2 < \ldots < x_n = b. \tag{2.1}$$

The *width* or *norm* of a partition is

$$\|P\| \stackrel{def}{=} \max\{x_i - x_{i-1} : 1 \le i \le n\}.$$

Definition 2.2 (Riemann sum)**.** Given a partition

$$P : a = x_0 < x_1 < x_2 < \ldots < x_n = b.$$

of the interval $[a,b]$, a *tag* for the interval $[x_{i-1}, x_i]$ with $1 \le i \le n$ is a number

$$\xi_i \in [x_{i-1}, x_i] \tag{2.2}$$

for each i. A partition with a tag for each interval $[x_{i-1}, x_i]$ is called a *tagged* partition. Given a tagged partition as described, and given a functions f on $[a,b]$, the corresponding Riemann sum is

$$S = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}).$$

One often writes $\Delta x_i = x_i - x_{i-1}$, so the Riemann sum can also be written as

$$S = \sum_{i=1}^{n} f(\xi_i)\,\Delta x_i.$$

The Riemann integral

$$\int_a^b f(x)\,dx$$

is defined as the limit of the Riemann sums S associated with the partition P as $\|P\| \to 0$, independently of the choice of the tags. While not important for our purposes, we will give a rigorous definition:

Definition 2.3 (Riemann integral)**.** If there is a real number A such that for every $\epsilon > 0$ there is a $\delta > 0$ such that for any Riemann sum S for f associated with a partition of width $< \delta$ of $[a,b]$ we have $|A - S| < \epsilon$, then we call A the *Riemann integral* of f on $[a,b]$, and we write $A = \int_a^b f$. In this case we call f *Riemann integrable* on $[a,b]$.

Intuitively, the Riemann sum approximates the “area under a curve” by the sum of areas of rectangles, where the height of each rectangle is the value of the function somewhere in the base of the rectangle. The integral is given as the limit these sum when we make sure that the base of each rectangle is small. We put “area under a curve” in quotation marks above, since this area is in fact defined in terms of the Riemann integral, to give a rigorous definition of the intuitive concept of area.

2.2 Area between curves

Given an interval $[a, b]$ and two functions f_1 and f_2 on the interval such that $f_1(x) \le f_2(x)$ for all $x \in [a, b]$, we may use integration to calculate the area over the interval $[a, b]$ between the graphs of the functions f_1 and f_2. That is, one is considering the area of the region

$$R = \{(x, y) : a \le x \le b \quad \text{and} \quad f_1(x) \le y \le f_2(x)\}$$

This area can be calculated by subtracting the area under the bottom curve from the area under the top curve. That is, assuming that f_1 and f_2 are Riemann integrable on $[a, b]$, this area A can be calculated as

$$A = \int_a^b f_2 - \int_a^b f_1 = \int_a^b \big(f_2(x) - f_1(x)\big)\, dx.$$

2.3 Volume of rotation with slices

Given an interval $[a, b]$ and a function f on this interval such that $f(x) \ge 0$ for all $x \in [a, b]$. consider the region

$$R = \{(x, y) : a \le x \le b \quad \text{and} \quad 0 \le y \le f(x)\}. \tag{2.3}$$

Let S be the solid swept out of the space obtained by rotation this region about the x-axis. The volume of this solid can be approximated by taking a partition as in (2.1); then over each interval $[x_{i-1}, x_i]$ of this partition, taking a rectangle of height $f(\xi_i)$, where $\xi \in [x_{i-1}, x_i]$ is the corresponding tag. Rotation this rectangle about the x-axis, we obtain a circular cylinder of height $\Delta x_i = x_i - x_{i-1}$, with $f(\xi_i)$ being the radius of the base. The volume of this cylinder,

$$\pi\big(f(\xi_i)\big)^2 \Delta x_i$$

approximates the volume of slice of the rotated solid over the interval $[x_{i-1}, x_i]$. The sum of these volumes is a Riemann sum approximating the integral

$$\int_a^b \pi\big(f(x)\big)^2\, dx,$$

assuming this integral exist. This integral expresses the volume of the solid S.

In a slightly more complicated situation, instead of rotating the region R, one is given two functions, f_1 and f_2 on the interval such that $0 \le f_1(x) \le f_2(x)$, one rotates the region

$$R' = \{(x, y) : a \le x \le b \quad \text{and} \quad f_1(x) \le y \le f(x)\}. \tag{2.4}$$

To obtain the volumee of the solid S' sweptg by rotating this region about the x-axis, we need subtract the volume of the solid swept by rotating the region under the graph of f_1 from the volume of the solid swept by rotating the region under the graph of f_2. That is, the volue of S' equals

$$\int_a^b \pi\big(f_2(x)\big)^2\, dx - \int_a^b \pi\big(f_1(x)\big)^2\, dx = \int_a^b \pi\Big(\big(f_2(x)\big)^2 - \big(f_1(x)\big)^2\Big)\, dx.$$

2.4 Volume of rotation with shells

Consider the solid T obtained by rotating the region R given in (2.3) about the y-axis, and, as before, consider the rectangle of height $f(\xi_i)$ over the interval $[x_{i-1}, x_i]$ of the partition P in (2.1). Rotating this rectangle about the y-axis sweeps out a cylindrical shell of inner radius x_{i-1}, outer radius x_i, and height $f(\xi_i)$. The volume of this cylindrical shell approximates the volume obtained by rotating the part of the region over the interval $[x_{i-1}, x_i]$. While it is not too difficult to calculate the volume of this shell exactly, we only need an approximation. Assume that this thin shell is made of a flexible material (paper will do), so one can streighten out this shell to obtain a thin rectangular box.[2.1] The volume of this rectangular box is nearly the same as that of the cylindrical shell. The two sides of this box are $f(\xi_i)$, approximately $2pi\xi_i$, the latter being the circumference of a circle of radius ξ_i, and its thickness is $\Delta x_i = x_i - x_{i-1}$. That is, the volume of this box is approximately

$$2\pi\xi_i\, f(\xi_i)\, \Delta x_i.$$

The sum of these volumes is a Riemann sum approximation the integral

$$\int_a^b 2\pi x f(x)\, dx,$$

assuming this integral exist. This integral expresses the volume of the solid T.

To obtain the volume of the the solid swept out by rotation the region R' given in (2.4) can be obtained by subtracting the the volume swept by rotating the region under the bottom curve about the y-axis from the volume swept by rotating the top curve. That is, this volume equals

$$\int_a^b 2\pi x f_2(x)\, dx - \int_a^b 2\pi x f_1(x)\, dx = \int_a^b 2\pi x\big(f_2(x) - f_1(x)\big)\, dx.$$

2.5 Work

Assume a point is moving from $x = a$ to $x = b$ along the x-axis as a result of a force $f(x)$ acting on the point when the point has has coordinates $(x, 0)$. Assume, this force is acting in the direction of the x axis; it is positive when the force points in the direction of the positive x axis, and otherwise negative.[2.2] The work performed by the force is the force multiplied by the displacement. That is, assuming $a < b$, so the point moves from left endpoint of the interval to the right endpoint always in the direction of the positive x axis, we partition the interval $[a, b]$, as given in (2.1) with tags as in (2.2), the work performed by moving the point from x_{i-1} to x_i is approximately

$$f(\xi_i)\, \Delta x_i,$$

[2.1] It is common experience that a sheet of paper, which can be considered a thin rectangular box, can be rolled up into a cylinder. The reason this is possible is that paper that paper is somewhat stretchable and shearable (i.e., it responds to shear forces that pulls only one side of the paper, in that the rectangular box deformes into a parallepiped, i.e. a similar solid that is not quite rectangular). In a cylindrical shell, the outer circumference is somewhat larger than the inner circumference, corresponding to the stretchability of the paper; it is also possible that in the cylindrical shell, inner circumference completely closes, while there is a gap in the outer circumference – corresponding to a shearing of the paper; that is, the layers the the paper slightly slip.

Glass is much more rigid than paper (even though glass is also stretchable, since it can bend before breaking if you lift a large sheet of glass only on one end), so a glass sheet cannot be rolled up into a cylindrical shell.

[2.2] If the force is not directed in the direction of the x-axis, then it needs to be decomposed into components parallel and perpendicular to the x-axis. Only the parallel component will have an effect, the perpendicular component can be ignored.

where $\Delta x_i = x_i - x_{i-1}$, as before. Adding up these amounts of work, we obtain a Riemann sum approximating the integral

$$\int_a^b f(x)\,dx.$$

This integral represents the total work when moving the point from $x = a$ to $x = b$. When discussing work in an example from physics, physical dimensions of the quantities need to be included. In the International System of Units, the unit of distance is a meter, the unit of force is a Newton, and the unit of work is a Newton meter (that is, a Newton times a meter).

2.6 Arc length

Writing $|AB|$ for the length of the line segment between the point A and B, the length of an arc $\widehat{PQ}$ between points P and Q is defined as the limit of the sums

$$\sum_{i=1}^{n} |P_{i-1}P_i|$$

for all integers $n > 1$ and all sequences of points along the arc $\widehat{PQ}$ such that $P_0 = P$, $P_n = Q$, and P_i is between P_{i-1} and P_{i+1} for all i with $0 < i < n$. If this limit does not exist, then the length of the arc $\widehat{PQ}$ is not defined. A curve that has a length is called *rectifiable*, meaning, the curve can be "straightened out."

For a rigorous definition, we need to define what a curve is and how to traverse a curve in a certain direction, so that the points P_i follow in a proper order along the curve. The easiest what to define (a finite) curve is as the set of points

$$\bigl\{\bigl(f(t), g(t)\bigr) : 0 \le t \le 1\bigr\},$$

where f and g are continuous functions on $[0,1]$; in other words, as the set

$$\{(x,y) : x = f(t),\ y = g(t),\ \text{and } 0 \le t \le 1\}.$$

The pair of equations $x = f(t)$ and $y = g(t)$ are called the paramentric equations of the curve. The curve is traversed in the positive direction if t increases from 0 to 1. The interval $[0,1]$ here can be replaced by any closed interval; the equations can easily be rewritten when this replacement is made.

For example, the equation of the unit circle can be written as

$$\{(x,y) : x = \cos t,\ y = \sin t \text{ and } 0 \le t \le 2\pi\}.$$

in parametric form. The equation of the whole unit circle cannot be given in explicit form. The implicit form of the equation is $x^2 + y^2 = 1$, but when we solve this equation, we do not get a single solution for y.

In the present notes, we will only consider curves given by an explicit equation, that is, curves of the form

$$\{(x,y) : y = f(x),\ \text{and } a \le t \le b\},$$

where $a < b$ and f is a continuous function on $[a,b]$. In fact, we will also assume that f is differentiable on (a,b). Taking a partition as given in (2.1), the length of the line segment between the points $\bigl(x_{i-1}, f(x_{i-1})\bigr)$ and $\bigl(x_i, f(x_i)\bigr)$ is

$$\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2},$$

where $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = f(x_i) - x_{i-1})$. By the mean-value theorem of differentiation, we have

$$\Delta y_i = f(x_i) - f(x_{i-1}) = f'(\xi_i)(x_i - x_{i-1}) = f'(\xi_i)\,\Delta x_i$$

for some $\xi_i \in (x_{i-1}, x_i)$. Choosing the tags ξ_i as the numbers satisfying these equations, the length of the line segment between the points just mentioned is

$$\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{(\Delta x_i)^2 + \big(f'(\xi_i)\,\Delta x_i\big)^2} = \sqrt{1 + \big(f'(\xi_i)\big)^2}\,\Delta x_i.$$

Adding up these lengths for i with $1 \le i \le n$, we obtain a Riemann sum

$$\sum_{i=1}^{n} \sqrt{1 + \big(f'(\xi_i)\big)^2}\,\Delta x_i$$

approximating the integral

$$\int_a^b \sqrt{1 + \big(f'(x)\big)^2}\,dx. \tag{2.5}$$

In a more concise notation, this arc length is also written as

$$L = \int_a^b \sqrt{1 + (y')^2}\,dx.$$

if this integral exists.

In the present subsection we made special choices for the tags ξ, whereas we did not do this in the earlier subsections. Regardless, if the integral in question exists, the Riemann sums converge to this integral for any choice of the tags. Another issue that we did not emphasize was that we did not assume that $f'(a)$ or $f'(b)$ exist; we only assumed that $f'(x)$ exists for x with $a < x < b$. So, technically, the integral given in equation (2.5) will not exist if $f'(a)$ for $f'(b)$ does not exist. However, if we replace f' with any function g that is defined on $[a, b]$ and $g(x) = f'(x)$, and the integral exists,

$$\int_a^b \sqrt{1 + \big(g(x)\big)^2}\,dx.$$

then its value does not depend on the choice of $g(a)$ and $g(b)$. This integral can be considered an *improper* Riemann integral, albeit improper in a very simple way. Improper integrals will be discussed extensively below.

2.7 Reading

Areas and volumes of revolution: [9, §§ 6.1–6.3, pp. 436–464]. Work (there will be no exam questions on work, so you can treat this lightly): [9, § 6.4, pp. 465–470], Arc length: [9, § 8.1, pp. 560–564],

2.8 Homework

I suggest that you only set up the integrals and do not try to evaluate them. Especially for arclength, there might be integrals that need techniques only to be learnt later.

Areas: [9, § 6.1, pp. 442], 1, 3, 9, 21, 27, 35. Volumes by slices: [9, § 6.2, pp. 456], 3, 5, 17, 21, 23. Volumes by shells: [9, § 6.3, pp. 464], 1, 5, 7, 15, 27. Work (if you are interested): [9, § 6.4, pp. 470], 7. Arc length: [9, § 8.1, pp. 565], 3, 7, 13, 23, 25.

3 Integration by parts

Integration by parts is a reversal of the product rule of differentiation. Before discussing it, we will review the change of variable (substitution) rule, since it will be used in several of the examples.

3.1 Change of variables in indefinite integrals

Let F and g be functions with continuous derivatives, and let $f = F'$. By the chain rule, we have

$$\frac{d}{dx} F(g(x) = f(g(x)g'(x).$$

Reversing this, i.e., using the fact that integration is the inverse operation to differentiation, we can write this as

$$\int f((g(x))\, g'(x)\, dx = F(g(x)) + C = F(t)\Big|_{t=g(x)} = \int f(t)\, dt\,\Big|_{t=g(x)}.$$

Omitting the two middle members, this becomes the substitution rule for indefinite integrals:

$$\int f((g(x))\, g'(x)\, dx = \int f(t)\, dt\,\Big|_{t=g(x)}. \tag{3.1}$$

One can think of this rule as follows: when one introduces a new variable $t = g(x)$, one needs to put $dt = g'(x)\, dx$.

3.2 Change of variables in definite integrals

The following result is a consequence of the Fundamental Theorem of Calculus:

Theorem 3.1 (Newton–Leibniz formula)**.** *Let ϕ and Φ be functions on the interval $[\alpha, \beta]$, where $\alpha < \beta$. Assume Φ is continous on $[\alpha, \beta]$, $\phi(x) = \Phi'(x)$ for $x \in (\alpha, \beta)$, and, further, ϕ is continuos on (α, β). Then*

$$\int_\alpha^\beta \phi(x)\, dx = \Phi(\beta) - \Phi(\alpha). \tag{3.2}$$

There are several comments to be made on this formulation of the Newton–Leibniz formula. First, if a function is differentiable at a point then it is also continuous there; so, as far as the continuity of Φ is concerned, it only needs to be assumed that Φ is continous from the right at α and continous from the left at β. As far as the assumption that ϕ is continous on (α, β), this can be relaxed to say that ϕ is Riemann integrable on $[\alpha, \beta]$, but the proof of that result does not rely on the Fundamental Theorem of Calculus; see [1, Theoem on p. 63]. It is customary to use the notation

$$\Phi(x)\Big|_{x=\alpha}^{x=\beta} = \Phi(x)\Big|_{x=\alpha}^{\beta} = \Phi(x)\Big|_{\alpha}^{\beta} = \Phi\Big|_{\alpha}^{\beta} \overset{def}{=} \Phi(\beta) - \Phi(\alpha);$$

the expression after the second and third equality signs are somewhat ambiguous, and should only be used if they do not lead to misunderstandings.

So, let F and g be functions, let $f = F'$, and assume that F, f and g' are continous.[3.1] Given an interval $[a, b]$, and noting that $(d/dx)\, F(g(x) = f(g(x))\, g'(x)$ by the chain rule, the Newton–Leibniz

[3.1] We were somewhat vague about these assumptions. These assumptions need to be made at every point that comes under consideration in the discussion next; at the end, we will say more.

formula implies that

$$\int_a^b f(g(x))\, g'(x)\, dx = F(g(b)) - F(g(a)).$$

From the assumption that $F' = f$, the Newton–Leibniz formula, used on the interval $[g(a), g(b)]$ if $g(a) < g(b)$ or on the interval $[g(a), g(b)]$ if $g(a) \geq g(b)$, we can also conculde that

$$\int_{g(a)}^{g(b)} f(t)\, dt = F(g(b)) - F(g(a)).$$

Since the right-hand side of these equations are the same, it follows that

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(t)\, dt. \tag{3.3}$$

Conditions under which these result is true are discussed in [1, Change of Variable Theorem, p. 67], That result, however, is not obtained from the Newton–Leibniz formula.

3.3 Integration by parts for indefinite integrals

Let f and g be functions with continuous derivatives. We have

$$(fg)' = f'g + fg'. \tag{3.4}$$

Integrating this, we have

$$\int (f'g + fg') = fg + C.$$

This can be rearranged as

$$\int fg' = fg - \int f'g, \tag{3.5}$$

where the constant of integration is omitted, since we assume that the constants each integrals produce cancel out. This is reasonable in that the formula is used to calculate the integral on the left, and the constant of integration will not be lost, since the right-hand side will produce a constant of integration.

The formula we obtained is the integration by parts formula for indefinite integrals.

3.4 Integration by parts for definite integrals

Assuming f and g and f' and g' are continous on $[a, b]$. Then, integrating equation (3.4) on $[a, b]$, we obtain

$$\int_a^b (f'g + fg') = fg\Big|_a^b.$$

Rearranging this, we obtain

$$\int_a^b f(x)g'(x)\, dx = f(x)g(x)\Big|_{x=a}^{b} - \int_a^b f'(x)g(x)dx. \tag{3.6}$$

This is the interation by parts formula for definite integrals. The assumptions on the f and g can be weakened to say that f and g are continous on $[a, b]$, differentiabe on (a, b), and the derivatives f' and g' are Riemann integrable on $[a, b]$; see [1, p. 65].[3.2]

3.5 Examples

Problem 3.1. Find

$$I = \int x \sin x \, dx.$$

Solution. Using the integration by parts formula (3.5) with $f(x) = x$ and $g'(x) = \sin x$, when $f'(x) = 1$ and $g(x) = -\cos x$,[3.3] We obtain

$$I = x(-\cos x) - \int 1 \cdot (-\cos x)\, dx = -x \cos x + \int \cos x \, dx = -x \cos x + \sin x + C.$$

Problem 3.2. Find

$$I = \int_0^{\pi/3} x^2 \sin x \, dx.$$

Solution. This is integral can be calculated in a way similar to the one in the previous formula, but we need to use integration by parts twice. We use formula (3.6). With $f(x) = x^2$, $g'(x) = \sin x$, we have $f'(x) = 2x$ and $g(x) = -\cos x$. We obtain

$$\begin{aligned} I &= \int_0^{\pi/3} x^2 \sin x \, dx = x^2(-\cos x)\Big|_{x=0}^{\pi/3} - \int_0^{\pi/3} 2x\,(-\cos x)\, dx \\ &= \frac{\pi^2}{3^2}\left(-\frac{1}{2}\right) + \int_0^{\pi/3} 2x\, \cos x \, dx = -\frac{\pi^2}{18} + \int_0^{\pi/3} 2x\, \cos x\, dx; \end{aligned}$$

to obtain the second equality, note that $\cos(\pi/3) = 1/2$. Performing another integration by parts with $f(x) = 2x$ and $g'(x) = \cos x$, when $f'(x) = 2$ and $g(x) = \sin x$, to evaluate the second integral, we obtain that

$$\begin{aligned} I &= -\frac{\pi^2}{18} + 2x\, \sin x\Big|_{x=0}^{\pi/3} - \int_0^{\pi/3} 2 \sin x \, dx = -\frac{\pi^2}{18} + 2 \cdot \frac{\pi}{3}\,\frac{\sqrt{3}}{2} + 2\cos x\Big|_{x=0}^{\pi/3} \\ &= -\frac{\pi^2}{18} + \frac{\pi}{\sqrt{3}} + (1-2) = -\frac{\pi^2}{18} + \frac{\pi}{\sqrt{3}} - 1; \end{aligned}$$

to obtain the second equation, note that $\sin(\pi/3) = \sqrt{3}/2$, and the third one, that $\cos(\pi/3) = 1/2$.

Problem 3.3. Find

$$I = \int x^3 e^{2x}\, dx.$$

[3.2] Since we are not assuming that f and g are differentiable at a of b, f' and g' may not be defined at a or b, and in that case their Riemann integrals are not defined. What we mean is that in this case, defining f' and g' arbitrarily at a or b should give a Riemann-integrable function.

[3.3] We have $g(x) = -\cos x + C$, but we are free to choose the constant C of integration. In the present case, we choose $C = 0$; this is usually the best choice – though there could be exceptional circumstances when other choices are preferable.

Solution. Using integration by parts, taking $f(x) = x^3$ and $g'(x) = e^{2x}$, when $f'(x) = 3x^2$ and $g(x) = (1/2)e^{2x}$,[3.4] We obtain

$$I = x^3\,\frac{e^{2x}}{2} - \int \frac{3}{2}x^2\,e^{2x}\,dx.$$

Two more integrations by parts along similar lines give the final result; we skip the details. We obtain

$$I = \frac{4x^3 - 6x^2 + 6x - 3}{8}\,e^{2x} + C.$$

Repeated integration by parts can be used to find $\int x^n e^x\,dx$, $\int x^n \sin x\,dx$, and $\int x^n \cos x\,dx$ can be evaluated for any positive integer n in a similar way. These integrals cannot be evaluated with our methots if n is a negative integer; more precisely, those integrals cannot be expressed in terms of elementary functions in a closed form (that is, without using some infinitary process, such as limits, infinite sums, etc.).

Problem 3.4. Find

$$I = \int \ln x\,dx.$$

Solution. We use integration by parts to remove the transcendental function $\ln x$. In formula (3.5) we take $f(x) = \ln x$, $g'(x) = 1$, when $f'(x) = 1/x$ and $g(x) = x$. That is

$$I = \int (\ln x)\cdot 1\,dx = (\ln x)\cdot x - \int \frac{1}{x}\cdot x\,dx = x\ln x - \int dx = x\ln x - x + C.$$

Problem 3.5. Find

$$I = \int x^\alpha \ln x\,dx.$$

for any real $\alpha \neq -1$

Solution. We use an approach similar to the one used in the previous example. While integrating by parts, we take $f(x) = \ln x$ and $g'(x) = x^\alpha$, when $f'(x) = 1/x$ and $g(x) = x^{\alpha+1}/(\alpha+1)$, given that $\alpha \neq -1$. We get

$$\begin{aligned} I &= \int (\ln x)\,x^\alpha\,dx = (\ln x)\,\frac{x^{\alpha+1}}{\alpha+1} - \int \frac{1}{x}\,\frac{x^{\alpha+1}}{\alpha+1}\,dx = \frac{x^{\alpha+1}\ln x}{\alpha+1} - \int \frac{x^\alpha}{\alpha+1}\,dx \\ &= \frac{x^{\alpha+1}\ln x}{\alpha+1} - \frac{x^{\alpha+1}}{(\alpha+1)^2} + C. \end{aligned}$$

Problem 3.6. Find

$$I = \int x^{-1}\ln x\,dx.$$

Solution. Taking a similar approach with integration by parts, we pick $f(x) = \ln x$ and $g'(x) = 1/x$. Then we have $f'(x) = 1/x$ and $g(x) = \ln x$. So, we obtain

$$I = \int (\ln x)\,\frac{1}{x}\,dx = (\ln x)(\ln x) - \int \frac{1}{x}\,\ln x\,dx = \ln^2 x - \int \frac{1}{x}\,\ln x\,dx$$

[3.4] We have $g(x) = \int e^{2x}\,dx$; to evaluate this integral, one uses the substitution $t = 2x$, when $dt = 2\,dx$, i.e., $dx = (1/2)\,dt$. The calculation is so simple that when can produce he result quickly, without writing anything down.

The same integral occurs on the right-hand side as the one we started with. This is not a problem at present, since it gives an equation for I. Writing $I + C'$ for the integral on the right-hand side with an arbitrary constant C' (we do this because the integral on the left and on the right may not produce the same constant of integration), we obtain an equation for I:

$$I = \ln^2 x - (I + C').$$

Solving this equation, we obtain

$$I = \frac{\ln^2 x}{2} + C,$$

where $C = -C'/2$. Since C' was an arbitrary constant, this just means that C is an arbitrary constant. Many other integrals will follow a similar pattern: the integral that we started with will reappear on the right-hand side, providing an equation for the integral we want to calculate.

Problem 3.7. Find

$$I = \int \arctan x \, dx.$$

Solution. As in the examples involving logarithm, we will eliminate the transcendental function $\arctan x$ by differentiation. Using the integration by parts formula (3.5) with $f(x) = \arctan x$ and $g'(x) = 1$. We then have $f'(x) = 1/(x^2 + 1)$ and $g(x) = x$, and so

$$I = \int (\arctan x)(1) \, dx = (\arctan x)x - \int \frac{1}{x^2+1} x \, dx = x \arctan x - \int \frac{x \, dx}{x^2+1}.$$

The integral on the right-hand side can be evaluated using the substitution $t = x^2 + 1$, when $dt = 2x \, dx$, i.e., $x \, dx = (1/2) \, dt$. Hence

$$\int \frac{x \, dx}{x^2+1} = \frac{1}{2} \int \frac{dt}{t} = \frac{1}{2} \ln |t| + C' = \frac{1}{2} \ln(x^2+1) + C',$$

where no absolute value is needed on the right-hand side, since $x^2 + 1$ is always positive. Thus,

$$I = \int \arctan x \, dx = x \arctan x - \frac{1}{2} \ln(x^2+1) + C,$$

where $C = -C'$ is an arbitrary constant.

Problem 3.8. Find

$$I = \int \arcsin x \, dx.$$

Solution. We use an approach similar to the one taken before. We integrate by parts while taking $f(x) = \arcsin x$ and $g'(x) = 1$, when we have $f'(x) = 1/\sqrt{1-x^2}$ and $g(x) = x$. We obtain

$$\begin{aligned} I &= \int (\arcsin x) \cdot 1 \, dx = (\arcsin x) \, x - \int \frac{1}{\sqrt{1-x^2}} \, x \, dx \\ &= x \arcsin x - \int \frac{x \, dx}{\sqrt{1-x^2}}. \end{aligned}$$

The integral on the right-hand side can be evaluated by makign the substitution $t = \sqrt{1-x^2}$.[3.5] We then have $t^2 = 1 - x^2$ and $2t\,dt = -2x\,dx$, i.e., $x\,dx = -t\,dt$. [3.6] We obtain

$$\int \frac{x\,dx}{\sqrt{1-x^2}} = -\int \frac{t\,dt}{t} = -\int dt = -t + C' = -\sqrt{1-x^2} + C'.$$

Hence

$$I = \int \arcsin x\,dx = x \arcsin x + \sqrt{1-x^2} + C,$$

where $C = -C'$ is an arbitrary constant.

Problem 3.9. Find

$$I = \int \cos^2 x\,dx.$$

Solution. We use the integration by parts formula (3.5) with $f(x) = \cos x$ and $g'(x) = \cos x$, when $f'(x) = -\sin x$ and $g(x) = \sin x$. We obtain

$$\begin{aligned} I &= \int (\cos x)\,(\cos x)\,dx = (\cos x)\,(\sin x) - \int (\sin x)\,(-\sin x)\,dx \\ &= \cos x\,\sin x + \int \sin^2 x\,dx = \cos x\,\sin x + \int (1 - \cos^2 x)\,dx \\ &= \cos x\,\sin x + x - \int \cos^2 x\,dx. \end{aligned}$$

The integral on the right-hand side is the same as the integral on the left, but perhaps with a different constant of integration. So, writing $I + C'$ for the integral on the right, we obtain the equation

$$I = \cos x\,\sin x + x - (I + C'),$$

from where we obtain that

$$I = \int \cos^2 x\,dx = \frac{\cos x\,\sin x + x}{2} + C,$$

where $C = -C'/2$ is an arbitrary constant.

From this result we can also conclude that

$$I = \int \sin^2 x\,dx = \int (1 - \cos^2 x)\,dx = x - \frac{\cos x\,\sin x + x}{2} + C = \frac{-\cos x\,\sin x + x}{2} + C. \tag{3.7}$$

These integrals could have also been obtained by using the identities

$$\cos^2 x = \frac{1 + \cos 2x}{2} \qquad \text{and} \qquad \sin^2 x = \frac{1 - \cos 2x}{2}. \tag{3.8}$$

These formulas are essentialy the half-angle identities, which are usually written as

$$\cos\frac{x}{2} = \pm\sqrt{\frac{1 + \cos x}{2}} \qquad \text{and} \qquad \sin\frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{2}}.$$

[3.5]The substitution $t = 1 - x^2$ also works, but the one we are using is more elegant.
[3.6]Differentiating the equation $t = \sqrt{1-x^2}$ directly will give the same result, but in a less elegant way.

Problem 3.10. Find

$$I = \int e^{3x} \sin 2x \, dx.$$

Solution. We use the integration by parts formula (3.5) with $f(x) = e^{3x}$ and $g'(x) = \sin 2x$, when $f'(x) = 3e^{3x}$ and $g(x) = -(1/2)\cos 2x$. We obtain

$$\begin{aligned} I &= \int e^{3x} \sin 2x \, dx = e^{3x} \left(-\frac{1}{2}\right) \cos 2x - \int 3e^{3x} \left(-\frac{1}{2}\right) \cos 2x \, dx \\ &= -\frac{1}{2} e^{3x} \cos 2x + \frac{3}{2} \int e^{3x} \cos 2x \, dx \end{aligned} \tag{3.9}$$

Write J for the integral on the right-hand side. To handle this integral, we do another integration by parts with $f(x) = e^{3x}$ and $g'(x) = \cos 2x$, when $f'(x) = 3e^{3x}$ and $g(x) = (1/2)\sin 2x$. We obtain

$$\begin{aligned} J &= \int e^{3x} \cos 2x \, dx = e^{3x} \frac{1}{2} \sin 2x - \int 3e^{3x} \frac{1}{2} \sin 2x \, dx \\ &= \frac{1}{2} e^{3x} \sin 2x - \frac{3}{2} \int e^{3x} \sin 2x \, dx. \end{aligned} \tag{3.10}$$

Noting that the integral on the right-hand side is the same as the integral I we want to evaluate, but perhaps with a different constant of integration. Writing $I + C'$ for this integral, and substituting the expression for the integral on the right-hand side of formula (3.9), we obtain the equation

$$I = -\frac{1}{2} e^{3x} \cos 2x + \frac{3}{2} \left(\frac{1}{2} e^{3x} \sin 2x - \frac{3}{2}(I + C') \right).$$

Solving this for I, we obtain

$$I = \int e^{3x} \sin 2x \, dx = \frac{e^{3x}(3 \sin 2x - 2 \cos 2x)}{13} + C, \tag{3.11}$$

where $C = -(3/13)C'$ is an arbitrary constant. Substituting this into (3.10), we also obtain the integral

$$J = \int e^{3x} \cos 2x \, dx = \frac{e^{3x}(2 \sin 2x + 3 \cos 2x)}{13} + C \tag{3.12}$$

with a different C, which will be also of interest in the discussion below.

In evaluating the integral I, we used two integration by parts, and both times we took $f(x)$ to be the exponential function and $g'(x)$ to be the trigonometric function. Instead, we could have taken $f(x)$ to be the trigonometric function both times, and $g'(x)$ to be the exponential function. What would not have worked is to mix these, and take $f(x)$ once the exponential function, and the other time the trigonometric function.

The two approaches work equally well, but they do not always involve the same amount of effort. For example, in integrating $e^{4x} \cos x$ one would take $f(x)$ to be the exponential function, to avoid fractions, while in integrating $e^{x} \sin 3x$, one would take $f(x)$ to be the trigonometric function both times, so as to avoid fractions.

Problem 3.11. Find

$$I = \int x^2 \, e^{3x} \sin 2x \, dx$$

We will not work out the solution of this problem, but we give an outline. In calculating this integral, we lower the power of x by repeated integration by parts, choosing $f(x)$ to be the appropriate power of x, and $g'(x)$ the remaining part of the integral, which will be an expression formed by (a sum of) one or two products of an exponential and a trigonometric function. In determining $g(x)$, the already calculated integrals in (3.11) and (3.12) can be used. The result is not pretty:

$$I = \int x^2 e^{3x} \sin 2x \, dx = \frac{e^{3x}}{2197}\big((507x^2 - 130x - 18) \sin 2x + (-338x^2 + 312x - 92) \cos 2x\big).$$

3.6 Reading

[9, §7.1, pp. 484–490].

3.7 Homework

[9, §7.1, p. 490], 1, 3, 9, 15, 21, 23, 27, 35, 43,

4 Trigonometric integrals

4.1 Integration of products of sine and cosine

The integral

$$\int \sin^m x \cos^n x \, dx$$

for integers $m, n \geq 0$ can be evaluated depending on whether or not m or n is odd. If m is odd, the substitution $t = \cos x$ works; if n is odd, the substitution $t = \sin x$ works. If both of them are odd, both substitutions work. If both are even, one can use the half-angle identities (3.8); other techniques, such as integration by parts may also work (see Problem 3.9).

Problem 4.1. Find

$$I = \int \sin^5 x \cos^4 x \, dx.$$

Solution. We have

$$I = \int \sin^4 x \cos^4 x \sin x \, dx = \int (\sin^2 x)^2 \cos^4 x \sin x \, dx = \int (1 - \cos^2 x)^2 \cos^4 x \sin x \, dx.$$

Using the substitution $t = \cos x$, we have $dt = -\sin x \, dx$, and so

$$\begin{aligned} I &= \int (1 - t^2)^2 t^4 \, (-dt) = -\int (1 - 2t^2 + t^4) t^4 \, dt = \int (-t^8 + 2t^6 - t^4) \, dt \\ &= -\frac{t^9}{9} + \frac{2t^7}{7} - \frac{t^5}{5} + C = -\frac{\cos^9 x}{9} + \frac{2\cos^7 x}{7} - \frac{\cos^5 x}{5} + C. \end{aligned}$$

The technique sometimes works also with negative exponets, as the following problem shows.

Problem 4.2. Find

$$I = \int \tan x \, dx.$$

Solution. We have

$$I = \int \frac{\sin x\, dx}{\cos x}.$$

Using the substitution $t = \cos x$, we have $dt = -\sin x\, dx$, and so

$$I = -\int \frac{dt}{t} = -\ln|t| = -\ln|\cos x| + C.$$

The following problem is somewhat similar, but it involves other complications:

Problem 4.3. Find

$$I = \int \sec x\, dx.$$

Solution.

$$I = \int \frac{1}{\cos x}\, dx = \int \frac{\cos x}{\cos^2 x}\, dx = \int \frac{\cos x\, dx}{1 - \sin^2 x}.$$

We will substitute $t = \sin x$, when $dt = \cos x\, dx$. We obtan

$$I = \int \frac{1}{1-t^2}\, dt = \frac{1}{2}\int \left(\frac{1}{1+t} + \frac{1}{1-t}\right) dt = \frac{1}{2}\left(\int \frac{1}{t+1}\, dt - \int \frac{1}{t-1}\, dt\right).$$

The second equation is a simple example of partial fraction decomposition, a powerful technique that will be discussed later in detail for the integration of rational functions in Section 7.

We can calculate the integrals on the right-hand side by the simple substitution $u = t + 1$ and $u = t - 1$. Thus,

$$\begin{aligned} I &= \frac{1}{2}(\ln|t+1| - \ln|t-1|) + C = \frac{1}{2}\ln\frac{|t+1|}{|t-1|} + C = \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C \\ &= \frac{1}{2}\ln\left|\frac{(1+\sin x)^2}{1-\sin^2 x}\right| + C = \ln\left|\frac{1+\sin x}{\sqrt{1-\sin^2 x}}\right| + C = \ln\left|\frac{1+\sin x}{\cos x}\right| + C; \end{aligned}$$

the third equation holds since $t = \sin x$, and the fifth equation is based on the identity $(1/2)\ln u = \ln\sqrt{u}$. Hence we have

$$I = \ln\left|\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right| + C = \ln|\sec x + \tan x| + C.$$

4.2 Integration of products of tangent and secant

The integral

$$\int \tan^m x \sec^n x\, dx$$

for integers $m, n \geq 0$, can be evaluated depending on whether or not m or n is odd or even. If $n > 0$ is even, the substitution $t = \tan x$ works; if m is odd and $n > 0$, the substitution $t = \sec x$ works. If both conditions are satisfied, then both methods work. If $m > 2$ and $n = 0$, the exponent of tangent can be reduced by using the equation $\tan^2 x = \sec^2 x - 1$. In other cases, different techniques, such as integration by parts, might work.

Problem 4.4. Find

$$I = \int \tan^4 x \sec^6 x \, dx$$

Solution. We will use the substitution $t = \tan x$. We have

$$\begin{aligned} I &= \int \tan^4 x \, (\sec^2 x)^2 \sec^2 x \, dx = \int \tan^4 x \, (\sec^2 x)^2 \sec^2 x \, dx \\ &= \int \tan^4 x \, (1 + \tan^2 x)^2 \sec^2 x \, dx. \end{aligned}$$

With $t = \tan x$, when $dt = \sec^2 x \, dx$, this becomes

$$\begin{aligned} I &= \int t^4 (1 + t^2)^2 \, dt = \int t^4 (1 + 2t^2 + t^4) \, dt = \int (t^4 + 2t^6 + t^8) \, dt \\ &= \frac{t^9}{9} + \frac{2t^7}{7} + \frac{t^5}{5} + C = \frac{\tan^9 x}{9} + \frac{2 \tan^7 x}{7} + \frac{\tan^5 x}{5} + C. \end{aligned}$$

Problem 4.5. Find

$$I = \int \tan^3 x \sec^5 x \, dx.$$

Solution. We will use the substitution $t = \sec x$. We have

$$I = \int \tan^2 x \sec^4 x \; \tan x \sec x \, dx = \int (\sec^2 x - 1) \sec^4 x \; \tan x \sec x \, dx$$

With $t = \sec x$, when $dt = \tan x \sec x \, dx$, we obtain

$$I = \int (t^2 - 1) t^4 \, dt = \int (t^6 - t^4) \, dt = \frac{t^7}{7} - \frac{t^5}{5} + C = \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} + C.$$

Problem 4.6. Find

$$I = \int \tan^5 x \, dx$$

Solution. This does not directly fit the pattern we described above. Nevertheless, using those ideas, we can lower the exponent of tangent. We have

$$\begin{aligned} I &= \int \tan^3 x \tan^2 x \, dx = \int \tan^3 x \, (\sec^2 x - 1) \, dx \\ &= \int \tan^3 x \sec^2 x \, dx - \int \tan^3 x \, dx = I_1 - I_2. \end{aligned}$$

We can calculate I_1 using the substitution $t = \tan x$, when $dt = \sec^2 x \, dx$. We have

$$I_1 = \int t^3 \, dt = \frac{t^4}{4} + C = \frac{\tan^4 x}{4} + C.$$

As for I_2, we can reduce the exponent of tangent:

$$\begin{aligned} I_2 &= \int \tan^3 x \, dx = \int \tan x \, (\sec^2 x - 1) \, dx \\ &= \int \tan x \sec^2 x \, dx - \int \tan x \, dx = I_3 - I_4. \end{aligned}$$

The second integral on the right-hand side was calculated in Problem 4.2:

$$I_4 = \int \tan x \, dx = -\ln|\cos x| + C.$$

As for I_3 we can use the substitution $t = \tan x$, when $dt = \sec^2 x \, dx$. Thus,

$$I_3 = \int \tan x \sec^2 x \, dx = \int t \, dt = \frac{t^2}{2} + C = \frac{\tan^2 x}{2} + C.$$

Putting all this together,[4.1] we have

$$I = I_1 - I_2 = I_1 - (I_3 - I_4) = I_1 - I_3 + I_4 = \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} - \ln|\cos x| + C.$$

Problem 4.7. Find

$$I = \int \sec^3 x \, dx.$$

Solution. We have

$$I = \int \sec x \sec^2 x \, dx.$$

Using the integration by parts formula (3.5) with $f(x) = \sec x$ amd $g'(x) = \sec^2 x$, when $f'(x) = \sec x \tan x$ and $g(x) = \tan x$, we obtain

$$\begin{aligned} I &= \sec x \tan x - \int \sec x \tan x \, \tan x \, dx = \sec x \tan x - \int \sec x \tan^2 x \, dx \\ &= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx = \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx. \end{aligned}$$

The first integral on the right-hand side is the same as the integral on the left-hand side, but perhaps producing a different constant of integration. and the second integral on the right-hand side was calculated in Problem 4.3. Thus, using the result of the problem just mentioned, and writing $I + C'$ for the integral on the right, we have the equation

$$I = \sec x \tan x - (I + C') + \ln|\sec x + \tan x|.$$

Solving this equation, we obtain

$$I = \frac{1}{2} (\sec x \tan x + \ln|\sec x + \tan x|) + C,$$

where $C = -C'/2$ is an arbitrary constant.

4.3 Product of sine and cosine of multiple angles

The formulas

$$\begin{aligned} 2\cos x \cos y &= \cos(x - y) + \cos(x + y), \\ 2\sin x \sin y &= \cos(x - y) - \cos(x + y), \\ 2\sin x \cos y &= \sin(x + y) + \sin(x - y), \end{aligned}$$

can be used to calculate integrals such as

$$\int \cos \alpha x \cos \beta x \, dx, \qquad \int \sin \alpha x \sin \beta x \, dx, \qquad \int \sin \alpha x \cos \beta x \, dx,$$

where α and β are real numbers.

[4.1] The several occurrences of C in the calculation above do not denote the same constant.

4.4 Reading

[9, §7.2, pp. 493–498].

4.5 Homework

[9, §7.2, p. 498], 1, 7, 11, 17, 21, 27, 43, 57.

5 Evaluation of integrals with square root of quadratics

We will consider the evaluations of integrals of form

$$\int R\left(x, \sqrt{ax^2+bx+c}\right)\,dx,$$

where R is a rational function. In doing so we will assume that $a \neq 0$. Indeed, if $a = 0$, the integral involves $\sqrt{bx+c}$. Assuming $b \neq 0$, such an integral can be evaluated by the substitution $t = \sqrt{bx+c}$.

There are several different methods to evaluate such integrals. In some of the methods, one completes the square of the expression under the square root, and then one uses a linear substitution $t = Ax + B$. This allows one to convert the integrals to one of the three forms

$$\int R\left(x, \sqrt{a^2-x^2}\right)\,dx,$$
$$\int R\left(x, \sqrt{x^2+a^2}\right)\,dx,$$
$$\int R\left(x, \sqrt{x^2-a^2}\right)\,dx;$$

here x and a are not the same as before; we may assume that that $a > 0$. It is also possible that the expression under the square root is a complete square; then one can remove the square root altogether. The fourth form $\int R\left(x, \sqrt{-x^2-a^2}\right)\,dx$ is also possible, but in that case the square root is never real, and we only discuss real-valued functions here.

5.1 Trigonometric substitutions

5.1.1 Integrals with $\sqrt{a^2-x^2}$

Integrals of form

$$\int R\left(x, \sqrt{a^2-x^2}\right)\,dx$$

can be handled with the substitution $t = \arcsin(x/a)$. In this case $x = a\sin t$ and $-\pi/2 \leq t \leq \pi/2$. Further, we have $dx = a\cos t\,dt$, and

$$\sqrt{a^2-x^2} = \sqrt{a^2(1-\sin^2 t)} = a\cos t,$$

noting that $a > 0$ and $\cos t \geq 0$ for $-\pi/2 \leq t \leq \pi/2$.

5.1.2 Integrals with $\sqrt{x^2+a^2}$

Integrals of form

$$\int R\left(x, \sqrt{x^2+a^2}\right)\, dx$$

can be handled with the substitution $t = \arctan(x/a)$. In this case $x = a\tan t$ and $-\pi/2 < t < \pi/2$. Further, we have $dx = a\sec^2 t\, dt$, and

$$\sqrt{x^2+a^2} = \sqrt{a^2(\tan^2 t + 1)} = a\sec t,$$

noting that $a > 0$ and $\sec t > 0$ for $-\pi/2 < t < \pi/2$.

5.1.3 Integrals with $\sqrt{x^2-a^2}$

Integrals of form

$$\int R\left(x, \sqrt{x^2-a^2}\right)\, dx$$

can be handled with the substitution $t = \operatorname{arcsec}(x/a)$. In this case $x = a\sec t$ and $0 \le t < \pi/2$ or $\pi \le t < 3\pi/2$. Further, we have $dx = a\sec t\tan t\, dt$, and

$$\sqrt{x^2-a^2} = \sqrt{a^2(\sec^2 t - 1)} = a\tan t, \tag{5.1}$$

noting that $a > 0$ and $\tan t > 0$ for $0 \le t < \pi/2$ or $\pi \le t < 3\pi/2$.

Note that this substitution depends on our definition of $\operatorname{arcsec} x$ given in Subsection 1.5 so as to ensure that the sign of tangent in equation (5.1) is positive. When the more widely accepted definition based on the principal value Sec_1 described in Subsection 1.5 is used, one needs to make slight changes; this is discussed in the notes [6].

5.2 Euler substitutions

As an alternative to the trigonometric substitution, the Euler substititions can also be used. For this, one does not need to complete the square in the expression ax^2+bx+c. There are three cases, according as $a > 0$, $c > 0$, or when the expression ax^2+bx+c can be factored into real linear factors.

These cases overlap, but when none of these apply, we have $ax^2+bx+c < 0$ for all x. Indeed, if the expression cannot be factored into real factors, the discriminant b^2-4ac must be negative. This can occur only in case of $a, c > 0$ or $a, c < 0$. If $c < 0$ then this expression is negative for $x = 0$, and if in case the discriminant is negative, then ax^2+bx+c never changes sign, so it stays negative for all x.

5.3 First Euler substitution

When $a > 0$, we may introduce t with the equation

$$\sqrt{ax^2+bx+c} = \pm x\sqrt{a} + t;$$

one can choose whether to use the + or the − sign.

5.4 Second Euler substitution

When $c > 0$, we may introduce t with the equation

$$\sqrt{ax^2 + bx + c} = xt \pm \sqrt{c};$$

one can choose whether to use the + or the − sign.

5.5 Third Euler substitution

If the polynomial $ax^2 + bx + c$ has real zeros α and β, we have $ax^2 + bx + c = a(x - \alpha)(x - \beta)$. In this case, we may introduce t with the equation

$$\sqrt{ax^2 + bx + c} = \sqrt{a(x - \alpha)(x - \beta)} = (x - \alpha)t.$$

One of the Euler substitutions is always applicable. Indeed, if the first Euler substitution is not applicable, then $a < 0$, in which case the third Euler substitution should be applicable. Indeed, if $ax^2 + bx + c$ has no real zeros, then this expression will never change signs. Since it is negative for large values of x, this means that the expression is always negative, so its square root is never real. On the other hand, if one is willing to work with complex numbers, then all Euler substitutions are applicable.

5.6 Sometimes there are simpler ways

In same cases, there are more direct ways than to use the substitutions described above, as shown by the following example.

Problem 5.1. Find

$$I = \int x\sqrt{x^2 + 1}\, dx.$$

Solution. Here the substitution $t = \sqrt{x^2 + 1}$ will work. We then have $t^2 = x^2 + 1$, $2t\, dt = 2x\, dx$, i.e., $x\, dx = t\, dt$. Thus,

$$I = \int \sqrt{x^2 + 1}\, x\, dx = \int t\, t\, dt = \int t^2\, dt = \frac{t^3}{3} + C = \frac{\left(\sqrt{x^2 + 1}\right)^3}{3} + C = \frac{(x^2 + 1)^{3/2}}{3} + C.$$

In this example, what made things easy was that x occured with an odd exponent outside the square root. This can help even if this exponent is negative, as shown by the next example.

Problem 5.2. Find

$$I = \int \frac{\sqrt{x^2 - 1}}{x}\, dx$$

Direct solution. We use the substitution $t = \sqrt{x^2 - 1}$, when $t^2 = x^2 - 1$, $2t\, dt = 2x\, dx$, i.e., $x\, dx = t\, dt$. Hence

$$\begin{aligned} I &= \int \frac{\sqrt{x^2 - 1}\, x\, dx}{x^2} = \int \frac{t\, t\, dt}{t^2 + 1} = \int \frac{t^2}{t^2 + 1}\, dt = \int \left(1 - \frac{1}{t^2 + 1}\right) dt \\ &= t - \arctan t + C = \sqrt{x^2 - 1} - \arctan \sqrt{x^2 - 1} + C \end{aligned}$$

Solution via a trigonometric substitution. We use the substitution $t = \operatorname{arcsec} x$, when $x = \sec t$, $dx = \sec t \tan t\, dt$, and, as we pointed out in equation (5.1), we have $\sqrt{x^2-1} = \tan t$. Thus,

$$\begin{aligned} I &= \int \frac{\tan t}{\sec t} \sec t \tan t\, dt = \int \tan^2 t\, dt \\ &= \int (\sec^2 t - 1)\, dt = \tan t - t + C = \sqrt{x^2-1} - \operatorname{arcsec} x + C. \end{aligned}$$

Solution via the first Euler substitution. Making the substitition for with t satisfying

$$\sqrt{x^2-1} = -x + t, \tag{5.2}$$

where we pick the $-$ sign of the possible $\pm$, we have $x^2 - 1 = x^2 - 2xt + t^2$, that is $2xt = t^2 + 1$, and so

$$x = \frac{t^2+1}{2t}.$$

Hence,

$$dx = \frac{2t \cdot 2t - (t^2+1)\,2}{4t^2}\, dt = \frac{2(t^2-1)\,2}{4t^2}\, dt = \frac{t^2-1}{2t^2}\, dt$$

So,

$$\sqrt{x^2-1} = -x + t = -\frac{t^2+1}{2t} + t = \frac{t^2-1}{2t}.$$

Thus, we have

$$\begin{aligned} I &= \int \frac{\sqrt{x^2-1}}{x}\, dx = \int \frac{\frac{t^2-1}{2t}}{\frac{t^2+1}{2t}} \frac{t^2-1}{2t^2}\, dt = \int \frac{(t^2-1)^2}{2t^2(t^2+1)}\, dt \\ &= \frac{1}{2} \int \frac{t^4 - 2t^2 + 1}{t^4 + t^2}\, dt = \frac{1}{2} \int \frac{(t^4+t^2) - 3t^2 + 1}{t^4+t^2}\, dt \\ &= \frac{1}{2} \int \left(1 - \frac{3t^2-1}{t^4+t^2}\right) dt = \frac{t}{2} - \frac{1}{2} \int \frac{3t^2-1}{t^2(t^2+1)}\, dt. \end{aligned} \tag{5.3}$$

Here there is a problem with an integral on the right-hand side that can be handled by the general method of partial fraction decomposition, to be discussed in Subsection 7. We will show how to do this in the present case, which will help the general discussion later. The first thing to note that only t^2 occurs in the integrand, t itself does not. So, instead of applying the method of partial fraction decomposition to the integrand as given, we can take $u = t^2$, and use the decomposition with linear factors in the denominator. As we will present the method later, we can write

$$\frac{3u-1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}, \tag{5.4}$$

where the coefficients A and B are to be determined. Multiplying by the denomintors, we obtain

$$3u - 1 = A(u+1) + Bu.$$

There is a very important point to be made here. The coefficients A and B are to be so chosen that equation (5.4) be valid for all values of u except for $u = 0$ and $u = -1$, when it is meaningless. So the question is, does the latter equation have to be valid in these cases? The answer is simple; the latter equation can be written as $(A + B - 3)u + A + 1 = 0$. This is a linear equation for u, so it can have at most one solution, unless it is an identity. So the fact that it must be true for at least two values of u means that it must be true for all values of u.[5.1]

These comments are helpful in that the simplest way to find out the values of A and B is to substitute $u = 0$ and $u = -1$ in the last displayed equation. The substitutuion $u = 0$ gives $A = -1$ and the substitution $u = -1$ gives $B = 4$. Thus, we have

$$\frac{3u-1}{u(u+1)} = \frac{-1}{u} + \frac{4}{u+1}.$$

Hence, we have

$$\int \frac{3t^2-1}{t^2(t^2+1)}\,dt = 4\int \frac{dt}{t^2+1} - \int \frac{dt}{t^2} = 4\arctan t + \frac{1}{t} + C.$$

Substituting this into equation (5.3), we obtain

$$I = \frac{t}{2} - 2\arctan t - \frac{1}{2t} + C.$$

What remains yet to do is to go back to the original variable x. This is fairly simple to do. By equation (5.2), we have $t = x + \sqrt{x^2-1}$, and so

$$I = \frac{x+\sqrt{x^2-1}}{2} - 2\arctan(x+\sqrt{x^2-1}) - \frac{1}{2(x+\sqrt{x^2-1})} + C.$$

5.6.1 Are the results really different?

The question arises, how come the results obtained by three methods are all different? At least they look different. To understand the issues, first note that

$$\arctan\sqrt{x^2-1} = \arccos\frac{1}{|x|}. \tag{5.5}$$

Note that the sides of the equation are defined only if $|x| \ge 1$; we might as well assume $x \ge 1$, since the equation says the same thing if we replace x by $-x$. Hence, we may drop the absolute value on the right-hand side, so that we will not have trouble with differentiation. The derivative of either side is $1/\big(x^2\sqrt{x^2-1}\big)$, and for $x = 1$ both sides are 0. There are also direct ways, relying on trigonometric identities to verify this, but then one may need to pay attention to the ranges of inverse trigonometric function.

$$\arccos\frac{1}{|x|} = \operatorname{arcsec} x + C.$$

For positive $x \ge 1$ we have $C = 0$. and for $x \le -1$, we have $C = \pi$. To see this in case of $x \ge 1$, if we write y for the left-hand side, we see that $\cos y = 1/|x| = 1/x$, and so $\sec y = 1/\cos y = x$. We leave it to the reader to verify that our definition of the inverse of secant implies that $\sec(-x) = \sec x + \pi$. This shows that

$$\arctan\sqrt{x^2-1} = \operatorname{arcsec} x + C. \tag{5.6}$$

[5.1] These considerations work in general. In general we obtain a polynomial equation, and this must not have more solutions than the degree of the polynomial. Given that it is valid at infinitely many places, it follows that the equation must be an identity, i.e., it must be true everywhere, even at places where the fractions in the originating equation had a zero in the denominator.

This equation can be used to show that the results in the first and second solutions of Problem 5.2 are essentially the same in view of the presence of an arbitrary constant. This constant need not be the same for $x \ge 1$ and $x \le -1$, since the two parts of the integral are not connected.

As for the third solution, first note that

$$\frac{1}{x+\sqrt{x^2-1}} = x - \sqrt{x^2-1}. \tag{5.7}$$

This is true simply because we have

$$\left(x-\sqrt{x^2-1}\right)\left(x+\sqrt{x^2-1}\right) = x^2 - (x^2-1) = 1.$$

Further, we have

$$2\arctan\left(x+\sqrt{x^2-1}\right) = \arctan\left(\sqrt{x^2-1}\right) \pm \frac{\pi}{2}, \tag{5.8}$$

where we have the $+$ sign for $x \ge 1$ and the $-$ sign for $x \le -1$. To show this equation, we first assume that $x \ge 1$. Writing $y = \arctan\left(\sqrt{x^2-1}\right)$, in order to show this equation, we need to show that

$$\tan\left(\frac{y}{2}+\frac{\pi}{4}\right) = x + \sqrt{x^2-1}.$$

By the defintion of y, we have $\tan y = \sqrt{x^2-1}$, and using equation (5.6). we have $secy = x$ (note that $C = 0$ in this equation if $x \ge 1$. Hence we have

$$\tan\left(\frac{y}{2}+\frac{\pi}{4}\right) = \sec y + \tan y = x + \sqrt{x^2-1}, \tag{5.9}$$

where the first equation is a well-known trigonometric identity (see below). This verifies equation (5.8) for $x \ge 1$. Its verification for $x \le -1$ will be left to the reader (hint: use equation (5.7 together with the result for $x \ge 1$). Equations (5.7) and (5.8) can be use to show that the result obtained by the third solution agrees with the solutions obtained by the first two methods.

To show the trigonometric identity mentioned in equation (5.9), our statring point is the half-angle formula

$$\tan\frac{y}{2} = \frac{1-\cos y}{\sin y}.$$

This formula is easy to verify. Writing $u = y/2$ and using the double angle formulas for sine and cosine, we have

$$\begin{aligned}\frac{1-\cos y}{\sin y} &= \frac{1-\cos 2u}{\sin 2u} = \frac{1-(\cos^2 u - \sin^2 u)}{\sin 2u} = \frac{1-\left((1-\sin^2 u)-\sin^2 u\right)}{\sin 2u}\\ &= \frac{1-(1-2\sin^2 u)}{2\sin u\cos u} = \frac{2\sin^2 u}{2\sin u\cos u} = \frac{\sin u}{\cos u} = \tan u = \tan\frac{y}{2}.\end{aligned}$$

Using the formulas $\sin(y+\pi/2) = \cos y$ and $\cos(y+\pi/2) = -\sin y$ in adition to the above half-angle formula, we obtain

$$\tan\left(\frac{y}{2}+\frac{\pi}{4}\right) = \tan\frac{y+\pi/2}{2} = \frac{1-\cos(y+\pi/2)}{\sin(y+\pi/2)} = \frac{1+\sin y}{\cos y} = \sec y + \tan y.$$

An instructive lesson to learn here that different methods of calculating an integral occasionally produce totally different looking answers. Incidentally, the computer algebra system Maxima gives the answer to the problem as

$$I = \sqrt{x^2-1} + \arcsin\frac{1}{|x|} + C.$$

To see that this agrees with the answer given in the first solution, one needs to use equation (5.5) together with equation (1.8). The integral in Problem 5.2 can also be evaluated by using a *hyoerbolic* substitution; see [4, Section 1, pp. 1-2].

6 Examples for trigonometiric substituition

Problem 6.1. Find

$$I = \int \frac{x^2}{\sqrt{9 - x^2}}\,dx.$$

Solution. We use the substitution $t = \arcsin(x/3)$, when $x = 3\sin t$, $\sqrt{9 - x^2} = 3\cos t$, and $dx = 3\cos t\,dt$. Hence we have

$$\begin{aligned} I &= \int \frac{9\sin^2 t}{3\cos t} 3\cos t\,dt = \int 9\sin^2 t\,dt \\ &= \frac{9}{2}(t - \sin t\cos t) + C = \frac{9}{2}\arcsin\frac{x}{3} - \frac{1}{2}x\sqrt{9 - x^2} + C; \end{aligned}$$

the third equation uses the calculation in equation (3.7).

Problem 6.2. Find

$$I = \int \frac{dx}{x^2\sqrt{x^2 + 16}}.$$

Solution. We use the substitution $t = \arctan(x/4)$, when $x = 4\tan t$, $\sqrt{x^2 + 16} = 4\sec t$, and $dx = 4\sec^2 t\,dt$. Thus, we have

$$\begin{aligned} I &= \int \frac{4\sec^2 t\,dt}{16\tan^2 t\, 4\sec t} = \frac{1}{16}\int \frac{\sec t}{\tan^2 t}\,dt = \frac{1}{16}\int \frac{1/\cos t}{\sin^2 t/\cos^2 t}\,dt \\ &= \frac{1}{16}\int \frac{\cos t}{\sin^2 t}\,dt = \frac{1}{16}\int \frac{du}{u^2}, \end{aligned}$$

where the last equation was obtained by the substitution $u = \sin t$, when $du = \cos t\,dt$. Therefore,

$$\begin{aligned} I &= -\frac{1}{16u} + C = -\frac{1}{16\sin t} + C = -\frac{1}{16\sin t/\cos t}\,\frac{1}{\cos t} + C \\ &= -\frac{1}{16\tan t}\sec t + C = -\frac{4\sec t}{16 \cdot 4\tan t} + C = -\frac{\sqrt{x^2 + 16}}{16x} + C \end{aligned}$$

Problem 6.3. Find

$$I = \int \frac{x^2}{\sqrt{x^2 - 25}}\,dx.$$

Solution. We will put $t = \operatorname{arcsec}(x/5)$, when $x = 5\sec t$. $\sqrt{x^2 - 25} = 5\tan t$, and $dx = 5\sec t\tan t$. We have

$$I = \int \frac{25\sec^2 t}{5\tan t} 5\sec t\tan t\,dt = \int 25\sec^3 t\,dt = \frac{25}{2}(\sec t\tan t + \ln|\sec t + \tan t|) + C;$$

the last equation was given in the solution to Problem 4.7. Hence, going back to x, we have

$$I = \frac{1}{2}\left(x\sqrt{x^2 - 25} + 25\ln\left|x + \sqrt{x^2 - 25}\right|\right) + C.$$

As for the argument of the logarithm in the last displayed formula, note that $x = 5\sec t$ and $\sqrt{x^2 - 25} = 5\tan t$, but making these substitutions in the scope of the logarithm converts the the factor 5 into an addtive constant that can be incorporated into C. That is, the C at the end of the last display is not the same as the C at the end of the preceding one.

6.1 Reading

[9, §7.3, pp. 500–505].

6.2 Homework

[9, §7.3, p. 505], 1, 3, 5, 7, 11, 17. 21, 31.

7 Integration of rational functions: partial fraction decomposition

The integration of rational function involves calculating integrals of the form

$$\int \frac{P(x)}{Q(x)}\,dx.$$

The problem is completely solved in a theoretical sense; in practice, however, the step of factoring the denominator presents difficulties, and can be only accomplished approximately for higher degree polynomials, at which point numerical integration is usually a better technique.

The first step is to make sure that the numerator $P(x)$ has lower degree than the denominator $Q(x)$. This can be accomplished by the polynomial long division algorithm that, given polynomials $F(x)$ and $D(x)$, where $D(x)$ is not a constant, allows one to determine polynomials $Q(x)$ and $R(x)$ such that

$$F(x) = Q(x)D(x) + R(x),$$

where $R(x)$ has lower degree than $D(x)$ (possibly $R(x) = 0$); here the constant polynomial is said to have degree 0. This method is discussed in elementary algebra courses. In fraction form, this equation can be written as

$$\frac{F(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)},$$

Using the polynomial division algorithm with $P(x)$ and $Q(x)$ replacing $F(x)$ and $D(x)$ in the description, we can ensure that in the new fraction $P(x)/Q(x)$ the numeratior has lower degree.[7.1] After we ensured that in the fraction $P(x)/Q(x)$ the denominator has lower degree, one may use the *Euclidean algorithm* for polynomials to determine the greatest common divisor of $P(x)$ and $Q(x)$; that is, we may determine the highest degree polynomial $D(x)$ such that we have $P(x) = P_1(x)D(x)$ and $Q(x) = Q_1(x)D(x)$; if there is such a nonconstant $D(x)$, then one can simplify the problem, since $P(x)/Q(x) = P_1(x)/Q_1(x)$. Whether or not one perform this step will not make a difference in the final result, but it may make the calculation easier.[7.2]

The next step is factoring the denominator. Every polynomial $Q(x)$ can be written in the form

$$Q(x) = A\prod_{k=1}^{N}(x-\alpha_k)^{m_k},$$

where A is the leading coefficient (i.e., in coefficient of the highest degree term) of $Q(x)$, and the complex numbers α_k are the distinct zeros of of $Q(x)$. One needs to write the factors corresponding

[7.1] That is, we want to keep the notation, even though the numerator changed by the application of the polynomial division algorithm. The reason is that the use of the polynomial division is a preliminary step, and not part of the main description.

[7.2] The final result will be the same whether or not one performs this step.

to the same α_k together. The number m_k is called the multiplicity of the zero α_k. Since the zeros of a polynomial cannot be exactly determined (there are solution formulas only up to degree 4 – for degree higher than 5 the zeros can only be determined exactly in special cases), this step can only be approximately performed in many cases.

If one starts with a polynomial with real coefficients, one usually desires to stay in the realm of real numberes. This is possible, since the complex zeros of a polynomial with real coefficients in pairs of conjugate complex numbers (of the same multiplicity). Given $\alpha = a + bi$ and its conjugate $\bar{\alpha} = a - bi$ (with a and b real), the numbers $\alpha + \bar{\alpha} = 2a$ and $\alpha\bar{\alpha} = a^2 + b^2$ are real, and so the coefficients of the quadratic polynomial

$$(x - \alpha)(x - \bar{\alpha}) = x^2 - (\alpha + \bar{\alpha})x + \alpha\bar{\alpha}$$

are real. Thus, one can convert the factorization above as a factorization into the product of linear and quadratic factors as

$$Q(x) = \Big(\prod_{k=1}^{N_1} (a_k x + b_k)^{m_k}\Big) \prod_{k=1}^{N_2} (c_k x^2 + d_k x + e_k)^{n_k}; \tag{7.1}$$

the notation here does not have a direct connection to the notation used above.[7.3] Here we assume that the coefficients a_k through e_k are real, and the quadratic factors have no real zeros.

The possibility of a partial fraction decomposition is described in the following theorem.

Theorem 7.1. *Let $P(x)/Q(x)$ be a ratio of two polynomials with real coefficients. Assume the degree of $P(x)$ is lower than the degree of $Q(x)$. Assume, further, that we are given a factorization of the denominator as described in equation (7.1), then $P(x)/Q(x)$ is described as a sum of fractions, where to each factor in the denominator corresponds a part of this sum. Specifically, to a linear factor $(a_k x + b)^k$ there corresponds a sum*

$$\sum_{l=1}^{m_k} \frac{A_{kl}}{(a_k x + b_k)^l};$$

to each quadratic factor $(c_k x^2 + d_k x + e_k)^{n_k}$ there corresponds a sum

$$\sum_{l=1}^{n_k} \frac{B_{kl}x + C_{kl}}{(c_k x^2 + d_k x + e_k)^l}.$$

The proof of this theorem is beyond the scope of these notes. In the next step, one needs to determine the coefficients in these numerators of these sums. The final step, to be discussed in the next section, is the integration of these fractions.

7.1 An example for form of partial fraction decomposition

Problem 7.1. Find the form of the partial fraction decomposition of the fraction

$$\frac{P(x)}{Q(x)} = \frac{3x^8 - 6x^5 + 7}{(2x+3)(5x-2)^4(3x^2-2x+4)^3}.$$

Do not determine the coefficients.

[7.3] There is of course a relationship, such as $N_1 + 2N_2 = N$, and the numbers n_k and m_k come out from the numbers n_k used before, but each m_k corresponds to two of the n_k above, but it is not worth describing the relationship exactly. As for the linear factors, it is more convenient to use the form $A_k x + B_k$, rather than the form $x - \alpha_k$. One reason is a famous theorem of Gauss, saying that if a polynomial with integer coefficients can be factored as a product of two polynomials with rational coefficients, then it is possible to factor is as a product of two polynomials with integer coefficients. In so far as possible, one tries to retain the integer coefficients.

Solution. There are a few things to check before we write out the form of the partial fraction decomposition First, we need to ascertain that the degree of the numerator is lower than that of the denominator. The degree of the numerator is 8; to get the degree of the denominator, we need to add the degrees of the factors: $1 + 4 \cdot 1 + 3 \cdot 2 = 11$. So the degree of the numeror is indeed lower. If this were not the case, then we would have to multiply out the denominator in order to perform the polynomial long division $P(x)/Q(x)$. For finding the partial fraction decomposition after that, we need to return to the factored form of the denominator.

The second thing to check is that the quadratic factor cannot be factored into real linear factors; in other words, to show that $3x^2 - 2x + 4$ has no real zeros. This is indeed true, since the discriminant[7.4] $2^2 - 4 \cdot 3 \cdot 4$ is negative. The third thing to check is that the factors in the denominator is indeed a proper factorization; that is, the same factor or its constant multiple doen not occur more than once; this is ineed the case. If they did, they would need to be combined into the power of a single factor.

After these checks, the form of the partial fracton decomposition can be given easily by a proper understanding of the statement of the theorem above:

$$\begin{aligned}\frac{P(x)}{Q(x)} &= \frac{3x^8 - 6x^5 + 7}{(2x+3)(5x-2)^4(3x^2-2x+4)^3} \\ &= \frac{A}{2x+3} \\ &\quad + \frac{B_1}{5x-2} + \frac{B_2}{(5x-2)^2} + \frac{B_3}{(5x-2)^3} + \frac{B_4}{(5x-2)^4} \\ &\quad + \frac{C_1x + D_1}{3x^2-2x+4} + \frac{C_2x + D_2}{(3x^2-2x+4)^2} + \frac{C_3x + D_3}{(3x^2-2x+4)^3}.\end{aligned}$$

7.2 Finding the coefficents with linear factors, no multiple zeros

Problem 7.2. Find the partial fraction decomposition of

$$\frac{4x^2 + 5x - 14}{(x+1)(x-2)(2x-3)}.$$

Solution. As for the form of the decomposition, we have

$$\frac{4x^2 + 5x - 14}{(x+1)(x-2)(2x-3)} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{2x-3}.$$

Multiplying by the denominator on the left, we obtain

$$4x^2 + 5x - 14 = A(x-2)(2x-3) + B(x+1)(2x-3) + C(x+1)(x-2). \tag{7.2}$$

Note that while the previous equation holds for all values of x except for $x = -1$, $x = 2$, and $x = 3/2$, the latter equation must hold even for these values; this was explained above on accont of equation (5.4). Substituting $x = -1$ gives $-15 = 15A$, i.e. $A = -1$, Substituting $x = 2$ gives $12 = 3B$, i.e. $B = 4$. Substituting $x = 3/2$ gives $5/2 = -(5/4)C$, i.e., $C = -2$. Thus,

$$\frac{4x^2 + 5x - 14}{(x+1)(x-2)(2x-3)} = \frac{-1}{x+1} + \frac{4}{x-2} + \frac{-2}{2x-3}.$$

[7.4] The discriminant of the equation $ax^2 + bx + c = 0$ with real coefficient is $b^2 - 4ac$, assuming $a \neq 0$. If this is positive, the equation has two real zeros; if it is 0, the equation has a double real zero. If it is negative, the equation has no real zeros.

As we saw in the solution of the above problem, one way to solve the equation is to substitute the values that make one of the denominators zero, since this eliminated all other coefficients.[7.5] For example, one can obtain an equation by substituting other values of x in equation(7.2). For example, substituting $x = 3$ gives

$$37 = 3A + 12B + 4C.$$

While this requires some effort to solve, this avoids having work with fractions, which we needed to do when substituting $x = 3/2$. One often prefers to substitute $x = 0$, but the equation this gives is not particularly simple, either:

$$-14 = 6A - 3B - 2C.$$

On the other hand, if one uses these equations when the values of A and B are known, they do not require much effort to solve.

Another way to obtain equations for A, B, and C is to multiply out the right-hand side of equation(7.2). This leads to

$$4x^2 + 5x - 14 = (2A + 2B + C)x^2 - (7A + B + C)x + (6A - 3B - 2C).$$

The polynomials on the sides of this equation are identical, so the coefficients must agree. This gives the equations $4 = 2A + 2B + C$, $5 = -(7A + B + C)$, $-14 = 6A - 3B - 2C$. Again, multiplying out the right-hand side of equation(7.2) equation does require some effort, but determining the leading coefficient (the coefficient of x^2) on the right-hand side is quite easy. In fact, to obtain the coefficient of x^2 is much easier than to substitue $x = 3/2$ into this equation.

7.3 Finding the coefficents with linear factors, multiple zeros

Problem 7.3. Find the partial fraction decomposition of

$$\frac{4x^3 - 5x^2 - 9x + 7}{(x + 2)(x - 1)^3}.$$

Solution. The partial fraction can be written as follows.

$$\frac{4x^3 - 5x^2 - 9x + 7}{(x + 2)(x - 1)^3} = \frac{A}{x + 2} + \frac{B}{x - 1} + \frac{C}{(x - 1)^2} + \frac{D}{(x - 1)^3}.$$

Multiplying through by the denominators, this gives the equation

$$4x^3 - 5x^2 - 9x + 7 = A(x - 1)^3 + B(x - 1)^2(x + 2) + C(x - 1)(x + 2) + D(x + 2). \tag{7.3}$$

Substituting $x = 1$, we obtain the equation $-3 = 3D$, giving $D = -1$. Substituting $x = -2$ gives $-27 = (-27)A$, giving $A = 1$. We have yet to determine B and C. For this, we will differentiate equation (7.3), once and twice, and then substitute $x = 1$. Differentiating it once, we obtain we obtain

$$12x^2 - 10x - 9 = B\,2(x - 1)(x + 2) + C\big((x + 2) + (x - 1)\big) + D + (x - 1)^2 \cdot \text{something}.$$

We did not determine the polynomial multiplying $(x - 1)^2$ after substituting $x = 1$; we kept $x - 1$, since we want to differentiate once more. Substituting $x = 1$, this equation gives $-7 = 3C + D$. Given that $D = -1$, this gives $C = -2$. Differentiating once more, we obtain

$$\begin{aligned} 24x - 10 &= B\,2\big((x - 1) + (x + 2)\big) + 2C + (x - 1) \cdot \text{something}, \\ &= B\,2(x + 2) + 2C + (x - 1) \cdot \text{something}. \end{aligned}$$

[7.5] This does not work in case of multiple zeros; we will discuss what to do in case of multiple zeros below.

where the first occurrence of $x-1$ in these equations was incorporated into the last term on the right-hand side. Substituting $x=1$, this gives $14=6B+2C$. Since $C=-2$, it follows that $B=3$. Thus,

$$\frac{4x^3-5x^2-9x+7}{(x+2)(x-1)^3}=\frac{1}{x+2}+\frac{3}{x-1}-\frac{2}{(x-1)^2}-\frac{1}{(x-1)^3}.$$

7.4 Quadratic factors

All the methods shown so far work even for quadratic factors. However, to zero out a quadratic factors we need to substitute complex values for x. We will give a simple example as to how to do this.

Problem 7.4. Find the partial fraction decomposition of

$$\frac{5x^2+x+8}{(x+2)(x^2-2x+5)}.$$

Solution. Note that the equation x^2-2x+5 has no real solutions. Indeed, the quadratic formula gives that its solutions are

$$1+2i \qquad \text{and} \qquad 1-2i. \tag{7.4}$$

The form of the partial fraction decomposition is

$$\frac{5x^2+x+8}{(x+2)(x^2-2x+5)}=\frac{A}{x+2}+\frac{Bx+C}{x^2-2x+5}.$$

Multiplying by the denominator on the left gives

$$5x^2+x+8=A(x^2-2x+5)+(Bx+C)(x+2) \tag{7.5}$$

Substituting $x=-2$ gives the equation $26=13A$, i.e., $A=2$. Next we want to substitute $x=1+2i$; as given in equation (7.4), this is one of the zeros of the quadratic polynomial in the denominator. We want to minimize the calculations with complex numbers, so first we rewrite this equation as

$$11x-17+5(x^2-2x+5)=A((x^2-2x+5))+B(x^2-2x+5)+(4B+C)x-5B+2C.$$

Since the substitution $x=1+2i$ makes the quadratic polynomial zero, we have an easier time to perform the substitution:

$$11(1+2i)-17=+(4B+C)(1+2i)-5B+2C.$$

The real and imatinary parts of this give two equations for B and C. The real part gives

$$-6=(4B+C)-5B+2C=-B+3C.$$

The imaginary part gives

$$22=8B+2C.$$

Multiplying the former equation by -2, we obtain $12=2B-6C$, and multiplying lte latter by 3, we ogtain $66=24B-6C$. Adding these equations, we obtain $78=26B$, i.e., $B=3$. Substituting this into the former equation, we obtain $-6=-3+3C$, that is, $-3=3C$, and so $C=-1$. Thus we obtain

$$\frac{5x^2+x+8}{(x+2)(x^2-2x+5)}=\frac{2}{x+2}+\frac{3x-1}{x^2-2x+5}.$$

Substituting complex values into equation (7.5) may not have been the easiest way to determine B and C. For example, comparing the coefficients of x^2 in this equation, we obtain $A+B=5$. Since we know at this point that $A=2$, this gives $B=3$. Substituting $x=0$ gives $8=5A+2C$, and given that $A=2$, this givee $C=-1$. In any case, we wanted to show how to substitute complex values to find coefficients. Sometimes, that may be the easiest way. The advantage of substituting complex numbers may be more clear when there are more fractions.

Further, note that we only used one of the zeros in equation (7.4). Using the other zero would have given the same equations.

8 Integration of rational functions

To accomplish the integration of rational functions, after partial fracton decomposition, one needs to integrate the partial fractions.

8.1 Integrating partial fractions with linear denominator

Integration fractions where the denominator is a power of a linear polynomial is a simple matter. In fact, assuming $a \ne 0$ and $n \ge 1$ is an integer, the calculation of the integral

$$\int \frac{1}{(ax+b)^n}\,dx$$

can be calculated with the substitution $t = ax+b$, when $dt = a\,dx$, i.e., $dx = (1/a)\,dt$. Hence

$$\int \frac{1}{(ax+b)^n}\,dx = \frac{1}{a}\int \frac{1}{t^n}\,dt,$$

and completing the integration is a simple matter.

8.2 Fractions with quadratic denominators: integrating the part with x in the numerator

When integrating a partial fraction with a power of a quadratic polynomial in the denominator, one first needs to remove the part involving x. This can be done as follows. Assuming $a \ne 0$ and $n \ge 1$ is an integer, we have

$$\int \frac{Ax+B}{(ax^2+bx+c)^n}\,dx = \frac{A}{2a}\int \frac{2ax+b}{(ax^2+bx+c)^n}\,dx + \left(B - \frac{Ab}{2a}\right)\int \frac{1}{(ax^2+bx+c)^n}\,dx.$$

The first integral on the right-hand side is a simple matter. Using the substitution $t = ax^2+bx+c$, we have $dt = (2ax+b)\,dx$. Hence

$$\int \frac{2ax+b}{(ax^2+bx+c)^n}\,dx = \int \frac{1}{t^n}\,dt,$$

and calculating the integral on the right-hand side is a simple matter. The evaluation of the second integral will be discussed next.

8.3 Fractions with quadratic denominators: integrating them with constant numerator

In integrals of the form

$$\int \frac{1}{(ax^2+bx+c)^n}\,dx \qquad (n \ge 1), \tag{8.1}$$

in case $n > 1$ the exponent n in the denominator n can be reduced to $n-1$. How to do this is discussed in case of $ax^2+bx+c = x^2+1$ in Subsection 9.1. The technique is not limited to this special form, it can be used with an arbitrary quadratic polynomial; and example is given in Subsection 9.3. In that example, the quadratic polynomial has real zeros, so the alternative method of partial fraction decomposition would also work.

In any case, if in the integral in equation (8.1), the denominator quadratic polynomial in the denominator has no real zeros, by a linear substitution, one can convert the integral to the form

$$\int \frac{1}{(x^2+1)^n}\,dx \qquad (n \ge 1).$$

Such a substitution in case $n = 1$ is discussed in Problem 1.2; the same ideas can also be used in case $n > 1$, After converting the integral to the latter form, the exponent in the denominator can be reduced step by step to 1, and in that case we can use the equation

$$\int \frac{1}{x^2+1}\,dx = \arctan x + C.$$

9 Integration techniques: reducing the exponent of a quadratic denominator

9.1 A recursive formula for integrals with quadratic denominators

Given a positive integer n, one way of integrating

$$\int \frac{1}{(x^2+1)^n}\,dx \tag{9.1}$$

is to reduce the exponent n in the denominator by 1 in each step until the exponent is 1, when one can use the basic integral

$$\int \frac{1}{x^2+1}\,dx = \arctan x + C.$$

This can be done as follows.

$$\int \frac{1}{(x^2+1)^n}\,dx = \int \frac{x^2+1}{(x^2+1)^{n+1}}\,dx = \int \frac{x^2}{(x^2+1)^{n+1}}\,dx + \int \frac{1}{(x^2+1)^{n+1}}\,dx. \tag{9.2}$$

The first integal on the right-hand side can be calculated by integration by parts:

$$\begin{aligned}\int \frac{x^2}{(x^2+1)^{n+1}}\,dx &= \int \frac{x}{2}\cdot\frac{2x}{(x^2+1)^{n+1}}\,dx = \int \frac{x}{2}\cdot\left(\frac{d}{dx}\frac{-1}{n(x^2+1)^n}\right)dx\\ &= -\frac{x}{2n(x^2+1)^n} + \frac{1}{2n}\int \frac{1}{(x^2+1)^n}\,dx.\end{aligned}$$

Substituting the right-hand side to the first integral on the right-hand side of (9.2), we obtain

$$\int \frac{1}{(x^2+1)^n}\,dx = -\frac{x}{2n(x^2+1)^n} + \frac{1}{2n}\int \frac{1}{(x^2+1)^n}\,dx + \int \frac{1}{(x^2+1)^{n+1}}\,dx.$$

Rearranging this equation, we obtain

$$\int \frac{dx}{(x^2+1)^{n+1}} = \frac{x}{2n(x^2+1)^n} + \frac{2n-1}{2n}\int \frac{1}{(x^2+1)^n}\,dx.$$

9.2 An example: trigonometric substitutions for integrals with quadratic denominators

The integral (9.1) can also be calculated by using the trigonometric substitution $t = \arctan x$. For example,

Problem 9.1. Evaluate the integral

$$I = \int \frac{1}{(x^2+1)^2}\,dx. \tag{9.3}$$

Solution. Using the substition $t = \arctan x$. In this case, $x = \tan t$, $dx = \sec^2 t\,dt$, and $1 + x^2 = 1 + \tan^2 t = \sec^2 t$. Thus, the integral becomes

$$I = \int \frac{\sec^2 t\,dt}{\sec^4 t} = \int \frac{1}{\sec^2 t}\,dt = \int \cos^2 t\,dt. \tag{9.4}$$

There are at least two ways to calculate this integral. One is using the half-angle formula[9.1]

$$\cos^2 t = \frac{1+\cos 2t}{2}$$

or by integration by parts. Here we make the latter choice. We will use x instead of t the integral, since the example is insructive on its own:

$$\begin{aligned} J &= \int \cos^2 x\,dx = \int \cos x \cdot \cos x\,dx = \int \cos x \left(\frac{d}{dx}\sin x\right) dx \\ &= \cos x \sin x + \int \sin x \sin x\,dx = \cos x \sin x + \int \sin^2 x\,dx = \cos x \sin x + \int (1-\cos^2 x)\,dx \\ &= \cos x \sin x + x - \int \cos^2 x\,dx = \cos x \sin x + x - (J + C'), \end{aligned}$$

where C' is a constant of integration; indefinite integrals involve a constant of integration, and the integral before the last equality need not involve the same constant of integration as the integral on the left. Solving this equation for J, we obtain

$$J = \int \cos^2 x\,dx = \frac{\cos x \sin x + x}{2} + C,$$

[9.1] The half-angle formulas are usually stated as

$$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}} \qquad \text{and} \qquad \sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$

where C is an arbitrary constant ($C = -C'/2$, but this is of no interest, since C' is also an arbitrary constant). Subtituting this with x replacee by t into (9.4), we obtain

$$\begin{aligned} I &= \int \frac{1}{(x^2+1)^2}\,dx = \frac{\cos t \sin t + t}{2} + C = \frac{1}{2}\left(\cos^2 t\,\frac{\sin t}{\cos t} + t\right) + C \\ &= \frac{1}{2}\left(\frac{1}{\sec^2 t}\,\tan t + t\right) + C = \frac{1}{2}\left(\frac{x}{1+x^2} + \arctan x\right) + C. \end{aligned}$$

9.3 An example: extending the idea of recursion with powers of quadratic denominators

The use of recursion in Subsection 9.1 can be extended to more general situations. Rather than working out the calculations in general, we give an example; the example is instructive in that it shows techniques that are usable also in other situations.

Problem 9.2. Evaluate the integral

$$\int \frac{1}{(x^2-x-2)^2}\,dx. \tag{9.5}$$

As for the quadratic polynomial in the denominator, we have $x^2 - x - 2 = (x+1)(x-2)$; hence the integral can be evaluated directly by partial fraction decomposition. Here we show how to reduce the exponent in the denominator before proceeding to partial fraction decomposition.

Solution. First we show how to use the ideas of Subsection 9.1 to reduce the exponent in the denominator. We multiply the integrals used in solving the problem by 4 in order to avoid fractions. We have

$$\begin{aligned} 4\int \frac{1}{x^2-x-2}\,dx &= 4\int \frac{x^2-x-2}{(x^2-x-2)^2}\,dx \\ &= \int 2x\,\frac{2x-1}{(x^2-x-2)^2}\,dx - \int \frac{2x-1}{(x^2-x-2)^2}\,dx - \int \frac{9}{(x^2-x-2)^2}\,dx. \end{aligned}$$

Note that the numerator in the first two integrals on the right-hand side is the derivative of the polynomial in the denominators, to facilitate easy integration. Evaluating the second integral the substitution $t = x^2 - x - 2$, when $dt = (2x-1)\,dx$, we obtain

$$4\int \frac{1}{x^2-x-2}\,dx = \int 2x\,\frac{2x-1}{(x^2-x-2)^2}\,dx + \frac{1}{x^2-x-2} - \int \frac{9}{(x^2-x-2)^2}\,dx. \tag{9.6}$$

The first integral on the right-hand side can be handled by integration by parts:

$$\begin{aligned} \int 2x\,\frac{2x-1}{(x^2-x-2)^2}\,dx &= \int 2x\left(\frac{d}{dx}\frac{-1}{x^2-x-2}\right)dx \\ &= -2x\,\frac{1}{x^2-x-2} + \int \frac{2}{x^2-x-2}\,dx. \end{aligned}$$

Substituting it into equation (9.6), we obtain

$$\begin{aligned} 4\int \frac{1}{x^2-x-2}\,dx &= -2x\,\frac{1}{x^2-x-2} + \int \frac{2}{x^2-x-2}\,dx \\ &\quad + \frac{1}{x^2-x-2} - \int \frac{9}{(x^2-x-2)^2}\,dx. \end{aligned}$$

Rearranging this equation, we obtain

$$\int \frac{9}{(x^2-x-2)^2}\,dx = -2\int \frac{1}{x^2-x-2}\,dx - \frac{2x}{x^2-x-2} + \frac{1}{x^2-x-2} \tag{9.7}$$

This reduces the exponent of the denominator in the integral in (9.3). Next, we determine the integral on the write hand side using partial fraction decomposition.

We have

$$\frac{1}{x^2-x-2} = \frac{1}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$$

with appropriate consants A and B. Multiplying by the denominator, this equation becomes

$$1 = A(x-2) + B(x+1).$$

Note that this equation is true for all x, even though the equation preceding it makes no sense if $x = -1$ or $x = 2$, since one of the denominators is zero for those values of x. But the last equation certainly holds if $x \neq -1$ and $x \neq 2$. However, this is possible only if the two sides are identical. Were this not the case, the difference of the two sides would be a nonzero polynomial (of degree 1 at present, but in other situations it might be of higher degree), and of polynomial of degree n can only have n zeros, so it cannot be zero at infinitely many places.

Substituting $x = 2$, the equation becomes $1 = 3B$, i.e., $B = 1/3$ Substituting $x = -1$, we obtain $1 = -3A$, i.e., $A = -1/3$. Hence

$$\frac{1}{x^2-x-2} = \frac{1}{(x+1)(x-2)} = -\frac{1/3}{x+1} + \frac{1/3}{x-2}$$

Hence

$$\int \frac{1}{x^2-x-2}\,dx = -\frac{1}{3}\int \frac{dx}{x+1} + \frac{1}{3}\int \frac{dx}{x-2} = -\frac{1}{3}\ln|x+1| + \frac{1}{3}\ln|x-2| + C.$$

Substituting this on the right-hand side of equation (9.7), we obtain

$$\int \frac{1}{(x^2-x-2)^2}\,dx = \frac{2}{27}\ln|x+1| - \frac{2}{27}\ln|x-2| - \frac{2x-1}{9(x^2-x-2)} + C.$$

9.4 Reading

[9, §7.4, pp. 507–514].

9.5 Homework

[9, §7.4, p. 515], 1, 3, 5, 9, 11, 23, 37, 41, 47, 51, 55.

10 More on integration: the Weierstrass substitution

The Weierstrass substitution can convert integrals that involve rational functions of $\sin x$ and $\cos x$, but not x itself, into an integral of a rational function. We use the substitution

$$t = \tan\frac{x}{2}.$$

Then we have

$$dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx = \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)\,dx = \frac{1+t^2}{2}\,dt,$$

that is

$$dx = \frac{2}{1+t^2}\,dt.$$

Further, writing $\alpha = x/2$, we have

$$\sin x = \sin 2\alpha = 2\sin\alpha\cos\alpha = 2\frac{\sin\alpha}{\cos\alpha}\cos^2\alpha = \frac{2\tan\alpha}{\sec^2\alpha} = \frac{2\tan\alpha}{1+\tan^2\alpha} = \frac{2t}{1+t^2},$$

and

$$\begin{aligned}\cos x &= \cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \cos^2\alpha - (1-\cos^2\alpha) = 2\cos^2\alpha - 1\\ &= \frac{2}{\sec^2\alpha} - 1 = \frac{2}{1+\tan^2\alpha} - 1 = \frac{2}{1+t^2} - 1 = \frac{2-(1+t^2)}{1+t^2} = \frac{1-t^2}{1+t^2}.\end{aligned}$$

Problem 10.1. Find

$$I = \int \frac{1}{3\sin x - 4\cos x}\,dx.$$

Solution. Using the Weierstrass substitution $t = \tan(x/2)$ described above, we have

$$I = \int \frac{1}{3\,\frac{2t}{1+t^2} - 4\,\frac{1-t^2}{1+t^2}}\,\frac{2}{1+t^2}\,dt = \int \frac{2}{4t^2+6t-4}\,dt = \int \frac{1}{2t^2+3t-2}\,dt.$$

The denominator can be factored by solving the quadratic equation $2t^2+3t-2=0$. We have

$$t = \frac{-3\pm\sqrt{9+16}}{4} = \frac{-3\pm 5}{4} = \begin{cases}1/2,\\ -2.\end{cases}$$

Hence,

$$2t^2+3t-2 = 2(t-1/2)\big(t-(-2)\big) = (2t-1)(t+2).$$

Hence, using partial fraction decomposition,

$$\frac{1}{2t^2+3t-2} = \frac{1}{(2t-1)(t+2)} = \frac{A}{2t-1} + \frac{B}{t+2}.$$

Multiplying by the denominators, this leads to the equation

$$1 = A(t+2) + B(2t-1).$$

The substitution $t = 1/2$ gives $1 = 5A/2$, i.e. $A = 2/5$, and the substitution $t = -2$ gives $1 = -5B$, and so $B = -1/5$. Thus,

$$\frac{1}{2t^2+3t-2} = \frac{2/5}{2t-1} - \frac{1/5}{t+2}.$$

Hence

$$\begin{aligned}I &= \int \frac{2/5}{2t-1}\,dt - \int \frac{1/5}{t+2}\,dt = \int \frac{1/5}{t-1/2}\,dt - \int \frac{1/5}{t+2}\,dt\\ &= \frac{1}{5}\ln|t-1/2| - \frac{1}{5}\ln|t+2| + C'.\end{aligned}$$

We have

$$\ln|t-1/2| = \ln\bigl(2|t-1/2|\bigr) - \ln 2 = \ln|2t-1| - \ln 2.$$

Therefore,

$$I = \frac{1}{5}\ln|2t-1| - \frac{1}{5}\ln|t+2| + C = \frac{1}{5}\ln\frac{|2t-1|}{|t+2|} + C = \frac{1}{5}\ln\frac{|2\tan\frac{x}{2}-1|}{|\tan\frac{x}{2}+2|} + C,$$

where $C = C' - (1/5)\ln 2$. In any case, C is an arbitrary constant.

10.1 Reading

[9, §7.5, pp. 517–521], [9, §7.7, pp. 529–538].

10.2 Homework

[9, §7.4, p. 516], 65, [9, §7.5, p. 521], 1, 5, 11, 19, 21, 29, 35, 37, 51, 69, 81. 1, 9, 15, 19, 27, 29, 37, 43, 45, 59, 77, 89.

11 The l'Hôpital rule

The rule named after l'Hôpital, or l'Hôpital, helps one calculate limits of the form $\lim_x f(x)/g(x)$ when $f(x)$ and $g(x)$ both tends to 0 or to $\pm\infty$. By $\lim_x$ we mean all various limits, such as $\lim_{x\to a}$, $\lim_{x\searrow a}$, $\lim_{x\nearrow a}$ for some real a, or $\lim_{x\to+\infty}$, $\lim_{x\to-\infty}$, $\lim_{x\to\pm\infty}$. We say that this limit is infinite if it is $+\infty$, $-\infty$, or $\pm\infty$; if the limit is infinite, we still say that the limit does not exist.

Theorem 11.1 (l'Hôpital's rule)**.** *Let f and g be real valued functions, assume that $\lim_x f(x)$ and $\lim_x g(x)$ are both zero or are both infinite. If the limit $\lim_x f'(x)/g'(x)$ exists or is infinite, we have*

$$\lim_x \frac{f(x)}{g(x)} = \lim_x \frac{f'(x)}{g'(x)}.$$

It is important to note here that if the limit on the right-hand side does not exist and is not infinite, then nothing is claimed: the limit on the left-hand side may still exist. We will prove one of the many cases listed in the theorem below; the proof will clarify why the limit on the left may still exist even if the one on the right does not. We need some preliminary results.

11.1 Rolle's theorem

The following is the well-known theorem of Rolle.

Theorem 11.2 (Rolle's theorem)**.** *Let $[a,b]\subset\mathbb{R}$ be a closed interval, and let f be a function that is continuous on $[a,b]$ and differentiable in (a,b). Assume that $f(a)=f(b)=0$. Then there is a number $\xi\in(a,b)$ such that $f'(\xi)=0$.*

Proof. The proof of this relies on the Maximum-Value Theorem,[11.1] saying that a function f that is continuous on the interval $[a,b]$ assumes its largest value somewhere in the interval, that is, there is a place $\xi\in[a,b]$ such that $f(\xi)\ge f(x)$ for all $x\in[a,b]$. It is also well known that at a maximum,

[11.1] The proof of the Maximum-Value Theorem requires more advanced techniques, and it is beyond the scope of these notes.

if it is inside the interval, we must have $f'(\xi) = 0$. Unfortunately, it is possible that the maximum is assumed at an endpoint. In this case, one looks for the mininum (which exists by the same Maximum-Value Theorem, applied to $-f$), and there we also have have $f'(\xi) = 0$, provided this place of minimum ξ is not an endpoint. In case both the the maximum and the minimum are assumed at an endpoint, the assumptions imply that $f(x) = 0$ for all $x \in [a, b]$. This shows that we can find a ξ as required. ☐

11.2 The Cauchy Mean-Value Theorem

We need the following generalization of the Mean-Value Theorem of differentiation given by Cauchy.

Theorem 11.3 (Cauchy's Mean-Value Theorem)**.** *Let f and g be continuous real-valued functions on the interval $[a, b]$ that are differentiable on (a, b). Then there is a number $\xi \in (a, b)$ such that*

$$f'(\xi)(g(b) - g(a)) = g'(\xi)(f(b) - f(a)).$$

Proof. Consider the function $F(x) = (f(x) - f(a))(g(b) - g(a)) - (g(x) - g(a))(f(b) - f(a))$. It is easy to see that F is continuous on $[a, b]$, differentiable in (a, b), and $F(a) = F(b) = 0$; thus, we can use Rolle's Theorem for the function F. We can conclude that there is a $\xi \in (a, b)$ such that $F'(\xi) = 0$. That is,

$$0 = F'(\xi) = f'(\xi)(g(b) - g(a)) - g'(\xi)(f(b) - f(a)).$$

The desired conclusion easily follows from this equality. The proof is complete ☐

For those familiar with determinants, the choice of $F(x)$ can be better explained. We put

$$F(x) = -\begin{vmatrix} f(x) & g(x) & 1 \\ f(a) & g(a) & 1 \\ f(b) & g(b) & 1 \end{vmatrix} = -\begin{vmatrix} f(x) - f(a) & g(x) - g(a) & 0 \\ f(a) & g(a) & 1 \\ f(b) - f(a) & g(b) - g(a) & 0 \end{vmatrix}.$$

The first determinant shows that $F(a) = F(b) = 0$, because determinant has two equal rows in case $x = a$ and $x = b$. The second deerminant was obtained by subtracting the second row from the other two rows; expanding it by the last column gives the representation of $F(x)$ given above.

Giving $F(x)$ in terms of determinants suggest a further generalization of the Mean-Value Theorem. Given functions f, g, and h that are continous in $[a, b]$ and differentiable in (a, b), writing

$$G(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix},$$

we have $G(a) = G(b) = 0$. Hence, by Rolle's theorem, there is a $\xi \in (a, b)$ such that

$$G'(\xi) = \begin{vmatrix} f'(\xi) & g'(\xi) & h'(\xi) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0.$$

Theorem 11.4 (Cauchy's Mean-Value Theorem, modified version)**.** *Let f and g continuous real-valued functions on the interval $[a, b]$ that are differentiable on (a, b). Assume, further, that $g'(x) \neq 0$ for every $x \in (a, b)$. Then there is a $\xi \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{g(a) - g(b)} = \frac{f'(\xi)}{g'(\xi)}.$$

Proof. If $g'(x) \neq 0$ holds for every $x \in (a,b)$ then $g(b) - g(a) \neq 0$, since by the usual (i.e., not Cauchy's) Mean-Value Theorem[11.2]there is an $\eta \in (a,b)$ such that

$$0 \neq g'(\eta) = g(b) - g(a).$$

That is, in this case we can divide both sides of the equation expressing Cauchy's Mean-Value Theorem by $g'(\xi)(g(b) - g(a)$ to obtain the result. □

We will show how to prove the following case of l'Hôpital's rule:

Theorem 11.5 (l'Hôpital's rule)**.** *Let f and g be real valued functions, and let a be a real number. Assume that $\lim_{x \searrow a} f(x) = 0$ and $\lim_{x \searrow a} g(x) = 0$. Assume, further, that f and g are differentiable If the limit $\lim_{x \searrow a} f'(x)/g'(x)$ exists or is infinite, we have*

$$\lim_{x \searrow a} \frac{f(x)}{g(x)} = \lim_{x \searrow a} \frac{f'(x)}{g'(x)}. \tag{11.1}$$

Proof. We may extend the definition of f and g to the point a by putting $f(a) = g(a) = 0$; then the functions f and g will be continuous on the interval $[a, b)$. Continuity at every $x \in (a,b)$ follows from differentiability, and right-handed continuity at a will hold since $\lim_{x \searrow a} f(x) = \lim_{x \searrow a} g(x) = 0$.

Write

$$\lim_{x \searrow a} \frac{f'(x)}{g'(x)} = L.$$

Let $x \in (a,b)$ be arbitrary. We claim that there is a $\xi \in (a,x)$ such that

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(\xi)}{g'(\xi)}. \tag{11.2}$$

Indeed, the first equality here holds since we made the stipulation $f(a) = g(a) = 0$, and the second equality holds by the (modified version of) Cauchy's Mean-Value Theorem 11.4. Since the right-hand side in equation (11.2) approaches L as x approaches a, so does the left-hand side, and so we also have

$$\lim_{x \searrow a} \frac{f(x)}{g(x)} = L$$

follows. This establishes equation (11.1), completing the proof. □

Note. It should be clear from this proof that in equation (11.1) we cannot conclude the existence of the limit on the right-hand side from that of the left-hand side. The reason for this is tha equation (11.2) does not talk about every number $\xi > a$ close to a. It only talks about those values of ξ that come up in the application of Cauchy's Mean-Value Theorem; it is instructive to think through a situation when these values of ξ do not comprise all values of $\xi > a$ close to a. For example, given $f(x) = x^2 \sin \frac{1}{x}$ and $g(x) = \sin x$, it is quite easy to see that

$$\lim_{x \searrow 0} \frac{f(x)}{g(x)} = \lim_{x \searrow 0} \frac{x^2 \sin \frac{1}{x}}{\sin x} = 0,$$

while an attempt at applying l'Hôpital's rule leads to

$$\lim_{x \searrow 0} \frac{f'(x)}{g'(x)} = \lim_{x \searrow 0} \frac{2x \sin \frac{1}{x} - \cos \frac{1}{x}}{\cos x},$$

and it is not hard to see that this limit does not exist.

[11.2]The usual Mean-Value Theorem is the special case of Cauchy's Mean-Value Theorem when $g(x) = x$.

11.3 Interchange of limits and continuos functions

While l'Hôpital's rule will be our main tool in discussing examples involving limits, in some cases we will need the following Lemma.

Lemma 11.1. *Let f and g be functions and L a real number such that $\lim_x g(x) = L$, and assume f is continuous at L. Then*

$$\lim_x f(g(x)) = f(L). \tag{11.3}$$

Here, as above, $\lim_x$ may mean any of $\lim_{x\to a}$, $\lim_{x\searrow a}$, $\lim_{x\nearrow a}$ for a real number a, or $\lim_{x\to+\infty}$, $\lim_{x\to-\infty}$, or $\lim_{x\to\pm\infty}$. Equation equation (11.3) may be more easy to remember if it is written as

$$\lim_x f(g(x)) = f(\lim_x g(x)); \tag{11.4}$$

this equation may be expressed in words by saying that the order of taking a limit and applying a continuous function can be interchanged.

In some cases, it is enough to assume that f is continous only from the left, or from the right, depending on how $g(x)$ approaches L. The lemma can also be extended to the case L is infinite, though we will not formulate it, since we do not want to discuss the concept of continuity at infinity. When we need the lemma in these cases, we will describe the situation explicitly. We will give a (some what informal proof). A rigorous proof would require a rigorous definition of limit and continuity, which has not yet been given.

Proof. For the sake of simplicity, settling on the case $x \to a$ when a is a real number, $\lim_{x\to a} g(x) = L$ means that for x close to a but $x \neq a$, $g(x)$ is close to L. The function f being continuous at L means that for y close to L, $f(y)$ is close to is close to $f(L)$.[11.3] That is, when $x \neq a$ is close to a, $y = g(x)$ is close to L, so $f(g(x)) = f(y)$ is close to $f(L)$. □

11.4 The exponential function compared to the power function

For any $\alpha > 0$ we have

$$\lim_{x\to+\infty} \frac{x}{e^{\alpha x}} = \lim_{x\to+\infty} \frac{1}{\alpha e^{\alpha x}} = 0,$$

where the first equation holds according to l'Hôpital's rule. Hence, for any real $\alpha, \beta > 0$ we have

$$\lim_{x\to+\infty} \frac{x^\alpha}{e^{\beta x}} = \lim_{x\to+\infty} \left(\frac{x}{e^{(\beta/\alpha)x}}\right)^\alpha = \left(\lim_{x\to+\infty} \frac{x}{e^{(\beta/\alpha)x}}\right)^\alpha = 0^\alpha = 0;$$

the second equation here follows from Lemma 11.1,[11.4] since the function $f(x) = x^\alpha$ is continuous from the right at $x = 0$. In words, any exponential function with base > 1 tends to infinity faster than any power function with a positive exponent.[11.5] More emphatically, even for a very large $\alpha > 0$, x^α tends to infinity more slowly than $e^{\beta x}$ for a very small $\beta > 0$.

[11.3] Here, of course, there is no need to exclude $y = L$, since $f(y) = f(L)$ for $y = L$, since $f(L)$ is close to itself. In case of $\lim_{x\to a} g(x)$, $g(a)$ need not even be defined, so it is necessary to exclude $x = a$.

[11.4] As extended to one-sided continuity in the comments afterward.

[11.5] For $a > 1$, $a^x = e^{x \ln a}$ with $\ln a > 1$.

11.5 The power function and the logarithm

Using l'Hôpital's rule, we have

$$\lim_{t\to+\infty}\frac{\ln t}{t}=\lim_{t\to+\infty}\frac{1/t}{1}=0.$$

Writing $t=x^\alpha$ with $\alpha>0$, we have $t\to+\infty$ when $x\to+\infty$. Hence,

$$\lim_{x\to+\infty}\frac{\ln x}{x^\alpha}=\lim_{x\to+\infty}\frac{\ln x^\alpha}{\alpha x^\alpha}=\frac{1}{\alpha}\lim_{t\to+\infty}\frac{\ln t}{t}=0.$$

Thus, again using the extended version of Lemma 11.1,for $f(x)=x^\alpha$ at $x=0$, for any $\alpha,\beta>0$, we have

$$\lim_{x\to+\infty}\frac{(\ln x)^\alpha}{x^\beta}=\lim_{x\to+\infty}\left(\frac{\ln x}{x^{\beta/\alpha}}\right)^\alpha=\left(\lim_{x\to+\infty}\frac{\ln x}{x^{\beta/\alpha}}\right)^\alpha=0. \tag{11.5}$$

Thus, even a very large power of the logarithm tends to infinity more slowly than a tiny positive power of x.

Noting that $\ln(1/x)=-\ln x$, we can convert this to a limit when $x\searrow 0$: for any $\alpha,\beta>0$ we have

$$\lim_{x\searrow 0}(-\ln x)^\alpha x^\beta=0.$$

The $-$ sign before the logarithm is necessary, since, for x with $0<x<1$, $\ln x$ is negative, and powers of negative numbers are not defined except for a few exceptional values of the exponent.[11.6]

11.6 Examples: l'Hôpital's rule is not just for fractions

Applying the l'Hôpital rule requires a fraction, but the original question may not be given in a fraction form, and it needs to be converted to a fraction for an application of the rule.

Problem 11.1. Find

$$L=\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right).$$

Solution. We have

$$L=\lim_{x\to 0}\frac{\sin x-x}{x\sin x}=\lim_{x\to 0}\frac{\cos x-1}{\sin x+x\cos x}=\lim_{x\to 0}\frac{-\sin x}{\cos x+(\cos x-x\sin x)}=\frac{0}{2}=0.$$

The second and third equations are obtained by using l'Hôpital's rule, showing that one may need to apply the rule repeatedly. In the fourth equation, we use the quotient rule for limits, and take the limits of the numerator and denominator separately; since these are continuous functions, there limits are simply the substitution values.[11.7]

Problem 11.2. Find

$$L=\lim_{x\to+\infty}x\left(\frac{\pi}{2}-\arctan x\right).$$

Solution. We have

$$L=\lim_{x\to+\infty}\frac{\frac{\pi}{2}-\arctan x}{1/x}=\lim_{x\to+\infty}\frac{-\frac{1}{1+x^2}}{-\frac{1}{x^2}}=\lim_{x\to+\infty}\frac{x^2}{1+x^2}=\lim_{x\to+\infty}\frac{1}{1/x^2+1}=1;$$

here we used l'Hôpital's rule to obtain the second equation.

[11.6] For $x<0$, x^α is defined only in case x is an integer or a fraction with an odd denominator.

[11.7] I.e., the values at the point where the limit is taken.

Problem 11.3. Given a real number a, find

$$L = \lim_{x\to 0}(1+ax)^{1/x}.$$

This example is quite important, and we will discuss this after giving the solution.

Solution. Using the notation $\exp x = e^x$,[11.8] we have

$$L = \lim_{x\to 0} \exp\left(\frac{1}{x}\ln(1+ax)\right) = \exp\left(\lim_{x\to 0}\frac{\ln(1+ax)}{x}\right) = \exp\left(\lim_{x\to 0}\frac{\frac{a}{1+ax}}{1}\right) = \exp a = e^a;$$

The second equation uses Lemma 11.1, and the third equation holds in view of l'Hôpital's rule.

11.7 The standard definition of the natural exponential function

In the solution of Problem 11.3 shows that

$$\lim_{x\to 0}(1+ax)^{1/x} = e^a.$$

Changing the notation, writing $y = 1/x$ and $x = a$, we can write this as

$$\lim_{y\to\pm\infty}\left(1+\frac{x}{y}\right)^y = e^x.$$

Choosing y to be a positive integer n, we will only have $n\to\infty$, but the limit will be the same. That is, we have

$$\exp x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n.$$

This equation can be used as the definition of the function $\exp x$. This definition has the advantage that is relies only on powers with positive integer exponents. This equation is widely used in the literature as the definition of the natural exponential function $e^x = \exp x$. This definition works even when x has complex values. A definition of e^x relying on this equation is given in [2].

11.8 Reading

[9, §4.4, pp. 309–316].

11.9 Homework

[9, §4.4, p. 316], 13, 21, 25, 31, 47, 51, 53, 59. 67.

[11.8] This symbol is widely used in mathematics. It is important in that if the exponent is a complicated expression, one does not need to write it "upstairs" in tiny letters.

12 Improper integrals

12.1 The definition of improper integrals

Riemann integrals can only be defined on finite intervals. Sometimes integration can be extended to infinite integrals by putting

$$\int_a^{+\infty} f \overset{def}{=} \lim_{A\to+\infty} \int_a^A f,$$
$$\int_{-\infty}^{b} f \overset{def}{=} \lim_{A\to-\infty} \int_A^b f,$$
$$\int_{-\infty}^{+\infty} f \overset{def}{=} \lim_{\substack{A\to-\infty\\ B\to+\infty}} \int_A^B f,$$

where a and b are real numbers (often, one writes ∞ instead of $+\infty$. As for the last type of integral, we need to have A must approach $-\infty$ and B must approach $+\infty$. To really understand what this means one needs to limits involving two variables.[12.1] so it may be simpler to define the last integral in terms of the first two integrals as

$$\int_{-\infty}^{+\infty} f \overset{def}{=} \int_{-\infty}^{c} f + \int_c^{+\infty} f$$

for an arbitrarily chosen c, in order to avoid limits involving two variables.

In other integrals, given $a, b \in \mathbb{R}$ with $a < b$, the Riemann integral does not exist on the interval $[a, b]$ because of some trouble at one or more points in the interval. We will focus on the case where a single point causes trouble; the cases of more trouble points can be handled similarly. The point $c \in [a, b]$ causing the trouble is called a singularity. The singularity may be that the the integrand is not defined at a c, or even if it is defined at c, the integrand has an infinite limit or some other kind of trouble at c.[12.2] Assuming $c \in [a, b]$ is the only singularity, we can put

$$\int_a^b f \overset{def}{=} \lim_{A\searrow a} \int_A^b f \qquad \text{if} \quad c = a$$
$$\int_a^b f \overset{def}{=} \lim_{B\nearrow b} \int_a^B f \qquad \text{if} \quad c = b$$
$$\int_a^b f \overset{def}{=} \lim_{A\nearrow c} \int_a^A f + \lim_{B\searrow c} \int_B^b f \qquad \text{if} \quad a < c < b.$$

The last improper integral could also have been defined in terms of the first two improper integrals as

$$\int_a^b f \overset{def}{=} \int_a^c f + \int_c^b f.$$

[12.1] While here we take limits at infinity, and so there can be no misunderstanding, if a and b are real numbers, the limits

$$\lim_{\substack{x\to a\\ y\to b}} \qquad \text{and} \qquad \lim_{(x,y)\to(a,b)}$$

are distinct concepts. In the first one, we want $x \neq a$ and $y \neq b$; in the latter, we only want $x \neq a$ or $y \neq b$.

[12.2] For the Riemann integral to exist, the integrand must be bounded; so, writing $f(x)$ for the integrand, we must have a real number M such that $|f(x)| \leq M$ for all $x \in [a, b]$.

If the limit corresponding to an improper integral exists, the integral is said to be *convergent*, and if the limit does not exist, it is said to be *divergent*. Divergent integrals have no value assigned to them, except that one may say the integral is $+\infty$, $-\infty$, or $\pm\infty$ in case the limit is one of these infinities.

12.2 One needs to be careful with improper integrals

One must always watch out to see whether an integral we need to calculate is a proper Riemann integral or an improper integral. For example, consider the following.

Problem 12.1. Evaluate

$$I = \int_{-1}^{3\sqrt{3}} \frac{1}{x^{4/3} + x^{2/3}}\, dx.$$

Solution. The integrand becomes infinite at $x = 0$, so it must be treated as an improper integral. The indefinite integral is not too hard to evaluate if one uses the substitution $t = x^{-1/3}$, $x = t^{-3}$ and $dx = -3t^{-4}\, dt$. Hence,

$$J = \int \frac{1}{x^{4/3} + x^{2/3}}\, dx = \int \frac{1}{t^{-4} + t^{-2}}\, (-3t^{-4})\, dt.$$

Multiplying the numerator and the denominator in the integrand on the right-hand by t^4, we obtain

$$J = -3 \int \frac{1}{1+t^2}\, dt = -3 \arctan t + C = -3 \arctan x^{-1/3} + C.$$

To evaluate the integral I, we need to remember that this is an improper integral, so it needs to be evaluated as such. We have

$$I = \int_{-1}^{3\sqrt{3}} \frac{1}{x^{4/3} + x^{2/3}}\, dx = \int_{-1}^{0} \frac{1}{x^{4/3} + x^{2/3}}\, dx + \int_{0}^{3\sqrt{3}} \frac{1}{x^{4/3} + x^{2/3}}\, dx = I_1 + I_2.$$

Here,

$$\begin{aligned} I_1 &= \int_{-1}^{0} \frac{1}{x^{4/3} + x^{2/3}}\, dx = \lim_{A \nearrow 0} \int_{-1}^{A} \frac{1}{x^{4/3} + x^{2/3}}\, dx \\ &= -3 \lim_{A \nearrow 0} \left(\arctan \frac{1}{\sqrt[3]{A}} - \arctan(-1) \right) = -3 \lim_{B \to -\infty} \left(\arctan B - \left(-\frac{\pi}{4} \right) \right) \\ &= -3 \left(\left(-\frac{\pi}{2} \right) - \left(-\frac{\pi}{4} \right) \right) = \frac{3\pi}{4}; \end{aligned}$$

here, after the fourth equation, $B = 1/\sqrt[3]{A} \to -\infty$ when $A \nearrow 0$. Further,

$$\begin{aligned} I_2 &= \int_{0}^{3\sqrt{3}} \frac{1}{x^{4/3} + x^{2/3}}\, dx = \lim_{A \searrow 0} \int_{A}^{3\sqrt{3}} \frac{1}{x^{4/3} + x^{2/3}}\, dx \\ &= -3 \lim_{A \searrow 0} \left(\arctan \frac{1}{\sqrt[3]{3\sqrt{3}}} - \arctan \frac{1}{\sqrt[3]{A}} \right) = -3 \lim_{B \to +\infty} \left(\arctan \frac{1}{\sqrt{3}} - \arctan B \right) \\ &= -3 \left(\frac{\pi}{6} - \frac{\pi}{2} \right) = \pi; \end{aligned}$$

here, after the fourth equation, $B = 1/\sqrt[3]{A} \to +\infty$ when $A \searrow 0$. Thus,

$$I = I_1 + I_2 = \frac{3\pi}{4} + \pi = \frac{7\pi}{4}.$$

If one ignores that the integral I above is improper and tries to use the Newton–Leibniz formula[12.3] one arrives at the following:

$$\begin{aligned} I &\stackrel{?}{=} -3\arctan x^{-1/3}\Big|_{-1}^{3\sqrt{3}} = -3\left(\arctan\frac{1}{\sqrt[3]{3\sqrt{3}}} - \arctan(-1)\right) \\ &= -3\left(\arctan\frac{1}{\sqrt{3}} - \arctan(-1)\right) = -3\left(\frac{\pi}{6} - \left(-\frac{\pi}{4}\right)\right) = -\frac{5\pi}{4}. \end{aligned}$$

This result is absurd on the face of in, since the integrand is always positive, except at the singularity $x = 0$, where it is undefined, showing that the equation marked with a question mark cannot be true. This calculation should underline the requirement that the integrand be continuous in the Newton–Leibniz formula.

12.3 Integrals of powers of x on infinite intervals

Lemma 12.1. *Let $a > 0$. Given a real number α, we have*

$$\int_a^\infty x^\alpha\,dx = \begin{cases} -a^{\alpha+1}/(\alpha+1) & \text{if } \alpha < -1, \\ +\infty & \text{if } \alpha \ge -1. \end{cases}$$

Proof. Assuming $\alpha \ne -1$, we have

$$\begin{aligned} \int_a^\infty x^\alpha\,dx &= \lim_{A\to\infty}\int_a^A x^\alpha\,dx = \lim_{A\to\infty}\frac{x^{\alpha+1}}{\alpha+1}\Big|_{x=a}^{x=A} = \lim_{A\to\infty}\left(\frac{A^{\alpha+1}}{\alpha+1} - \frac{a^{\alpha+1}}{\alpha+1}\right) \\ &= \lim_{A\to\infty}\frac{A^{\alpha+1}}{\alpha+1} - \frac{a^{\alpha+1}}{\alpha+1} = \begin{cases} -a^{\alpha+1}/(\alpha+1) & \text{if } \alpha < -1, \\ +\infty & \text{if } \alpha > -1. \end{cases} \end{aligned}$$

This establishes the lemma except in case $\alpha = -1$, Assuming $\alpha = -1$, we have

$$\begin{aligned} \int_a^\infty x^\alpha\,dx &= \int_a^\infty x^{-1}\,dx = \lim_{A\to\infty}\int_a^A x^{-1}\,dx = \lim_{A\to\infty}\ln x\Big|_{x=1}^{x=A} \\ &= \lim_{A\to\infty}(\ln A - \ln a) = +\infty = \lim_{A\to\infty}\ln A - \ln a = +\infty, \end{aligned}$$

establishing the lemma also in case $\alpha = -1$. □

12.3.1 Between $\alpha = -1$ and $\alpha < -1$

Lemma 12.2. *Let $b > 1$. Given a real number $\beta > 0$, we have*

$$\int_b^\infty \frac{1}{x(\ln x)^\beta}\,dx = \begin{cases} 1/\big((\beta-1)(\ln b)^{\beta-1}\big) & \text{if } \beta > 1, \\ +\infty & \text{if } \beta \le 1. \end{cases}$$

[12.3] See Theorem 3.1, which requires that the integrand be continuous inside the interval of integration (i.e., continuity of the integrand at the endpoints is not required), and this requirement is not satisfied. This is the reason that the calculation we are going to show will give the wrong answer.

We want lower limit in the integral here to be greater than 1 to avoid the singularity at $x = 1$, since $\ln 1 = 0$. Further, we cannot allow $x < 1$ since then $\ln x < 0$, and powers of negative numbers are only defined for rational exponets with an odd denominator.

Proof. We have

$$\int_b^\infty \frac{1}{x(\ln x)^\beta}\,dx = \lim_{A\to+\infty}\int_b^A \frac{1}{x(\ln x)^\beta}\,dx = \lim_{A\to+\infty}\int_{\ln b}^{\ln A} \frac{dt}{t^\beta}\,dt = \int_{\ln b}^\infty \frac{dt}{t^\beta}\,dt;$$

the second equation follows by substituting $t = \ln x$, when $dt = dx/x$. Further, the for $x = b$ we have $t = \ln b$ and for $x = A$ we have $t = \ln A$. The third equation follows since we have $\ln A \to \infty$ when $A \to \infty$. Thus, the result follows from Lemma 12.1 with $a = \ln b$ and $\alpha = -\beta$. □

12.4 Comparison test for integrals with a limit at infinity

When faced with an improper integral, before making an effort at calculating its value one may want to make sure that it is convergent. We have the following

Theorem 12.1 (Comparison test)**.** *Let a be a real number, and let f and g be functions integrable on all finite subintervals of the interval $[a,\infty)$ such that $0 \le f(x) \le g(x)$ for all $x \ge a$. Assume that the integral $\int_a^\infty g$ is convergent. Then the integral $\int_a^\infty f$ is also convergent.*

Proof. For $A > a$ we have Write

$$\int_a^A f \le \int_a^A g \le \int_a^\infty g.$$

Hence we cannot have

$$\lim_{A\to\infty}\int_a^A f = +\infty.$$

As $\int_a^A f$ increases as A increase, and it must have a finite limit.[12.4] □

12.5 Limit comparison test for integrals with a limit at infinity

The limit comparison test is a simple consequence of the comparison test given in Theorem 12.1, but it is often much easier to apply.

Theorem 12.2 (Limit comparison test)**.** *Let a be a real number, and let $f(x) \ge 0$ and $g(x) \ge 0$ be functions integrable on all finite subintervals of the interval $[a,\infty)$. Assume the limit $L = \lim_{x\to\infty} f(x)/g(x)$ exist.*[12.5] *Assume, further, that the integral $\int_a^\infty g$ is convergent. Then the integral $\int_a^\infty f$ is also convergent.*

It is important to remember that a limit that is infinite does not exist.

[12.4] Admittedly, this argument is a little sloppy, and it cannot be properly justified at this point. The issue will be revisited with a rigorous argument in Lemma 16.1 below.

[12.5] For this limit to exist we must have $g(x) \neq 0$ for all large enough x. But it is not necessary to have $g(x) \neq 0$ for all $x \ge a$.

Proof. Write $L = \lim_{x\to\infty} f(x)/g(x)$. Then there is a positive real M such that $f(x)/g(x) < L+1$ for $x \geq M$. [12.6] Since the integral

$$\int_M^\infty (L+1)g(x)\,dx = (L+1)\int_M^\infty g(x)\,dx$$

is convegent and $f(x) < (L+1)g(x)$, the comparison test (Theorem 12.1) implies that

$$\int_M^\infty f$$

is convergent. Hence,

$$\int_a^\infty f = \int_a^M f + \int_M^\infty f$$

is also convergent. □

Problem 12.2. Given the integral

$$I = \int_1^\infty \frac{\ln x}{x^\alpha}\,dx,$$

where $\alpha > 0$, decide for which values of α is the integral convergent, and for which values it is divergent.

Solution. We will use the Limit Coparison Test (Theorem 12.2) to show that the integral is convergent if $\alpha > 1$, and it is divergent if $\alpha \leq 1$.

First assume that $\alpha > 1$. In using the Limit Coparison Test (Theorem 12.2),

$$f(x) = \frac{\ln x}{x^\alpha} \quad \text{and} \quad g(x) = x^\beta$$

for some β with $-1 > \beta > -\alpha$. As $\beta < -1$, $\int_1^\infty g$ is convergent according to Lemma 12.1. We have

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\frac{\ln x}{x^\alpha}}{x^\beta} = \lim_{x\to\infty} \frac{\ln x}{x^{\alpha+\beta}} = 0,$$

given that $\alpha + \beta > 0$ (cf. (11.5)). Hence $\int_1^\infty f$ is convergent according to the Limit Coparison Test.

Now assume that $\alpha \leq 1$. We will then show that the integral in question is divergent. To this end, now let

$$g(x) = \frac{\ln x}{x^\alpha} \quad \text{and} \quad f(x) = x^\beta,$$

for some β with $-1 \leq \beta \leq -\alpha$. We have

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^\beta}{\frac{\ln x}{x^\alpha}} = \lim_{x\to\infty} \frac{x^{\alpha+\beta}}{\ln x} = 0,$$

given that $\alpha + \beta \leq 0$.

Assuming $\int_1^\infty g$ is convergent, it follows that $\int_1^\infty f$ is convergent by the Limit Coparison Test. This is, however not the case according to Lemma 12.1, since $\beta \geq -1$ in this case. This is a contradiction, showing that the assumption that $\int_1^\infty g$ is convergent cannot be correct. That is, the integral in question is indeed divergent in this case.

[12.6] This is intuitive clear, but it can be shown rigorously by a rigorous definition of limit. Such a definition will be given for sequences (but not for functions) in Definition 16.1.

12.6 Integrals with lower limit at zero

Lemma 12.3. *Given a real number α, we have*

$$\int_0^1 x^\alpha\,dx = \begin{cases} 1/(\alpha+1) & \text{if } \alpha > -1, \\ +\infty & \text{if } \alpha \le -1. \end{cases}$$

Proof. Assuming $\alpha \ne -1$, we have

$$\int_0^1 x^\alpha\,dx = \lim_{A \searrow 0} \int_A^1 x^\alpha\,dx = \lim_{A \searrow 0} \frac{x^{\alpha+1}}{\alpha+1}\bigg|_{x=A}^{x=1} = \lim_{A \searrow 0} \left(\frac{1}{\alpha+1} - \frac{A^{\alpha+1}}{\alpha+1} \right)$$
$$= \frac{1}{\alpha+1} - \lim_{A \searrow 0} \frac{A^{\alpha+1}}{\alpha+1} = \begin{cases} 1/(\alpha+1) & \text{if } \alpha > -1, \\ +\infty & \text{if } \alpha < -1. \end{cases}$$

This establishes the lemma except in case $\alpha = -1$, Assuming $\alpha = -1$, we have

$$\int_0^1 x^\alpha\,dx = \int_1^\infty x^{-1}\,dx = \lim_{A \searrow 0} \int_A^1 x^{-1}\,dx = \lim_{A \to \infty} \ln x\bigg|_{x=A}^{x=1} = -\lim_{A \searrow 0} \ln A = +\infty,$$

establishing the lemma also in case $\alpha = -1$. □

12.7 Comparison tests for integrals with lower limit at zero

A comparison test and a limit comparison test with lower limit at 0 can be formulated in a way similar to the analogous tests described above. We have

Theorem 12.3 (Comparison test)**.** *Let $a > 0$ be a real number, and let f and g be functions integrable on all intervals $[c, a]$ with $0 < c < a$ such that $0 \le f(x) \le g(x)$ for all x with $0 < x \le a$. Assume that the integral $\int_0^a g$ is convergent. Then the integral $\int_0^a f$ is also convergent.*

Similarly,

Theorem 12.4 (Limit comparison test)**.** *Let $a > 0$ be a real number, and let $f(x) \ge 0$ and $g(x) \ge 0$ be functions on the interval $(0, a]$ integrable on intervals $[c, a]$ for c with $0 < c < a$. Assume the limit $L = \lim_{x \searrow 0} f(x)/g(x)$ exist.*[12.7] *Assume, further, that the integral $\int_a^\infty g$ is convergent. Then the integral $\int_a^\infty f$ is also convergent.*

The proofs are similar to the proofs of the similar theorems above.

12.8 Reading

[9, §7.8, pp. 542–549].

12.9 Homework

[9, §7.8, p. 549], 1, 5, 7, 9, 11, 17, 33.

[12.7] For this limit to exist we must have $g(x) \ne 0$ for all $x > 0$ close to 0. But it is not necessary to have $g(x) \ne 0$ for all x with $0 < x \le a$.

13 Polar coordinates

13.1 Representation of points in polar coordinates

The Cartesian coordinate sets up a one-to-one correspondence between *ordered pairs*[13.1] or real numbers and points in the plane. This representation is useful for many purposes, but there are calculations that are more easily performed using other ways of prepresenting points. In polar coordinates, ordered pairs of real numbers are used to represent points in the plane, but the correspondence is not one-to-one: there are many pairs that represent the same point. Given an ordered pair (r, θ), the first number is called the polar radius, and the second one is the polar angle. One usually superimposes a polar coordinate system on a Cartesian coordinate system with the same center and the same unit distance, so as to compare the Cartesian and the polar representation of the same point. Calling the center O, for an arbitrary point P in the plane, $r = \pm OP$.

If $r > 0$, then $\theta = \angle xOP$, i.e., the angle between the positive x-axis and the half-line OP. This angle is measured by the amount that the positive x-axis need to be rotated to go through the point P. Counter-clockwise rotation is considered positive, clockwise rotation is considered negative. Angles are normally measured in radian.[13.2] Thus, if θ is a measurement of an angle, for any integer k (positive, negative, or zero), $\theta + 2k\pi$ signifies the same angle.

If $r < 0$, then the we have $\theta = \angle xOP + \pi$, i.e., the angle of the half-line pointing in the direction opposite to the half-line OP is measured. If $OP = 0$, i.e., if the points P and O coincide, then θ can assume any values.

The conversion from polar coordinates to Cartesian coordinates is simple. If we are given a point $P(r, \theta)_{\text{polar}}$ in polar coordinates, the Cartesian coordinates $P(x, y)_{\text{Cart}}$ can be described by the formulas

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta. \tag{13.1}$$

There are no good formulas for the reverse conversion. We have

$$r = \pm\sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \arctan(y/x), \tag{13.2}$$

but one gets in trouble with this formula if $x = 0$, and the formula only gives a polar angle in the range $-\pi/2 < \theta < \pi/2$ (or possibly allowing the limits $\pm\pi/2$ if one is allowed to onsider $\arctan(\pm\infty)$. Many programming languages have a function `atan2`, which takes two variables x and y, and assigns a polar angle in the range $(-\pi, \pi]$.

13.2 Parametric description of curves

In Cartesian coordinates, curves are usually described by an equation, often in explicit form, such as $y = x^2$, sometimes in implicit form, such as $x^2 + y^2 = 1$. The former represent the set of points represented by the set of pairs

$$\{(x, x^2) : x \in \mathbb{R}\} = \{(x, y) : x, y \in \mathbb{R} \text{ and } y = x^2\},$$

[13.1] An ordered pair of items lumps two items together in a way that on lumps together two items in a way that one can tell which is a first member and which is the second member. The ordered pair of two things a and b is usually denoted as (a, b). In mathematics, there are various representations of ordered pairs; the most widely accepted set-theoretical definition is the Kuratowski representation: $(a, b) = \{\{a\}, \{a, b\}\}$. We will not need to get involved in the subtleties of this representation.

[13.2] When one uses degrees to express angles, one may consider 1° just denoting the number $\pi/180$.

while the latter, the set of points represented by the set of pairs

$$\{(x,y) : x, y \in \mathbb{R} \text{ and } x^2 + y^2 = 1\},$$

in Cartesian coordinates. Implicit descriptions are often ill-suited for calculations, but in may cases explicit descriptions are either undesirable or unobtainable. A parametric representation describes a set of pairs in the form

$$\{\bigl(f(t), g(t)\bigr) : t \in S\} = \{(x,y) : t \in S \text{ and } x = f(t),\ y = g(t)\},$$

where $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ are functions, and S is a set, usually an interval in $\mathbb{R}$. For example, the curve $x^2 + y^2 = 1$ can be desribed in parametric form as

$$\begin{aligned}\{(x,y) : x = \cos t,\ y = \sin t,\ t \in \mathbb{R}\} &= \{(x,y) : x = \cos t,\ y = \sin t,\ 0 \le t < 2\pi\} \\ &= \{(\cos t, \sin t) : 0 \le t < 2\pi\}.\end{aligned}$$

13.3 Description of curves in polar coordinates

The set

$$\{(r,\theta) : r = 2\cos\theta,\ r,\theta \in \mathbb{R}\} = \{(2\cos\theta, \theta) : r,\theta \in \mathbb{R}\}$$

represents a circle that in Cartesian coordinates can be describen by the equation

$$(x-1)^2 + y^2 = 1.$$

In fact this equation can be written as $x^2 - 2x + 1 + y^2 = 1$, that is $x^2 + y^2 = 2x$. Using formulas (13.1), and the first formula in (13.2) this equation can be written as $r^2 = 2r\cos\theta$. This latter equation can also be described as $r = 2\cos\theta$ or $r = 0$.[13.3] The equation $r = 0$ represents the origin of the coordinate system; however, the equation $r = 2\cos\theta$ already represents the origin as the point $(0, \pi/2)$ for $\theta = \pi/2$. Thus the polar equation $r = \cos\theta$ represents the circle in question.

13.3.1 Thales's theorem

The theorem of Thales asserts that, given two points A and B and a circle with diameter AB, and there is a third point C (different from A and B) on the circle. Then the angle $\angle ACB$ is a right angle. The converse is also true: if C is not on the circle, then the angle $\angle ACB$ is not a right angle.

To prove this theorem, write O for the center of the circle. Then $\angle CAO = \angle OCA$ since $OC = OA$ (the radii of a circle are equal!). Similarly, $\angle OBC = \angle BCO$. Thus, the angle of the triangle ABC at C is

$$\angle BCA = \angle OCA + \angle BCO = \angle CAO + \angle OBC. = \angle CAB + \angle ABC.$$

where the last equation just names the same angles differently. Since the angle sum of the triangle is π, we have

$$\pi = (\angle CAB + \angle ABC) + \angle BCA = 2\angle BCA.$$

So $\angle BCA = \pi/2$.

As for the converse of Thales's theorem, assume that $\angle ACB = \pi/2$, yet C is not on the circle, and let $C' \ne A$ be the point on the circle that is also on the line AC; there is such a point since $\angle CAB \ne \pi/2$ since a triangle can have at most one right angle). Then $\langle AC'B = \pi/2$ by Thales's

[13.3] One way to see is by noting that the equation $r^2 - 2r\cos\theta$ can be written as $r(r - 2r\cos\theta) = 0$, and a product can be zero only if one of the factors is zero.

theorem. is a right-angle by Thales's theorem. This is a contradiction, since the triangle $\triangle BCC'$ would have two right angles.

Thales's theorem directly shows that the equation $r = 2\cos\theta$ describes a circle in polar coordinates with center $(1, 0)$ and a diameter containing the points $O(0, 0)$ and $A(2, 0)$ (everything given in polar coordinates).[13.4] Indeed, $P(r, \theta)$ is a point on this circle if and only if $r = 2\cos\theta$, in view of Thales's theorem and its converse.

13.4 Sets and their graphs

The sets

$$\{(r, \theta) : r = 2\cos\theta,\ r, \theta \in \mathbb{R}\}$$

and

$$\{(x, y) : x = 2\cos y,\ x, y \in \mathbb{R}\}$$

represents the same set of ordered pairs, but the notation suggests that the first set is meant to be graphed in polar coordinates, and the latter, in Cartesian, and the graphs are very different. The first one is a circle, the second one is the a cosine function, graphed verically. Graphing a set of ordered pairs means applying a function from the set of all ordered pairs of reals to the plane, and this function describes the coordinate system used. One can call one of these functions F_{Cart} and the other F_{polar}.

13.5 Reading

[9, §10.3, pp. 684–691].

13.6 Homework

[9, §10.3, p. 692], 1, 3, 7, 9, 11, 55. Problem 56 is quite challenging, and it is recommended rather than requirement. To solve it, try to find the easiest match, and then try to match the rest, one by one.

14 Area and arclength in polar coordinates

14.1 Area

In Figure 14.1 we show a figure of a region in polar coordinates. In order to find its area, we divide the region into small sectors; These sectors can be described by a partition of the the interval $[a, b]$:

$$P : a = \theta_0 < \theta_1 < \theta_2 < \ldots < \theta_n = b.$$

Figure 14.2 shows this partition with $n = 5$. Note that the parts need not have equal length; that is, the corresponding angles need not be equal. Figure 14.3 shows the picture of a single sector. The area of such a secton can be approximated by the area of a circular sector determined by the rays $\theta = \theta_{i-1}$ and $\theta = \theta_i$ having radius $r_i = f(\xi_i)$ for some $\xi_i \in [\theta_{i-1}, \theta_i]$; if the angle $\Delta\theta_i = \theta_i - \theta_{i-1}$ itself is small and f is continuous, then any choice of ξ in the interval $[\theta_{i-1}, \theta_i]$ will give approximately

[13.4] The Cartesian coordinates of these three points are the same.

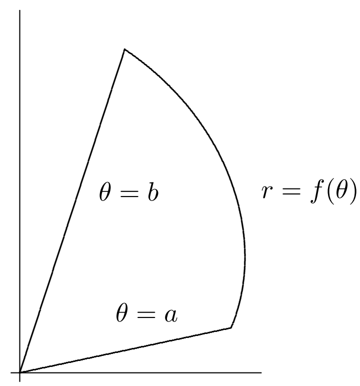

Figure 14.1: A region in polar coordinates

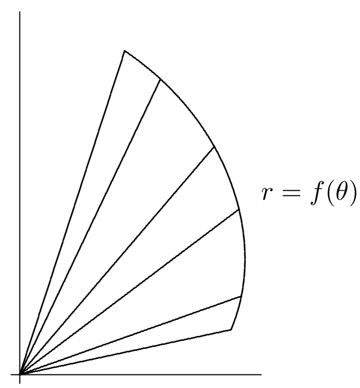

Figure 14.2: A region partitioned in polar coordinates

the same area. For small $\Delta\theta_i$, the circular sector itself can be approximated by an isosceles triangle of base $r_i\Delta\theta_i$ and altitude $r_i = f(\xi_i)$. This area is

$$\Delta A_i = \frac{1}{2} r_i \Delta\theta_i r_i = \frac{1}{2}\big(f(\xi_i)\big)^2(\theta_i - \theta_{i-1}).$$

We can add up these areas to obtain the total approximate area:

$$A \approx \sum_{i=1}^{n} \frac{1}{2}\big(f(\xi_i)\big)^2(\theta_i - \theta_{i-1}).$$

This is an approximating sum, called Rieman sum, to the integral

$$A = \int_a^b \frac{1}{2}\big(f(\theta)\big)^2\, d\theta.$$

More precisely, for the partition P defined above, call the length of its longest interval its norm, (or width):

$$\|P\| = \max\{(\theta_i - \theta_{i-1} : 1 \le i \le n\}.$$

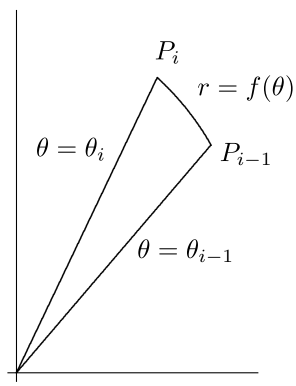

Figure 14.3: A single sector in the partition

The integral A is then defined as the limit of the approximating Riemann sums if $\|P\| \to 0$, whatever the choice of the numbers $\xi_i \in [x_{i-1}, x_i]$ (these numbers are called the *tags* of the partition); for $\|P\| \to 0$, the number n of the intervals in the partition must tend to infinity (but that alone, of course, is not enough). If the limit exits, then the function after the integral is called *integrable* (more precisely, *Riemann integrable*) on the interval $[a, b]$.[14.1]

As for integrability, it can be shown that every function continuous on an interval is integrable on that interval; so in calculating areas of bounded regions it is unlikely for us to meet a function that is not integrable. Note that the continuity on the closed interval is important; for example, the function $f(x) = 1/x$ is continuous on the interval $(0, 1]$ but it is not integrable there.[14.2] The above formula for the area can shortly be written as

$$A = \int_a^b \frac{1}{2} r^2 \, d\theta.$$

14.2 Arclength

Calculating the length of the curve shown in Figure 14.1 uses the same kind of partition used above, but instead of adding up areas, we add up lengths. For example, the length of the arc $\widehat{P_{i-1}P_i}$ in Figure 14.1 we use the length of the straight line segment $\overline{P_{i-1}P_i}$. This is easy to calculate by the Pythagorean theorem. Figure 14.4 shows the same sector as Figure 14.3, with a circular arc with radius OP_{i-1} and center O added. The curved triangle $P_{i-1}P_iQ$ can be thought of as approximately a right triangle $\triangle P_{i-1}P_iQ$ with straight line segments as sides. We have

$$\overline{P_{i-1}Q} \approx \widehat{P_{i-1}Q} = f(\theta_{i-1})(\theta_i - \theta_{i-1}) = r_{i-1}\Delta\theta_i.$$

Further,

$$\overline{QP_i} = f(\theta_i) - f(\theta_{i-1}) = r_i - r_{i-1} = \Delta r_i.$$

[14.1] This definition of the integral, called the *Riemann integral*, was first presented in a talk by Bernhard Riemann in 1854, and in first appeared in print in 1868.

[14.2] The Riemann integral is defined only on closed intervals, but even if we redefine f by putting $f(0) = 0$ and $f(x) = 1/x$, so that it is defined everywhere, f is still not Riemann integrable on $[0, 1]$.

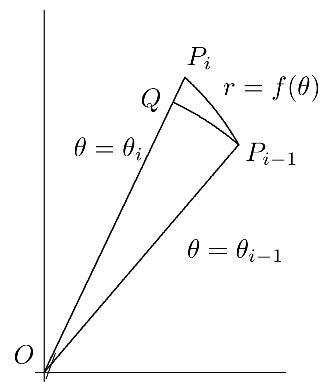

Figure 14.4: A single sector in the partition

Assuming that f is differentiable on $[a, b]$, by the Mean-Value Theorem of differentiation we have

$$f(\theta_i) - f(\theta_{i-1}) = f'(\xi_i)(\theta_i - \theta_{i-1})$$

for some $\xi_i \in (\theta_{i-1}, \theta_i)$. Since $f'(\theta_{i-1}) \approx f'(\xi_i)$, assuming f' is continuous and the interval (θ_{i-1}, θ_i) is sufficiently short, we can write

$$\overline{QP_i} = f(\theta_i) - f(\theta_{i-1}) \approx f'(\xi_i)(\theta_i - \theta_{i-1}) \approx f'(\xi_i)\Delta\theta_i.$$

Hence, the Pythagorean theorem gives

$$\overline{P_{i-1}P_i} = \sqrt{\overline{P_{i-1}Q}^2 + \overline{QP_i}^2} \approx \sqrt{(r_{i-1}\Delta\theta_i)^2 + (f'(\xi_i)\Delta\theta_i)^2}$$

By the continuity of f we have $r_{i-1} = f(\theta_{i-1}) \approx f(\xi_i)$, this gives

$$\overline{P_{i-1}P_i} \approx \sqrt{(r_{i-1}\Delta\theta_i)^2 + \big(f'(\xi_i)\Delta\theta_i\big)^2} \approx \sqrt{\big(f(\xi_i)\big)^2 + \big(f'(\xi_i)\big)^2}\Delta\theta_i.$$

Noting that $\Delta\theta_i = \theta_i - \theta_{i-1}$, the sum of these line segments approximate the total arclength:

$$\sum_{i=1}^{n} \overline{P_{i-1}P_i} = \sum_{i=1}^{n} \sqrt{\big(f(\xi_i)\big)^2 + \big(f'(\xi_i)\big)^2}(\theta_i - \theta_{i-1}).$$

As the the norm $\|P\|$ of the partition P (the length of its longest intervals) tends to 0, this sum will tend to the integral

$$L = \int_a^b \sqrt{\big(f(\theta)\big)^2 + \big(f'(\theta)\big)^2}\, d\theta.$$

This is the length of the arc in question. The formula can be written more concisely as

$$L = \int_a^b \sqrt{r^2 + r'^2}\, d\theta.$$

14.3 Intersection of curves in polar coordinates

To determine the intersection of two curves $y = f(x)$ and $y = g(x)$ one needs to find the solution of the system of equation consisting of these two equations. The problem in polar coordinates is more complicated. This is because the point $(0, \theta)$ in polar coordinates represents the origin of the coordinate system for any value of θ, and for $r \neq 0$, the point (r, θ) can also be represented as $(r, \theta + 2k\pi)$ for every integer (positive, negative, or zero) k and also as $\big(-r, \theta + (2k+1)\pi\big)$.

Thus, given the equation $r = f(\theta)$ of a curve where θ is not restricted to a specific interval or set of reals, that is the curve represented by all points

$$\{(f(\theta), \theta) : \theta \in \mathbb{R}\}$$

given in polar coordinates (where $\mathbb{R}$ denotes the set of all real nubers), the same curve, i.e., the same set of points, is represented by the equations $r = f(\theta + 2k\pi)$ and $r = -f\big(\theta + (2k+1)k\big)$ for any integer k. That is, when determining the intersection of the curves $r = f(\theta)$ and $r = g(\theta)$, we need to consider all equations representing one of the curves, but not both.[14.3] That is, leaving the first equation unchanged, we can find all points of intersection by solving the systems of equations

$$r = f(\theta),\ r = g(\theta + 2k\pi) \qquad \text{and} \qquad r = f(\theta),\ r = -g\big((\theta + (2k+1)\pi\big)$$

for every integer k (positive, negative, or zero).

14.4 An example

Consider the equations $r = 1 + \cos\theta$ and $r = 1 - \sin\theta$. These curves are shown in Figure 14.5, the

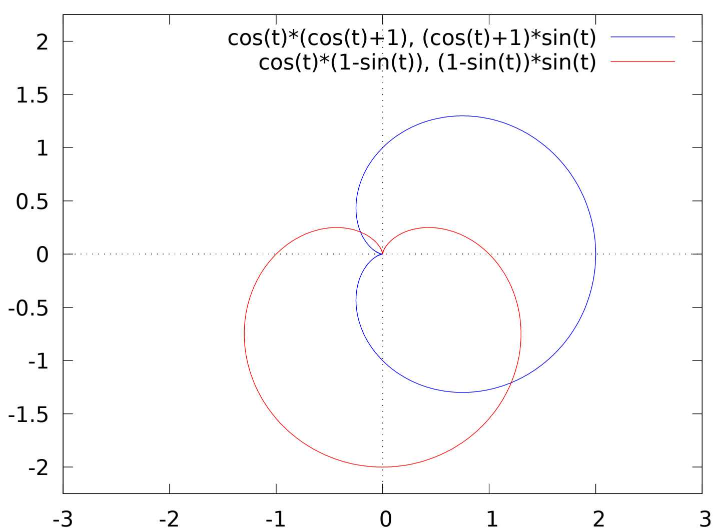

Figure 14.5: Graphs of $r = 1 + \cos\theta$ (blue) and $r = 1 - \sin\theta$ (red)

[14.3]Solving the system of equations $r = f(\theta)$ and $r = g(\theta)$ gives the same set of points as solving the system of equations $r = -f(\theta + \pi)$ and $r = -g(\theta + \pi)$, but with a different representation. On the other hand, the system $r = f(\theta)$ and $r = -g(\theta + \pi)$, but may give a different set of points.

first curve in blue, and the second one in red.[14.4] While the graphs show the points of intersection of these curves, we will show how to find these points without graphing. First we note that the origin is a point of intersection, represented in the first curve as the point $(0, \pi)$ in polar coordinates, while on the second curve it is represented as the point $(0, -\pi/2)$; of course, there are other coordinate representations of the origin on both curves. To find the other points of intersection, we do not need to add multiples of 2π to the argument θ, since both sine and cosine have periods 2π.[14.5] So, the points of intersection other than the origin of the coordinate system are given by the systems of equations

$$r = 1 + \cos\theta,\ r = 1 - \sin\theta \qquad \text{and} \qquad r = 1 + \cos\theta,\ r = -1 - \sin(\theta + \pi)$$

By equating the right-hand sides, the first system of equation gives $\cos\theta = -\sin\theta$; dividing both sides by $\cos\theta$, we obtain $\tan\theta = -1$, i.e., $\theta = -\pi/4 + 2k\pi$, when $r = 1 + \cos(-\pi/4) = 1 + 1/\sqrt{2}$, or $\theta = 3\pi/4 + 2k\pi$, when $r = 1 - 1/\sqrt{2}$. This gives the two points of intersection $(1 + 1/\sqrt{2}, -\pi/4 + 2k\pi)$ and $(1 - 1/\sqrt{2}, 3\pi/4 + 2k\pi)$. Noting that $\sin(\theta + \pi) = -\sin\theta$, equating the right-hand sides in the second system of equations, we obtain $1 + \cos\theta = -1 + \sin\theta$, i.e., $\sin\theta - \cos\theta = 2$. This equation has no solutions; the simplest way to see this is by squaring the equation; we obtain $\sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta = 4$. Noting that $\sin^2\theta + \cos^2\theta = 1$ and $2\sin\theta\cos\theta = \sin 2\theta$, this leads to the equation $-\sin 2\theta = 3$. This of course has no solution, since $-1 \le \sin t \le 1$ for all t. Thus, the second system of equation gives no points of intersection.

Next, we will determine the area inside the first curve and outside the second curve. To do this, we need to subtract the area inside the second curve from the area inside the first curve in the interval $[-\pi/4, 3\pi/4]$. That is, the area in question is

$$\begin{aligned} A &= \int_{-\pi/4}^{3\pi/4} \frac{1}{2}(1 + \cos\theta)^2\, d\theta - \int_{-\pi/4}^{3\pi/4} \frac{1}{2}(1 - \sin\theta)^2\, d\theta \\ &= \frac{1}{2}\int_{-\pi/4}^{3\pi/4} (2\cos\theta + 2\sin\theta + \cos^2\theta - \sin^2\theta)\, d\theta \\ &= \int_{-\pi/4}^{3\pi/4} (\cos\theta + \sin\theta)\, d\theta + \frac{1}{2}\int_{-\pi/4}^{3\pi/4} \cos 2\theta\, d\theta = A_1 + A_2, \end{aligned}$$

where A_1 and A_2 denote the two ineqgrals before the last equation sign; to obtain the *penultimate*[14.6] equation, we used the double angle formula for cosine. In order to find the limits for the integration, we used the polar coordinates of the points of intersection determined above. We have

$$A_1 = (\sin\theta - \cos\theta)\Big|_{\theta=-\pi/4}^{\theta=3\pi/4} = \left(\frac{1}{\sqrt{2}} - \frac{-1}{\sqrt{2}}\right) - \left(\frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = \frac{4}{\sqrt{2}} = 2\sqrt{2}.$$

To calculate the second integral, we make the substitution $t = 2\theta$, when $dt = 2\,d\theta$. As for the limits, if $x = -\pi/4$ we have $t = -\pi/2$ and if $x = 3\pi/4$ we have $t = 3\pi/2$. Hence

$$A_2 = \frac{1}{4}\int_{-\pi/2}^{3\pi/2} \cos t\, dt = \sin t\Big|_{t=-\pi/2}^{t=3\pi/2} = \sin 3\pi/2 - \sin(-\pi/2) = (-1) - (-1) = 0.$$

[14.4] The computer algebra system Maxima was used to produce this figure (the earlier figures were created by the TEX package PICTEX in the incarnation as a LATEX package, though originally PICTEX was not specifically created for LATEX). The formulas at the top of the figure show that the curves were entered as parametric equations in Cartesian coordinates with the parameter t being the polar angle θ; this was easier than to use features that would graph a curve in directly in polar coordinates.

[14.5] That is, $\sin(x + 2\pi) = \sin x$, and $\cos(x + 2\pi) = cosx$.

[14.6] The one before the last one

Hence, the area is $A = A_1 + A_2 = 2\sqrt{2}$.

Next, we will determine the length of the first curve. Its equation is $r = 1 + \cos\theta$, and to obtain the length of the whole curve, we will integrate on the interval $[0, 2\pi]$. We have

$$L = \int_0^{2\pi} \sqrt{r^2 + r'^2}\, d\theta = \int_0^{2\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\, d\theta$$
$$= \int_0^{2\pi} \sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta = \int_0^{2\pi} \sqrt{2 + 2\cos\theta}\, d\theta = \int_0^{2\pi} 2\left|\cos\frac{\theta}{2}\right| d\theta;$$

the last equation was obtained by using the half angle formula for cosine. The absolute value is necessary, since we need to use the positive value of the square root, while $\cos(\theta/2)$ is negative of parts of the interval $[0, 2\pi]$. While we can deal with the absolute value sign by splitting the intervl $[0, 2\pi]$ into two parts such that on each of the parts $\cos(\theta/2)$ has constant sign, it would have been easier to choose another interval of length 2π instead of the interval $[0, 2\pi]$. Indeed, exactly the same calculation shows that

$$L = \int_{-\pi}^{\pi} \sqrt{r + r'^2}\, d\theta = \int_{-\pi}^{\pi} 2\cos\frac{\theta}{2}\, d\theta,$$

and now the absolute value is not necessary, since $\cos(\theta/2)$ is nonnegative in the interval $[-\pi, \pi]$. Making the substitution $t = \theta/2$, we obtain

$$L = \int_{-\pi/2}^{\pi/2} 4\cos t\, dt = 4\sin t\,\Big|_{t=-\pi/2}^{t=\pi/2} = 4\big(1 - (-1)\big) = 8.$$

14.5 Reading

[9, §10.4, pp. 694–698].

14.6 Homework

[9, §10.4, p. 699], 13, 15, 19, 21, 27, 29. 37, 49.

15 The completeness of real numbers

It is commonly understood that the numberline has no gaps. For example, if the side of a square is 1 (measured in some units), its diagonal is $\sqrt{2}$. How can be assured that $\sqrt{2}$ can be placed on the numberline; that is, maybe where $\sqrt{2}$ should appear, there is a missing point, a gap, in the numberline. This has been a problem even for the ancient Greeks, who knew about 2500 years ago that $\sqrt{2}$ is irrational, that is, it cannot be expressed as a fraction of two integers. About 200 years later, the Greek mathematician Eudoxus found a satisfactory solution to this problem in his theory of ratios (the Greeks thought in terms of ratios, such as ratios of distances, areas, or volumes, rather than real numbers). The German mathematician Richard Dedekind essentially revived Eudoxus's solutions by introducing a definition of real numbers in terms of what is now called Dedekind cuts in the second half of the 19th century. A cut based approach, simpler than Dedekind's, to real numbers is given in [1, §3–4, pp. 6–10]. Here we give an axiomatic approach. The key is to find a replacement for a maximum, or largest element, of a set, since not all sets have largest element. For a start, we will give a definition of the largest element, or maximum, of a set. The definition is quite obvious; nevertheless, we want to state it precisely, so that there is absolutely no misunderstanding as to what we mean.

Definition 15.1. Given a set A of the real numbers $\mathbb{R}$, we say that $x \in A$ is the maximum of A if for all $y \in A$ we have $x \geq y$.

Clearly, the interval $[0, +\infty)$ has no maximum (infinity is not a number). The maximum of the interval $[0, 1]$ is 1; on the other hand, the interval $[0, 1)$ has no maximum (1 will not be the maximum, since it is required that the maximum be an element of the set, but $1 \notin [0, 1)$. Yet one is not quite satisfied with this answer, since, while 1 is not the maximum, in a sense one feels that 1 should be a good replacement for the maximum. We will describe how this can be remedied.

15.1 Upper bound, supremum

Definition 15.2. Let $A \subset \mathbb{R}$,[15.1] We say that the number x is an *upper bound* of A if for every number $y \in A$ we have $y \leq x$. If A has an upper bound, it is called *bounded from above*.

For example, the set of upper bounds of the interval $[0, 1]$ is the interval $[1, +\infty)$; the set of upper bounds of the interval $[0, 1)$ is the same interval $[1, +\infty)$, while the set $[0, \infty)$ has no upper bounds (so it is not bounded, or unbounded, from above).

Definition 15.3. Let $A \subset \mathbb{R}$ be a set. We call the number x a *supremum* of A if x is the least among its upper bounds. That is, x is an upper bound of A, and if y is also an upper bound of A then $x \leq y$.

The supremum is also called *least upper bound*. We denote the supremum of A by $\sup A$; the notation $\operatorname{lub} A$ (for Least Upper Bound) is also used. From what we said above it is easy to see that $\sup[0, 1] = 1$ and $\sup[0, 1) = 1$. Recall that we said above that the former set has 1 as its maximum, while the latter set has no maximum. In a sense, the supremum can be considered a replacement for maximum. On the other hand, neither the interval $[0, +\infty)$ nor the empty set $\emptyset$ has a supremum. Yet one occasionally writes that $\sup[0, +\infty) = +\infty$ and $\sup \emptyset = -\infty$, while remembering that these sets do not have suprema.[15.2] That there are no gaps in the numberline is codified in the

The Axiom of Completeness. Every nonempty set of reals that is bounded from above has a supremum.

Such an axiom is necessary for a rigorous discussion of real numbers, unless one adopts an equivalent alternative approach, such as Dedekind's mentioned above, or the one give in the already quoted notes [1, §3–4, pp. 6–10]. With the aid of the Axiom of Completeness, one can prove that there is a positive number whose square is 2. Indeed, take the set

$$S = \{x : x \geq 0 \quad \text{and} \quad x^2 \leq 2\}.$$

This set is certainly not empty (since $1 \in S$), and it is also bounded from above (3 is an upper bound of S, since S does not have any element $x > 3$). Let $u = \sup S$. Then we must have $u^2 = 2$.

[15.1] That is, let A be a subset of the real numbers $\mathbb{R}$; in other words, let A be a set of reals. It is important to remember that $\in$ and $\subset$ mean very different things. $x \in A$ means that x is an element of the set A, which $A \subset B$ means that A is a subset of B, that is, every element of A is also an element of B. As an illustration of the difference, writing $\emptyset$ for the empty set, i.e., for the set with no elements, we have $\emptyset \notin \emptyset$, since nothing is an element of $\emptyset$, while $\emptyset \subset \emptyset$ is (vacuosly) true, since nothing is an element of the emptyset, so there is no requirement is made by requiring that these things also be an element of the emptyset on the right.

[15.2] The Latin plural of supremum, similarly to maxima and minima being the plural of maximum and minimum.

Indeed, we cannot have $u^2 < 2$. Indeed, assuming that $u^2 < 2$, let ϵ with $0 < \epsilon \leq 1$ such that $u^2 + \epsilon \leq 2$.[15.3] Then

$$\left(u + \frac{\epsilon}{4}\right)^2 = u^2 + \frac{\epsilon}{2} + \frac{\epsilon^2}{16} \leq u^2 + \frac{\epsilon}{2} + \frac{\epsilon}{16} \leq u^2 + \epsilon \leq 2;$$

the second inequality holds since $\epsilon^2 \leq \epsilon$, given that $0 < \epsilon \leq 1$. This inequality shows that $u+\epsilon/4 \in S$, so u is not an upper bound of S.

A similar argument shows that we cannot have $u^2 > 2$. Indeed, assuming $u^2 > 2$, choosing ϵ with $\epsilon \leq u^2 - 2$ and $0 < \epsilon < 2u$,[15.4] we have

$$\left(u - \frac{\epsilon}{2}\right)^2 = u^2 - \epsilon + \frac{\epsilon^2}{4} > u^2 - \epsilon \geq 2.$$

So, for any $x \in S$ we have $u - \epsilon/2 \geq x$ (since $x^2 \leq 2$ and $(u - \epsilon/2)^2 \geq 2$).[15.5] Thus, $u - \epsilon/2$ is also an upper bound of S, showing that u is not the least upper bound of S. This shows that we must have $u^2 = 2$.

15.2 Lower bound, infimum

Definition 15.4. Let $A \subset \mathbb{R}$,We say that the number x is an *lower bound* of A if for every number $y \in A$ we have $y \geq x$. If A has an lower bound, it is called *bounded from below*. If A is bounded both from above and below, then A is called *bounded*.

Definition 15.5. Let $A \subset \mathbb{R}$ be a set. We call the number x an *infimum* of A if x is the greatest among its lower bounds. That is, x is an lower bound of A, and if y is also an lower bound of A then $x \geq y$.

The infimum is also called *greates lower bound*. We denote the infimum of A by $\inf A$; the notation $\operatorname{glb} A$ (for Greatest Lower Bound) is also used. From what we said above, we have $\inf[0,1] = 0$ and $\inf(0,1] = 0$. The former set has 0 as its minimum, while the latter set has no minimum. In a sense, the infimum can be considered a replacement for minimum. Neither the interval $([-\infty, 0)$ nor the empty set $\emptyset$ has an infimum. Yet one occasionally writes that $\inf(-\infty, 0] = -\infty$ and $\inf\emptyset = +\infty$, while remembering that these sets do not have infima.[15.6]

16 Sequences and limits

16.1 Sequences and subsequences

A sequence (of real numbers) is usually thought of as an infinite list numbers; for example

$$1, 1/2, 1/3, 1/4, \ldots, 1/n, \ldots.$$

[15.3] For this, we could take $\epsilon = 2 - u^2$ except that we would have to show that $u^2 \geq 1$ to guarantee that $\epsilon \leq 1$. This would be easy enough, but it is unnecessary with our choice of ϵ.

[15.4] Again, we could chose $\epsilon = u^2 - 4$, but then we would need to show that $u - \epsilon/2 \geq 0$. This would be easy enough to do, but we can avoid having to do this in our approach.

[15.5] Here we needed to know that $u - \epsilon/2 \geq 0$. Without this, we could not have concluded that $x \leq u - \epsilon/2$ from the inequality $x^{\leq}(u - \epsilon/2)^2$.

[15.6] The Latin plural infimum, similarly to maxima and minima being the plural of maximum and minimum.

A more formal way to think of a sequence as a function on the set $\mathbb{N} = \{1, 2, 3, 4 \ldots, m \ldots\}$.[16.1] With this in mind, the above sequence can be thought of as the function $f : \mathbb{N} \to \mathbb{R}$ such that $f(n) = 1/n$. A subsequence of the above sequence is usually thought of as taking only certain members of the sequence; one needs to take an infinite number of members, and one must keep the original order. Thus, a subsequence of the above sequence is

$$1/3, 1/7, 1/12, 1/18, \ldots.$$

With the function description of a sequence, this can formally be described as the composition $f \circ \pi$, where π is an *increasing* (to be defined) function $\pi : \mathbf{N} \to \mathbf{N}$.[16.2] In the given example, we need to have $\pi(1) = 3$, $\pi(2) = 7$, $\pi(3) = 12$, $\pi(4) = 18$, ... A function (on a set of reals or on a set of integers) is called increasing if for any x and y in the domain of f, if $x < y$ then we have $f(x) < f(y)$. The assumption that π in the subsequence $f \circ \pi$ of the sequence f is increasing ensures that the members of f are not rearranged in the subsequence $f \circ \pi$.

One often uses the notation $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$ for the sequence described by the function $f(n) = a_n$. This is like writing the argument of the function as a subscript, i.e., writing a_n instead of the function notation $a(n)$. When using the notation $\{a_n\}$ for the sequence, but still does not call the sequence itself a, whereas when one uses the description $f(n) = a_n$, the sequence itself would naturally be the function f, and not $f(n)$ or $\{f(n)\}$.

A subsequence the sequence $\{a_n\}$ is often described as $\{a_{n_k}\}$, where $\{n_k\}$ itself is a sequence. The function description is clearer, and in some contexts is much more useful, even though it is not used often. For example, if we write $f(n) = a_n$ and $\pi(k) = n_k$, we have $f \circ \pi(k) = a_{n_k}$.

16.2 Limits

In an informal discussion, one says that L is the limit of the sequence $\{a_n\}$, in symbols $L = \lim_{n\to\infty} a_n$, if for large n the number a_n is close to L. The problem with this description is that the words "large" and "close" do not have clear mathematical meanings. To correct this deficiency, we will call the number n N-large if $n \geq N$. Here N is usually thought of as a large (whatever that means) integer, but in fact no restriction needs to be put on N, so N need not be large, nor an integer (but it must be a real number) for N-large to make sense. Similarly, for $\epsilon > 0$, we will say that the number L is ϵ-close to the number x if $|L - x| < \epsilon$. Here one usually thinks of ϵ being small (whatever that means), so that ϵ-close really means close, but this again not a requirement for ϵ-close to make sense, and no restrictions other than $\epsilon > 0$ are put on ϵ.

This gives a clue as to how to make the definition of limit precise. What we want is to ensure that a_n is as close to L as we want by making sure that n is large enough. That is, L is called the limit of the sequence $\{a_n\}$ if for every $\epsilon > 0$ we can find an N such that if n is N-large then a_n is ϵ-close to L.

This is almost the final form, though it assumes that a sequence can only have one limit (this is true, but it is better left as a statement to be proved than to include it in the definition), and then we need to write the meaning of N-large and ϵ-close directly into the definition. That is,

Definition 16.1. We say that L is a limit of the sequence $\{a_n\}$ if for every $\epsilon > 0$ there is an N such that $|L - a_n| < \epsilon$ if $n > N$.

[16.1] $\mathbb{N}$ is called the set of natural numbers, hence the symbol used. Here we did not take 0 to be a natural number; in some contexts, it is better to also include 0. In particular, in discussions of the fundation of mathematics, 0 is also considered a natural number.

[16.2] The letter π is customarily used to denote the ratio of the circumference and the diameter of a circle; $\pi \approx 3.1415926\ldots$. Here, of course, π is used in a completely different sense, It is easy to show that a sequence can only have one limit. To see this, assume that L_1 and L_2 are both limits of $\{a_n\}$, where $L_1 \neq L_2$ and let $\epsilon = |L_2 - L_1|/2$. Clearly, $\epsilon > 0$, so there must be N_1 and N_2 such that $|L_1 - a_n| < \epsilon$ for $n \geq N_1$ and $|L_2 - a_n| < \epsilon$ for $n \geq N_2$. Then pick an n such that both $n \geq N_1$ and $n \geq N_2$ hold. Then

$$2\epsilon = |L_2 - L_1| = |(L_2 - a_n) + (a_n - L_1)| \leq |L_2 - a_n| + |a_n - L_1| < \epsilon + \epsilon = 2\epsilon;$$

here, the first inequality holds since $|a+b| \leq |a| + |b|$ for any two reals a and b.[16.3] We obtained that $2\epsilon < 2\epsilon$; this is a contradiction, showing the uniqueness of limits. Having shown the uniqueness of limit, we can use the notation $L = \lim_{n\to\infty} a_n$ to indicate that L is the limit of the sequence $\{a_n\}$.

A sequence that has a limit is called *convergent*; one that does not have a limit is called *divergent*. A sequence $\{a_n\}_{n=1}^{\infty}$ is called *nondecreasing* if $a_n \leq a_m$ whenever $1 < n < m$.[16.4] The sequence $\{a_n\}_{n=1}^{\infty}$ is called bounded from above if the associated set $\{a_n : n \geq 1\}$ is bounded from above. An upper bound of this set is also called an upper bound of the sequence, and the supremum of this set is called the supremum of the sequence. The supremum of the sequence $\{a_n\}_{n=1}^{\infty}$ is usually denoted as $\sup a_n$ or $\sup_{n\geq 1} a_n$. We have the following

Lemma 16.1. *Let $\{a_n\}_{n=1}^{\infty}$ be a nondecreasing sequence that is bounded from above. Then $\{a_n\}_{n=1}^{\infty}$ has a limit. More precisely, its supremum is also its limit.*

Proof. By the Axiom of Completeness given in Subsection 15.1. the secquence $\{a_n\}_{n=1}^{\infty}$ has a supremum; let this be L. Then, given an arbitrary $\epsilon > 0$, $L - \epsilon$ is not an upper bound of the set $\{a_n : n \geq 1\}$. Hence, there is an $N \in \mathbb{N}$ such that $a_N > L - \epsilon$. Since the sequence $\{a_n\}_{n=1}^{\infty}$ is nondecreasing, we have $a_n \geq a_N$ for all $n \geq N$. Hence we have $L - \epsilon < a_n \leq L$ for all $n \geq N$; the second inequality holds since L is an upper bound of the set $\{a_n : n \geq 1\}$. Thus $L - \epsilon < a_n < L + \epsilon$, i.e, $|L - a_n| < \epsilon$, for all $n \geq N$. Since $\epsilon > 0$ was arbitrary, this shows that $\lim_{n\to\infty} a_n = L$. □

A similar result can be formulated for nonincreasing sequences. A sequence that has a limit is called *convergent*; one that does not have a limit is called *divergent*. A sequence $\{a_n\}_{n=1}^{\infty}$ is called *nonincreasing* if $a_n \geq a_m$ whenever $1 < n < m$.[16.5] The sequence $\{a_n\}_{n=1}^{\infty}$ is called bounded from below if the associated set $\{a_n : n \geq 1\}$ is bounded from below. A lower bound of this set is also lower bound of the sequence, and the infimum of this set is called the infimun of the sequence. The infimum of the sequence $\{a_n\}_{n=1}^{\infty}$ is usually denoted as $\inf a_n$ or $\inf_{n\geq 1} a_n$. We have the following

Lemma 16.2. *Let $\{a_n\}_{n=1}^{\infty}$ be a nonincreasing sequence that is bounded from below. Then $\{a_n\}_{n=1}^{\infty}$ has a limit. More precisely, its infimum is also its limit.*

For the proof, note that under the assumptions, the sequence $\{-a_n\}_{n=1}^{\infty}$ is nondecreasing and bounded from above, and so it has a limit. The result then follows from the limit rules discussed in Theorem 16.2 below.

16.2.1 A subsequence of a convergent sequence is convergent

If we have a convergent sequence, then any of its subsequences converges to the same limit. We would like to formulate this rigorously and give a rigorous proof. We recall that, in this rigorous interpretation, a sequence is a function $f : \mathbb{N} \to \mathbb{R}$. Having a proper notation for the sequence,

[16.3] Indeed, equality holds unless a and b have different signs.

[16.4] This sequence is called *increasing* if $a_n < a_m$ whenever $1 < n < m$. One used to say increasing in the wider sense instead of nondecreasing, and strictly increasing instead of increasing.

[16.5] This sequence is called *deccreasing* if $a_n > a_m$ whenever $1 < n < m$. One used to say decreasing in the wider sense instead of nonincreasing, and strictly decreasing instead of deccreasing.

namely f, its limit $\lim_{n\to\infty} f(n)$ can be more conveniently written as $\lim f$; after all, n plays no explicit role here.

Theorem 16.1 (Convergence of a subsequence)**.** *Let $f : \mathbb{N} \to \mathbb{R}$ be a sequence, and let $\pi : \mathbf{N} \to \mathbf{N}$ be an increasing sequence. Assume that $\lim f = c$ for some $c \in \mathbb{R}$. Then $\lim f \circ \pi = c$*

Proof. According to Definition 16.1, $\lim f = c$ means that, given an arbitrary $\epsilon > 0$, there is an integer $N > 0$ such that for every $n > N$, we have $|L - f(n)| < \epsilon$. For such an n, the inequality $\pi(n) \geq n > N$ holds, so we certainly have $\bigl|L - f\bigl(\pi(n)\bigr)\bigr| < \epsilon$. Hence, again according to Definition 16.1, $\lim f \circ \pi = c$ holds. □

16.3 Limit rules

After this point, most of the discussion will return to an intuitive level, and rigorous arguments of the type given above will rarely be used. Yet it is important for the appreciation of the material to follow that the arguments used in the the justification of the statements can also be done on a rigorous level.

The well-known rules about limits of functions are also true for limits of sequences. We have

Theorem 16.2. *Let $\{a_n\}$ and $\{b_n\}$ be sequences, and let c be a number, and assume that the limits $A = \lim_{n\to\infty} a_n$ and $B = \lim_{n\to\infty} b_n$ exist. Then the following limits exist and satisfy the equations given:*

$$\lim_{n\to\infty}(a_n + b_n) = A + B, \qquad \lim_{n\to\infty}(a_n - b_n) = A - B, \qquad \lim_{n\to\infty} ca_n = cA,$$
$$\lim_{n\to\infty} a_n b_n = AB, \quad \text{and} \quad \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B},$$

the last one under the assumption that $B \neq 0$.

We omit the proof.

16.4 Reading

[9, §11.1, pp. 724–735].

16.5 Homework

[9, §11.1, p. 735], 5, 11, 17, 21, 25, 27, 41, 49, 57.

17 Series

A formal sum

$$a_1 + a_2 + a_3 + a_4 + \dots,$$

where the addition goes on *ad infinitum*,[17.1] or, with a more compact notation,

$$\sum_{n=1}^{\infty} a_n$$

[17.1]Latin for "to infinite."

is called a series; the sum is formal in the sense that it is impossible to add up infinitely many terms. The sum of this series is defined as the limit

$$\lim_{m\to\infty}\sum_{n=1}^{m} a_n.$$

That is, the sum of the series is defined as the limit of the sequence

$$\Big\{\sum_{n=1}^{m} a_n\Big\}_{m=1}^{\infty}.$$

The terms of this sequence are called *partial sums* of the series. More specifically, the mth term of, this sequence is called the mth partial sum. Thus, one can say that the sum of a series is the limit of its partial sums. If this limit exists, the series is called *convergent*, and if it does not exist, the sum is called *divergent*. If the limit is $+\infty$ or $-\infty$ then one says that the sum is $+\infty$ or $-\infty$, but in these cases one still says that the series is divergent.[17.2]

By a common abuse of language, one also refers to the sum of a series as the series itself (as in saying that this series equals 2, instead of saying that its sum is 2).

17.1 The geometric series

The series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \tag{17.1}$$

is called the geometric series.[17.3] The partial sums of this series can easily be written in a *closed form*, i.e. as a short expression without the summation notation. Indeed, we have

$$\begin{aligned}(1-x)\sum_{n=0}^{m} x^n &= \sum_{n=0}^{m}(1-x)x^n = \sum_{n=0}^{m}(x^n - x^{n+1}) = (1-x) + (x-x^2)\\ &+ (x^2-x^3) + (x^3-x^4) + \dots + (x^{m-3}-x^{m-2})\\ &+ (x^{m-2}-x^{m-1}) + (x^{m-1}-x^m) + (x^m - x^{m+1}) = 1 - x^{m+1}, \qquad (m \ge 0)\end{aligned} \tag{17.2}$$

where the last equation is obtained by the obvious cancelations before the equation sign.[17.4] Dividing both sides by $1-x$, we obtain that

$$\sum_{n=0}^{m} x^n = \frac{1-x^{m+1}}{1-x} \qquad (x \ne 1).$$

[17.2] Recall that when the limit of a function or a sequence is $+\infty$, $-\infty$, or $\pm\infty$, one still says that the limit does not exist. As for series, one rarely says that the sum of a series is $\pm\infty$.

[17.3] In the discussion above, we started the summation above with $n = 1$; of course, the summation can start with any other integer. Also note that 0^0 is not defined; however, in situations as in the series above, we take $x^0 = 1$ even in case $x = 0$, in order to avoid complications in the notation.

[17.4] Recall that

$$\sum_{n=k}^{m} a_n$$

for $k > m$ is taken to be 0, and is called an empty sum. Thus the sum written in the $\sum$ notation is clearly 0 if $m < 0$. When we wrote out the sum using "..." instead of the summation notation, there are extra terms for $m < 8$ that really do not belong. This shows the advantabe of the $\sum$ notation.

This is the sum formula for the *geometric progression*. Taking limits when $m \to \infty$, we obtain

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad (|x| < 1). \tag{17.3}$$

The limit clearly does not exists when $|x| \geq 1$.[17.5]

The kind of sum in formula (17.2) is called a telescoping sum or a collapsing sum: the cancelations when adding successive term make the sum collapse, like a telescope made of successively smaller tubes near the eye piece. More on telescoping sums can be found in the notes [8].

17.2 More telescoping sums

Telecoping sums abound. Examples are

$$\sum_{n=1}^{m} \frac{1}{n(n+1)} = \sum_{n=1}^{m} \left(\frac{1}{n} - \frac{1}{n+1} \right),$$

$$\sum_{n=1}^{m} \frac{1}{n(n+1)(n+2)} = \sum_{n=1}^{m} \frac{1}{2} \left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)} \right),$$

$$\sum_{n=1}^{m} \frac{1}{n(n+1)(n+2)(n+3)} = \sum_{n=1}^{m} \frac{1}{3} \left(\frac{1}{n(n+1)(n+2)} - \frac{1}{(n+1)(n+2)(n+3)} \right).$$

It is best if the reader tries to determine these sums on her/his own, that these sums are discussed in [8], where there sums are determined. Making $m \to \infty$ in these sums, the limits are easily determined, showing that the sums of the corresponding infinite series are 1, 1/2, and 1/3, *respectively*.[17.6]

Similar telescoping sums are

$$\sum_{n=1}^{m} n = \sum_{n=1}^{m} \frac{1}{2} \Big(n(n+1) - (n-1)n \Big),$$

$$\sum_{n=1}^{m} n(n+1) = \sum_{n=1}^{m} \frac{1}{3} \Big(n(n+1)(n+2) - (n-1)n(n+1) \Big),$$

$$\sum_{n=1}^{m} n(n+1)(n+2) = \sum_{n=1}^{m} \frac{1}{4} \Big(n(n+1)(n+2)(n+3) - (n-1)n(n+1)(n+2) \Big).$$

Again, these sums are easily evaluated, and their evaluation is best left to the reader. They are also discussed in [8]. These sums do, of course, not lead to infinite series, but they are very useful. On can, for example, to use the equation

$$\sum_{n=1}^{m} n^2 = \sum_{n=1}^{m} n(n+1) - \sum_{n=1}^{m} n$$

to evaluate the sum on the left-hand side.

[17.5] Observe that for $x = 1$, while the sum formula in the previous displayed formula does not apply, the series on the left-hand side is obviously divergent.

[17.6] That is, the first sum is 1, the second one is 1/2, and the third one is 1/3. The use of the word "respectively" is common in mathematics. It is also used in common parlance, but it is hard to find out its exact meaning is difficult to figure out from the dictionaries: it means that in two lists, the items in the corresponding places concern each other. The phrase "in turn" is also used in a similar sense. For example, we could have said that "the sums of the corresponing infinite series are, *in turn*, 1, 1/2, and 1/3." These terms are used in a very precise sense in mathematics, but it is difficult to know for a mathematician whether colloquial uses also reflects this precision.

17.3 Limit rules for series

We have

Theorem 17.1. *Assume that the series* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are convergent. Then the equations*

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n, \qquad \sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n,$$

$$\text{and} \qquad \sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$$

hold.

This is a direct consequence of the corresponding limit rules for sequences in Theorem 16.2, since the analogous equations clearly hold for partial sums of these series.

We can make a couple more observations. If $\sum_{n=1}^{\infty} a_n$ is a series, then, for any $k \ge 1$, the series $\sum_{n=k}^{\infty} a_n$ is called one of its tails. It can be easily seen that a series is convergent if and only if one of its tails is convergent. Indeed, for $1 \le k \le m$ we have

$$\sum_{n=1}^{m} a_n = \sum_{n=1}^{k-1} a_n + \sum_{n=k}^{m} a_n;$$

note that this equation is true even if $k = 1$, since in that case the first sum, being an empty sum, is 0. Making $m \to \infty$, we obtain the result about a series and one of its tails being convergent at the same time. Another observation is that if $\sum_{n=1}^{\infty} a_n$ is convergent then $a_n \to 0$. Indeed, we have

$$a_n = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} a_k \qquad (n \ge 1).$$

Writing $L = \sum_{n=1}^{\infty} a_n$, both sums on the right-hand side are partial sums of this series, so making $n \to \infty$ we obtain

$$\lim_{n\to\infty} a_n = L - L = 0.$$

That is, this equation is necessary for the convergence of the series in question. It is certainly not sufficient, as we will see on many examples. These two observations, about the tail and about the limit of a_n will be often used without explicitly calling attention to them.

17.4 The Basel problem

In 1741. the Swiss mathematician The problem of finding the sum of the series on the left-hand side of the equation

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \tag{17.4}$$

is called the Basel problem, posed by Pietro Mengoli in 1644, and solved by the Swiss mathematician Leonhard Euler in 1734, though he was not able to justify his arguments rigorously until 1741. It was certainly known in 1644, even before the inventors of calculus, Newton and Leibniz, were born that the series on the left was convergent. This illustrates the point that finding out the sum of a series is a much more difficult problem than deciding that it is convergent. When one knows that a seeries is convergent, then one can approximate the sum by evaluating a long enough partial sum,[17.7] and if a series is not convergent, one usually should not evem try to evaluate its partial sums.[17.8] We will discuss convergence tests in the next few sections.

17.5 Examples

Problem 17.1. Find out if the series

$$\sum_{n=4}^{\infty}(-1)^n\frac{2^{3n-10}}{3^{2n-7}}$$

is convergent. If it is converent, find its sum.

Solution. It is easy to see that the series will lead to a geometric series as in (17.1) with $x = -2^{3n}/3^{2n} = -8/9$. Since this value of x is less than 1 in absolute value, it follows that this series is convergnent.[17.9] As for the sum of the series, introducing the variable $k = n-4$, k goes from 0 to ∞, and $n = k+4$. Thus, the series becomes

$$\begin{aligned}\sum_{k=0}^{\infty}(-1)^{k+4}\frac{2^{3(k+4)-10}}{3^{2(k+4)-7}} &= \sum_{k=0}^{\infty}(-1)^k\frac{2^{3k+2}}{3^{2k+1}} = \sum_{k=0}^{\infty}(-1)^k\frac{2^2}{3^1}\frac{2^{3k}}{3^{2k}} = \frac{2^2}{3^1}\sum_{k=0}^{\infty}(-1)^k\frac{(2^3)^k}{(3^2)^k}\\ &= \frac{4}{3}\sum_{k=0}^{\infty}(-1)^k\frac{8^k}{9^k} = \frac{4}{3}\sum_{k=0}^{\infty}\left(-\frac{8}{9}\right)^k = \frac{4}{3}\frac{1}{1-(-8/9)} = \frac{4}{3+8/3} = \frac{4\cdot 3}{9+8} = \frac{12}{17};\end{aligned}$$

to obtain the sixth equality, we used the sum formula (17.3) for the geometric series.

Problem 17.2. Find the sum of the series

$$\sum_{n=1}^{\infty}\frac{1}{n(n+5)}.$$

Solution. This is reminiscent of the first telescoping sum given in Subsection 17.2, but we show a technique that is somewhat different from the technique described for telescoping sums in [8]. We have

$$5\cdot\sum_{n=1}^{\infty}\frac{1}{n(n+5)} = \sum_{n=1}^{\infty}\frac{5}{n(n+5)} = \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+5}\right)$$

Taking a partial sum of the sum on the right-hand side, we have

$$\sum_{n=1}^{N}\left(\frac{1}{n}-\frac{1}{n+5}\right) = \sum_{n=1}^{N}\frac{1}{n} - \sum_{n=1}^{N}\frac{1}{n+5} = \sum_{n=1}^{N}\frac{1}{n} - \sum_{k=6}^{N+5}\frac{1}{k},$$

[17.7] Even though it may be a difficult problem as to how long a partial sum to take when one knows that a series is convergent. That is, in addition to knowing that a series is convergent, often it is important to know how fast it converges. This question will also be discussed to some extent below.

[17.8] This is not quite true because there are series, called asymptotic expansions, that are helpful for evaluating certain quantities of interest. In this situation, taking a certain partial sum will give a good approximation, but this approximation will become worse if one takes too long a partial sum.

[17.9] If you do not understand this comment, just ignore it and read the rest of the solution. Then come back to it. On second reading, you will understand it. The point is, it is helpful to decide whether or not the series is convergent before doing the main part of the work, since it is divergent, that work does not need to be done.

where to obtain the last equality we substituted $k = n + 5$, when k runs from 6 to $N + 5$. Replacing k with n in the second sum on the right-hand side, we find that the right-hand side equals

$$\sum_{n=1}^{N} \frac{1}{n} - \sum_{n=6}^{N+5} \frac{1}{n} = \sum_{n=1}^{5} \frac{1}{n} - \sum_{n=N+1}^{N+5} \frac{1}{n} \qquad (N \geq 5);$$

the equation holds since the terms from $n = 6$ to $n = N$ cancel.[17.10] Making $N \to \infty$, the second sum tends to 0. Thus we have

$$5 \cdot \sum_{n=1}^{\infty} \frac{1}{n(n+5)} = \sum_{n=1}^{5} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{137}{60}.$$

The sum in the question is 1/5th of this; hence the sum equals 137/300.

17.6 Reading

[9, §11.2, pp. 738–747].

17.7 Homework

[9, §11.2, p. 747], 5, 7, 9, 13, 17, 19, 23, 27, 31, 37, 39, 47. 49, 61.

18 The comparison test and the integral test

18.1 The comparison test

We have the following

Theorem 18.1 (Comparison test)**.** *Let a_n and b_n be real numbers for n with $1 \leq n < \infty$ such that $0 \leq a_n \leq b_n$ for all $n \geq 1$. Assume that the series $\sum_{n=1}^{\infty} b_n$ is convergent. Then the series $\sum_{n=1}^{\infty} a_n$ is also convergent.*

Proof. Write

$$s_n = \sum_{k=1}^{n} a_k \qquad \text{and} \qquad S_n = \sum_{k=1}^{n} b_k$$

for the partial sums of these series. Then both $\{s_n\}$ and $\{S_n\}$ are nondescreasing sequences. Writing

$$S = \sum_{n=1}^{\infty} b_n,$$

we have $S = \lim_{n\to\infty} S_n = \sup_{n\geq 1} S_n$; the first equation holds in view of the definition of the sum of a series as the limit of its partial sums, and the second equation holds by the last sentence in Lemma 16.1. Since we have $s_n \leq S_n \leq S$, the nondecreasing sequence $\{s_n\}$ is bounded from above, hence it is convergent by Lemma 16.1. □

[17.10] For $N \leq 5$ there is no cancelation, since the ranges of the two sums do not overlap. This does not cause any trouble for $N = 5$, but it does for $N < 5$, invalidating the equation. This is unimportant, since we will make $N \to \infty$, except we want to make sure that the statemens we make are correct even in unimportant cases. In order to understand the nature of the trouble, consider the case $N = 4$. In this case, the second sum on the left-hand side starts with $n = 6$; on the right-hand side, the second sum starts with $n = N + 1 = 5$, and this term does not belong. For $N \geq 5$ this will not happen.

Since we have

$$0 \le \frac{1}{n^2} \le \frac{2}{n(n+1)},$$

and the series

$$\sum_{n=1}^{\infty} \frac{2}{n(n+1)},$$

is convergent (see Subsection 17.2), this shows that the sum on the left-hand side of (17.4) is convergent.

18.2 The integral test

Let f be a function on the interval $[1, \infty)$ that is integrable on every interval $[1, A]$ for all $A > 1$. Assume that $f(x) \ge 0$ for all $x \ge 1$. Then the improper integral $\int_1^\infty f(x)\,dx$ is convergent if and only if the sequence $\left\{\int_1^n f(x)\,dx\right\}_{n=1}^{\infty}$ is convergent. If they are convergent, their limits are the same.

In simple terms this means the following: Given L, the integral $\int_1^A f(x)\,dx$ is close to L for large real numbers A if and only if the integral $\int_1^n f(x)\,dx$ is close to L for all large integers n. This is clearly true, since if $n \le A \le n+1$ then

$$\int_1^n f(x)\,dx \le \int_1^A f(x)\,dx \le \int_1^{n+1} f(x)\,dx,$$

given that $f(x) \ge 0$ for $x \ge 1$.

A function f is called *nonincreasing* on an interval I if $f(x) \ge f(y)$ whenever $x \le y$ for $x, y \in I$.[18.1]

Theorem 18.2 (Integral test). *Let f be a nonincreasing function on $[1, \infty)$. Then the integral $\int_1^\infty f(x)\,dx$ is convergent if and only if the series $\sum_{n=1}^\infty f(n)$ is convergent.*

Proof. First, note that

$$\sum_{n=1}^{\infty} f(n+1) = \sum_{k=2}^{\infty} f(k) = \sum_{n=2}^{\infty} f(n);$$

the first equation follows by taking $k = n+1$ for the summation variable, and the second one by writing n instead of k. Given that f is nonincreasing, for $n \ge 1$ and for x with $n \le x \le n+1$ we have $f(n) \ge f(x) \ge f(n+1)$. Hence

$$f(n) = \int_n^{n+1} f(n)\,dx \ge \int_n^{n+1} f(x)\,dx \ge \int_n^{n+1} f(n+1)\,dx = f(n+1) \ge 0. \tag{18.1}$$

Thus by Theorem 18.1, given the three series $\sum_{n=1}^\infty f(n)$, $\sum_{n=1}^\infty \int_n^{n+1} f(x)\,dx$, and $\sum_{n=1}^\infty f(n+1) = \sum_{n=2}^\infty f(n)$, if the first series converges, then so does the second, and if the second series converges, then so does the third. Since the third series is a tail of the first one, they converge at the same time. Thus, all three series are convergent, or all three series are divergent. Recalling that the second series converges exactly if the improper integral $\int_1^\infty f(x)\,dx$ converges, the proof is complete. □

We established earlier that the integral

$$\int_1^\infty \frac{1}{x^\alpha}\,dx$$

[18.1] Instead of "nonincreasing" one might say "dercreasing in the wider sense."

is convergent if and only if $\alpha > 1$. Hence it follows that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$$

is convergent if and only if $\alpha > 1$. Similarly, we have shown that the

$$\int_2^\infty \frac{1}{x(\ln x)^\alpha}\, dx$$

is convergent if and only if $\alpha > 1$. Hence the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^\alpha}$$

is converges if and only if $\alpha > 1$.

18.3 Estimating the partial sums and the tails of a series

Inequality (18.1) can also be written as

$$\int_n^{n+1} f(x)\, dx \le f(n) \le \int_{n-1}^{n} f(x)\, dx; \tag{18.2}$$

the first inequality here is true for $n \ge 1$, and is equivalent to the first inequality in 18.1, and the second one is true for $n \ge 2$, and is equivalent to the second inequality in 18.1, with $n-1$ replacing n. Hence, for $N \ge 1$, summing for $n = N+1$ to ∞, we have

$$\begin{aligned} \int_{N+1}^\infty f(x)\, dx &= \sum_{n=N+1}^{\infty} \int_n^{n+1} f(x)\, dx \le \sum_{n=N+1}^{\infty} f(n) \\ &\le \sum_{n=N+1}^{\infty} \int_{n-1}^{n} f(x)\, dx = \int_N^\infty f(x)\, dx. \end{aligned} \tag{18.3}$$

we obtain an estimate for the tail of the sum; the tail describes the error we have if instead of summing from 1 to ∞ we sum only from 1 to N.

To estimate the partial sums from below, we will sum from 1 to $N-1$ the first inequality in (18.2). We obtain

$$\int_1^{n+1} f(x)\, dx = \sum_{n=1}^{N} \int_n^{n+1} f(x)\, dx \le \sum_{n=1}^{N} f(n)$$

To estimate the partial sums from above, we will sum from 2 to N the second inequality in (18.2):

$$\sum_{n=2}^{N} f(n) \le \sum_{n=2}^{\infty} \int_{n-1}^{n} f(x)\, dx = \int_1^N f(x)\, dx.$$

Putting these two inequalities together, we obtain

$$\int_1^{n+1} f(x)\, dx \le \sum_{n=1}^{N} f(n) = f(1) + \sum_{n=2}^{N} f(n) = f(1) + \int_1^N f(x)\, dx. \tag{18.4}$$

We will give some examples. Inequality (18.3) can only be used for convergent series. Taking $f(x) = 1/x^2$, for $a, b > 0$ we have

$$\int_a^b \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_{x=a}^{x=b} = -\frac{1}{b} + \frac{1}{a}.$$

Making $b \to \infty$, we obtain

$$\int_a^\infty \frac{1}{x^2}\,dx = \frac{1}{a}.$$

Hence, inequality (18.3) implies

$$\frac{1}{N+1} \le \sum_{N+1}^{\infty} \frac{1}{n^2} \le \frac{1}{N}.$$

Taking $f(x) = 1/x$, for $a > 0$ we have

$$\int_1^a \frac{1}{x}\,dx = \ln x\,\Big|_{x=1}^{x=a} = \ln a.$$

Hence, according to (18.4) we have

$$\ln(N+1) \le \sum_{n=1}^{N} \frac{1}{n} \le 1 + \ln N. \tag{18.5}$$

18.4 Euler's γ

Accoring to (18.5), we have

$$\begin{aligned} 0 \le \ln\left(1 + \frac{1}{N}\right) &= \ln \frac{N+1}{N} = \ln(N+1) - \ln N \\ &\le c_N \overset{def}{=} \sum_{n=1}^{N} \frac{1}{n} - \ln N = 1 + \sum_{n=2}^{N} \int_{n-1}^{n} \left(\frac{1}{n} - \frac{1}{x}\right) dx; \end{aligned}$$

the second equation here also shows that the sequence $\{c_N\}_{N=1}^{\infty}$ is decreasing, since the integrand is always negative or 0. Therefore, the limit

$$\gamma \overset{def}{=} \lim_{N\to\infty} \Big(\sum_{n=1}^{N} \frac{1}{n} - \ln N \Big) = \lim_{N\to\infty} c_N \tag{18.6}$$

exists according to Lemma 16.2. The number γ is called Euler's constant or Euler's γ. Its numerical value is $\gamma \approx 0.577, 215, 664, 9$.

18.5 Reading

[9, §11.3, pp. 751–758] for the Integral Test, and [9, §11.4, pp. 760–764] for the Comparison and Limit Comparison Tests. We will discuss the the Limit Comparison Test in Section 19.

18.6 Homework

[9, §11.3, p. 758], 3, 9, 13, 25, 31, 39 for the integral Test

19 The limit comparison test

The limit comparison test is a simple consequence of the comparison test given in Theorem 18.1, but it is often much easier to apply.

Theorem 19.1 (Limit comparison test)**.** *Let* $a_n \ge 0$ *and* $b_n > 0$ *be real numbers for* n *with* $1 \le n < \infty$ *such that the limit* $\lim_{n\to\infty} a_n/b_n$ *exist. Assume that the series* $\sum_{n=1}^{\infty} b_n$ *is convergent. Then the series* $\sum_{n=1}^{\infty} a_n$ *is also convergent.*

It is important to remember that a limit that is infinite does not exist.

Proof. Write $L = \lim_{n\to\infty} a_n/b_n$. Then there is a positive integer N such that $a_n/b_n < L+1$ for $n > N$. [19.1] Since the series

$$\sum_{n=N+1}^{\infty} (L+1)b_n = (L+1)\sum_{n=N+1}^{\infty} b_n$$

is convegent and $a_n < (L+1)b_n$ for $n > N$, the comparison test (Theorem 18.1) implies that

$$\sum_{n=N+1}^{\infty} a_n$$

is convergent. Hence,

$$\sum_{n=1}^{\infty} a_n$$

is also convergent. □

Note that it is not required that

$$\lim_{n\to\infty} a_n/b_n \ne 0. \tag{19.1}$$

This requirement is absolutely superfluous, and it seriously weakens the applicability of the test.[19.2] Yet the book [9, §11.4 pm p. 729] does in fact make this requirement. Assuming (19.1), the limit

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \frac{1}{a_n/b_n} = \frac{1}{\lim_{n\to\infty} a_n/b_n}$$

also exists and is not equal 0, according to the quotient rule for limits. Hence, if one of the two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ is convergent, then so is the other, according to Theorem 19.1. Therefore, requiring (19.1) makes the test symmetric in that it makes a_n and b_n interchangeable in the test. However, making the test symmetric is totally pointless. After all, when one applies this test then one usually knows which one of two the series is convergent, and then uses this fact to show that the other one is also convergent.

[19.1] This is intuitive clear, but it can be shown rigorously by Definition 16.1. In that definition we may assume that N is an integer. According to this definition, for all $\epsilon > 0$ there is (an integer) N such that inequality $|L - a_n/b_n| < \epsilon$ holds for all $n > N$. This inequality is equlivalent to the inequality

$$-\epsilon < L - a_n/b_n < \epsilon.$$

The inequality on the left can be rearranged to say that $a_n/b_n < L + \epsilon$. With $\epsilon = 1$, this means that $a_n/b_n < L + 1$ for some $n > N$.

[19.2] Even our requirement that the limit on the left-hand side exist is more than what is necessary. It is enough to require that the set $\{a_n/b_n : n \ge 1\}$ is bounded from above, but we wanted to avoid the discussion of such subtleties.

19.1 An example

An example showing that it is counterproductive to make the test symmetric is the following.

Problem 19.1. Show that the series

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$$

is convergent.

Solution. For example, let $b_n = n^{-3/2}$ and and $a_n = \ln n/n^2$. Then we know by the Integral Test that the series

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} n^{-3/2}$$

is convergent. So, by our version of the comparison test given in Theorem 19.1, we can conclude that the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\ln n}{n^2}$$

is also convergent. Indeed, we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n/n^2}{n^{-3/2}} = \lim_{n\to\infty} \frac{\ln n}{n^2 n^{-3/2}} = \lim_{n\to\infty} \frac{\ln n}{n^{2-3/2}} = \lim_{n\to\infty} \frac{\ln n}{n^{1/2}} = 0;$$

the last equation can be obtained by l'Hospital's rule, for example. Hence, it follows by Theorem 19.1 that the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\ln n}{n^2}$$

converges.

This example shows that a symmetric comparison test would be unhelpful in this case. Namely, the first series is clealy convergent by the Integral Test, so there is no need to use a comparison test to find this out. Furthermore, the Integral Test is difficult to use for the second series, since convergence of the integral

$$\int_1^{\infty} \frac{\ln x}{x^2}\,dx$$

is not immediately clear.[19.3]

The first question about this example is, how does one know to use the series

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} n^{-3/2}$$

[19.3] Except by using a limit comparison test for improper integrals, comparing it to the integral

$$\int_1^{\infty} x^{-3/2}\,dx$$

but then one goes from a simple quesion to a complicated question, and use the latter to solve a simple question. A pointless exercise. But the book [9] seems not to realize this.

in the Limit Comparison Test? The answer is that one could have used any of the series

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} n^{-\alpha}$$

for α with $1 < \alpha < 2$. Indeed, for the choice $\alpha > 1$, this series is convergent, and we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n/n^2}{n^{-\alpha}} = \lim_{n\to\infty} \frac{\ln n}{n^2 n^{-\alpha}} = \lim_{n\to\infty} \frac{\ln n}{n^{2-\alpha}} = 0,$$

where the last equation holds for $\alpha < 2$, as one can see, for example, by the l'Hospital rule.

19.2 Homework

The problems for the Comparison Test and the Limit Comparison Tests are not separated in the list [9, §11.4, p. 764], 7, 9, 11, l5, 17, 21, 27, 33, 43, 48, 49. Here, Problem 48 describes missing case of the Limit Comparison Test that was included in our version, and Problem 49 describes its *contrapositive.*[19.4]

20 The alternating series test

A typical series whose convergence is established by this test is

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \pm \dots.$$

As one can see, the signs of the successive terms alternate. The following theorem concerns this type of series.

Theorem 20.1 (Alternating Series Theorem)**.** *Let* $\{b_n\}_{n=1}^{\infty}$ *be a nonincreasing sequence with non-negative terms. That is, we have*

$$b_n \ge b_{n+1} \ge 0 \qquad \text{for all} \quad n \ge 1 \tag{20.1}$$

Then the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 \pm \dots.$$

is convergent if and only if

$$\lim_{n\to\infty} b_n = 0. \tag{20.2}$$

Proof. Clearly, condition (20.2) is clearly necessary. Indeed, for any series to be convergent, it is necessary that its nth term tends to zero as $n \to \infty$. We will show that it is also sufficient, that is,

[19.4] Given a statement of form if A then B, then its contrapositive is the statement if not B then not A. A statement and its contrapositive are equivalent, i.e., they say the same thinkg in different ways. In the present contex, a convergence test would say that if series s is convergent, then series σ is also convergent; its contrapositive then says that if series σ is divergent then series s is also divergent.

it ensures he convergence of the series in question. To this end, write s_n for the nth partial sum of the series; that is,

$$s_n = \sum_{k=1}^{n} (-1)^{k-1} b_k.$$

We then have

$$s_{2n+2} = s_{2n} + b_{2n+1} - b_{2n} = s_{2n} + (b_{2n+1} - b_{2n+2}) \geq s_{2n} \qquad (n \geq 1);$$

the inequality on the right holds because $b_{2n+1} \geq b_{2n}$. Similarly,

$$s_{2n+1} = s_{2n-1} - b_{2n} + b_{2n+1} = s_{2n-1} - (b_{2n} - b_{2n+1}) \leq s_{2n-1} \qquad (n \geq 1);$$

the inequality on the right holds because $b_{2n} \geq b_{2n+1}$. Further, note that

$$s_{2n+1} = s_{2n} + b_{2n+1} \geq s_{2n};$$

the inequality on the right holds because $b_{2n+1} \geq 0$.

Therefore, we have

$$s_2 \leq s_4 \leq s_6 \leq \ldots \leq s_{2n} \leq s_{2n+1} \leq s_{2n-1} \leq \ldots \leq s_5 \leq s_3 \leq s_1. \tag{20.3}$$

Thus the sequence $\{s_{2n}\}_{n=1}^{\infty}$ is nondecreasing and it is bounded from above by s_1, or, in fact, by s_{2k} for any k;[20.1] so it is convergent in view of Lemma 16.1. Now, according to (20.2) we have

$$0 = \lim_{n\to\infty} b_{2n+1} = \lim_{n\to\infty} (s_{2n+1} - s_{2n})$$

Thus

$$\begin{aligned} \lim_{n\to\infty} s_{2n+1} &= \lim_{n\to\infty} \big((s_{2n+1} - s_{2n}) + s_{2n}\big) \\ &= \lim_{n\to\infty} (s_{2n+1} - s_{2n}) + \lim_{n\to\infty} s_{2n} = 0 + \lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} s_{2n} \end{aligned}$$

Thus, the even and odd partial sums have the same limit. This same limit is the limit of the sequence. □

20.1 Error of the alternating series

Writing s for the limit of Theorem 20.1, by (20.3) we have $s_{2n} \leq s \leq s_{2m+1}$ for any $m, n \geq 1$. Thus, we have

$$s_{2n} \leq s \leq s_{2n+1} = s_{2n} + b_{2n+1}$$

and

$$s_{2n-1} \geq s \geq s_{2n} = s_{2n-1} - b_{2n}$$

Thus the sum of an alternating series is always between its two adjacent partial sums. Thus for the absolute value error of a partial sum of an alternating series we have $|s_n - s| \leq b_{n+1}$. In other words, when approximating the sum of an alternating series with a partial sum is less than, in absolute value, is less than the first term we fail to add to, or subtract from, the partial sum.

[20.1] Indeed, the inequalities just displayed mean that any even partial sum is less than or equal any odd partial sum.

20.2 An example

Problem 20.1. Is the sum

$$\sum_{n=1}^{\infty}(-1)^n \frac{n^3-3n^2+7n-1}{2n^4-211}$$

convergent?

Solution. Writing

$$b_n = \frac{n^3-3n^2+7n-1}{2n^4-211} = \frac{n^3}{n^4}\,\frac{1-3n^{-1}+7n^{-2}-n^{-3}}{2-211n^{-4}} \tag{20.4}$$

it is easy to see that equation (20.2) is satisfied, yet it is not immediately clear Theorem 20.1 is applicable.

There are several issues with applying Theorem 20.1 directly. First, $b_1 < 0$, but $b_n > 0$ for large n; the latter is clear from formula (20.4), since both the numerator and the denominator are positive for large n. This also means that even the first inequality in 20.1 cannot be satisfied for all $n \ge 1$. Yet, from general considerations, it follows that Theorem 20.1 implies convergence after one notices that b_n is meaningful for all $n \ge 1$. Namely, the denominator is never 0, for the simple reason that it is an odd number for all n, and 0 is an even number.

In fact, one can make the following observation: *If in Theorem 20.1 b_n is a rational function of n for $n \ge 1$ such that (20.2) is satisfied, then the series in the theorem is convergent even if condition (20.1) is not satisfied.*

To see this, note that b_n, being a rational function of n, has form $P(n)/Q(n)$ for some polynomials P and Q, where $Q(n) \ne 0$ for $n \ge 1$ (so that b_n is defined for all $n \ge 1$). Let

$$f(x) = \frac{P(x)}{Q(x)}.$$

Then

$$f'(x) = \frac{P'(x)Q(x) - Q'(x)P(x)}{Q^2(x)}.$$

In this fraction, both the numerator and the denominator and are polynomials, so they are nonzero except for finitely many values of x.[20.2] Thus, there is an integer $N \ge 0$ such that for $x \ge N$ the function $f'(x)$ is defined and it does not change its sign. Then, according as $f'(x) > 0$ for $f'(x) < 0$ for all $x \ge 0$, $f(x)$ itself is decreasing or increasing. Since $\lim_{x\to\infty} f(x) = 0$ in view of (20.2), in the former case $f(x) > 0$ and in the latter case $f(x) < 0$. Then the alternating series Theorem 20.1 shows that the tail

$$\sum_{n=N}^{\infty}(-1)^{n-1} b_n$$

of the series in question converges. This implies that the series in question also converges.

20.3 The Dirichlet convergence test

We will not even state the Dirichlet Convergence Test. We mention it here only because it is an important generalization of the Alternating Series Test in Theorem 20.1. For a discussion, see see the Subsection on The Dirichlet convergence criterion in [3], currently Subsection 28.1 on pp. 110–111, and [5, Problem 7, pp. 5–7].

[20.2] Unless $b_n = 0$ for all $n \ge 1$, in which case $f(x) = f'(x) = 0$ for all x, a case which is of no interest. Observe that we cannot have b_n to be a nonzero constant for $n \ge 1$, since (20.2) needs to be satisfied.

20.4 Reading

[9, §11.5, pp. 765–772].

20.5 Homework

[9, §11.5, p. 772], 3, 7, 11, 17, 37, 39, 41.

21 Absolute convergence; comparing to geometric series

We start with

Definition 21.1 (Absolute convergence)**.** The series

$$\sum_{n=1}^{\infty} a_n$$

is called *absolutely convergent* if the series

$$\sum_{n=1}^{\infty} |a_n|$$

is convergent.

The following lemma implies that absolute convergence implies convergence.

Lemma 21.1. *If the series*

$$\sum_{n=1}^{\infty} |a_n|$$

is convergent, then so is the series

$$\sum_{n=1}^{\infty} a_n$$

A series that is convergent but not absolutely convergent is called *conditionally convergent*. The proof that follows works only in case the numbers a_n are real, but the result is also true in case these numbers are complex.

Proof. We have

$$|a_n| + a_n = \begin{cases} 2|a_n| & \text{if} \quad a_n > 0, \\ 0 & \text{if} \quad a_n \le 0. \end{cases}$$

Thus, the series

$$\sum_{i=1}^{\infty} \big(|a_n| + a_n\big)$$

is convergent by the Comparison Test, comparing it to the series

$$\sum_{n=1}^{\infty} 2|a_n| = 2\sum_{n=1}^{\infty} |a_n|,$$

since the latter series is convergent by our assumptions. Hence, the series

$$\sum_{n=1}^{\infty} a_n = \sum_{i=1}^{\infty} \big(|a_n| + a_n\big) - \sum_{n=1}^{\infty} |a_n|,$$

completing the proof. □

21.1 Conditionally convergent series cannot be rearranged

Absolutely convergent series can be rearranged, and still have the same limit. This is not true for conditionally convergent series. In fact, there is a fairly simple theorem of Dirichlet saying that, given a conditionally convergent series, it can be rearranged to diverge to $+\infty$, $-\infty$, or converge to any given real number. While we will not prove this result, we will explain the reason why it is true on an example. The series $\sum_{n=1}^{\infty} 1/n$ is divergent by the integral test, yet the series

$$\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \pm \ldots.$$

is convergent by the Alternating Series Test. The reason is that if we take a partial sum of this series, the sum of the positive terms is balanced by the sum of the negative terms. A rearrangement of this series is

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \frac{1}{15} - \frac{1}{8} \pm \ldots.$$

In this series, the positive and negative terms do not balance, since in a partial sum of this series there are about twice as many positive terms as negative once. It is not difficult to show that this series diverges to $+\infty$.

21.2 The Ratio Test

The ratio test and the root test amount to using a convergent geometric series in a comparison test. They are about the least sensitive among the convergence tests. Their advantage is that they are easy to apply in situations when other convergence tests would be more complicated to apply. We have

Theorem 21.1 (Ratio Test). *Let a_n be numbers for $n \ge 1$, let $N \ge 1$ be an integer, and assume that $a_n \ne 0$ for $n \ge N$.*

If there is a real number q with $0 < q < 1$ such that

$$\left|\frac{a_{n+1}}{a_n}\right| \le q \qquad \text{for} \quad n \ge N \tag{21.1}$$

then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. If

$$\left|\frac{a_{n+1}}{a_n}\right| \ge 1 \qquad \text{for} \quad n \ge N \tag{21.2}$$

then $\sum_{n=1}^{\infty} a_n$ is divergent.

Proof. Assume (21.1) holds. Then

$$\begin{aligned}|a_{N+k}| &= |a_N| \left|\frac{a_{N+1}}{a_N}\right| \left|\frac{a_{N+2}}{a_{N+1}}\right| \left|\frac{a_{N+3}}{a_{N+2}}\right| \dots \left|\frac{a_{N+k}}{a_{N+k-1}}\right| \\ &= |a_N| \prod_{i=1}^{k} \left|\frac{a_{N+i}}{a_{N+i-1}}\right| \le |a_N| \prod_{i=1}^{k} q = |a_N| q^k\end{aligned}$$

for $k \ge 0$; Therefore, the series

$$\sum_{n=N}^{\infty} |a_n| = \sum_{k=0}^{\infty} |a_{N+k}|$$

is *dominated*[21.1] by the convergent geometric series

$$\sum_{k=0}^{\infty} |a_N| q^k.$$

Hence, the series is convergent by the Comparison Test. Therefore, the series

$$\sum_{n=1}^{\infty} |a_n|$$

is also convergent.

Now, assume (21.2) holds. Then

$$\begin{aligned}|a_{N+k}| &= |a_N| \left|\frac{a_{N+1}}{a_N}\right| \left|\frac{a_{N+2}}{a_{N+1}}\right| \left|\frac{a_{N+3}}{a_{N+2}}\right| \dots \left|\frac{a_{N+k}}{a_{N+k-1}}\right| \\ &= |a_N| \prod_{i=1}^{k} \left|\frac{a_{N+i}}{a_{N+i-1}}\right| \ge |a_N| \prod_{i=1}^{k} 1 = |a_N|.\end{aligned}$$

Thus, $|a_n| \ge |a_N| \ne 0$ for all $n \ge N$. Hence, $\lim_{n\to\infty} a_n \ne 0$. which implies that the series

$$\sum_{n=1}^{\infty} |a_n|$$

diverges. □

21.2.1 The Limit Ratio Test

To make the Ratio Test easier to apply, one often formulates a consequence, called the Limit Ratio Test.

Theorem 21.2 (Limit Ratio Test). *Let a_n be numbers for $n \ge 1$. Assume the limit $L = \lim_{n\to\infty} |a_{n+1}/a_n|$ exists or equals $+\infty$[21.2] If $L < 1$ then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, and if $L > 1$ then the series is divergent.*

[21.1] That is, the terms of the first series are less than the corresponding terms of the second series; this assumes that the terms of the first series are nonnegative. So, if the second series is convegent, then so is the first series, by the Comparison Test.

[21.2] In the latter case, we still say that the limit does not exist. In order for the limit to be meaningful, whether it exists or equals $+\infty$, we implicitly make the assumption that $a_n \ne 0$ for large n.

The theorem does not mention what happens if $L = 1$; that is because then the Ratio Test cannot help. For example, among the series

$$\sum_{n=1}^{\infty} \frac{1}{n}, \qquad \sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n}, \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2},$$

the first one is divergent, the second one is conditionally convergent, and the third one is absolutely convergent.[21.3] In all three cases, we have $L = 1$, so the Limit Ratio Test is inconclusive.

Proof. If $L < 1$, then there is an integer N such that

$$\left|\frac{a_{n+1}}{a_n}\right| < \frac{L+1}{2} < 1 \qquad \text{for all} \quad n > N,$$

so Theorem 21.1 implies that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, If $L > 1$, then there is an integer N such that

$$\left|\frac{a_{n+1}}{a_n}\right| > 1 \qquad \text{for all} \quad n > N,$$

so Theorem 21.1 implies that $\sum_{n=1}^{\infty} a_n$ is divergent. □

The Limit Ratio Test does not reflect the full strength of the Ratio Test, since the existence of the limit L is not assumed in the Ratio Test. The part of Theorem 21.1 that asserts convergence in case inequality (21.1) holds can be formulated in its full strength by using limit superior (see the section on Upper and lower limit in [1, currently §5, pp. 11–13]). The assumption $L < 1$ needs to be replaced by the assumption

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1. \tag{21.3}$$

The assumption of (21.2) cannot be formulated in its full form in a limit form, but the assumption

$$\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1 \tag{21.4}$$

does guarantee divergence, and still weaker than the assumption $L > 1$, since it does not assume the existence of the limit L.

21.3 The Root Test

As we mentioned above, the Root Test is also one of the weakest convence test, and it is also based on an implicit comparison to a geometric series. It is, however, the most useful test for understanding power series, the next major topic in these notes. Similarly, the Ratio Test is also useful for this purpose, but somewhat less so. We have

Theorem 21.3 (Root Test)**.** *Let a_n be numbers for $n \ge 1$, let $N \ge 1$ be an integer. If there is a real number q with $0 < q < 1$ such that*

$$\sqrt[n]{|a_n|} < q \qquad \textit{for} \quad n \ge N \tag{21.5}$$

[21.3] Of course, any convergent series with positive terms is absolutely convergent, though it may be a kind of sophistry to point this out.

then $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent. if*

$$\sqrt[n]{|a_n|} \geq 1 \qquad \text{for} \quad n \geq N \tag{21.6}$$

then $\sum_{n=1}^{\infty} a_n$ *is divergent.*

This result is much easier to show than the the ratio test given above in Theorem 21.1. The main reason to formulate it is that it makes the discussion of its limit form simpler.

Proof. Condition (21.5) means that $|a_n| \leq q^n$ for $n \geq N$. Therefore, the convergence of the series

$$\sum_{n=N}^{\infty} |a_n| = \sum_{k=0}^{\infty} |a_{N+k}|$$

by comparing it to the convergent geometric series

$$\sum_{k=0}^{\infty} q^k.$$

On the other hand, Condition (21.6) means that $|a_n| \geq 1$ for $n \geq N$; so $\lim_{n\to\infty} a_n \neq 0$. Therefore, the series $\sum_{n=1}^{\infty} a_n$ is divergent. □

21.3.1 The Limit Root Test

This test is the most useful test for studying power series. We have

Theorem 21.4 (Limit Root Test)**.** *Let a_n be numbers for $n \geq 1$. Assume the limit $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists or equals $+\infty$.*[21.4] *If $L < 1$ then the series then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, and if $L > 1$ then the series is divergent.*

Proof. If $L < 1$, then there is an integer N such that

$$\sqrt[n]{|a_n|} < \frac{L+1}{2} < 1 \qquad \text{for all} \quad n > N,$$

so Theorem 21.3 implies that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, If $L > 1$, then there is an integer N such that

$$\sqrt[n]{|a_n|} > 1 \qquad \text{for all} \quad n > N,$$

so Theorem 21.3 implies that $\sum_{n=1}^{\infty} a_n$ is divergent. □

The Limit Root Test does not reflect the full strength of the Root Test, since the existence of the limit L is not assumed in the Ratio Test. The part of Theorem 21.3 that asserts convergence in case inequality (21.5) holds can be formulated in its full strength by using limit superior (see the section on Upper and lower limit in [1, currently §5, pp. 11–13]). The assumption $L < 1$ needs to be replaced by the assumption

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1. \tag{21.7}$$

The assumption of (21.6) cannot be formulated in its full form in a limit form, but the assumption

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} > 1 \tag{21.8}$$

does guarantee divergence, and still weaker than the assumption $L > 1$, since it does not assume the existence of the limit L.

[21.4] In the latter case, we still say that the limit does not exist. In order for the limit to be meaningful, whether it exists or equals $+\infty$, we implicitly make the assumption that $a_n \neq 0$ for large n.

21.4 Reading

[9, §11.6, pp. 774–777].

21.5 Homework

[9, §11.6, p. 778], 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 31, 33.

22 Miscellaneous convergence problems

Problem 22.1. Show that the series

$$\sum_{n=1}^{\infty}(3^{1/n}-1)$$

is divergent.

Solution. [Solution] We will use the limit comparison test. We will follow the notation used in Theorem 19.1. Write $a_n = 1/n$ and $b_n = 3^{1/n} - 1$. We have

$$L = \lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1/n}{3^{1/n}-1} = \lim_{x\searrow 0}\frac{x}{3^x-1} = \lim_{x\searrow 0}\frac{1}{3^x\ln 3} = \frac{1}{\ln 3};$$

we will give an explanation why these equalities hold. In the second equality, we replaced $1/n$ by x. Both $1/n$ and x approach 0 from the right. The difference is that $1/n$ assumes only specific values. Now, if the expression on the right is close to $1/\ln 3$ for all $x > 0$ close to 0, then this is certainly true for the specific values of form $1/n$. This means that if the limit on the right-hand side of the second equality exists, then the limit on its left-hand side also exists. The third equality uses l'Hospital's rule,[22.1] The last equation follows, because the expression in the limit before represents a function continuous at 0; the value of the limit can be obtained simply by substituting $x = 0$ in the expression.

Given that the limit L exist, Theorem 18.1 says that if the series

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}(3^{1/n}-1)$$

converges, then the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\frac{1}{n}$$

also converges. Since this latter series is not convergent, we must conclude that the former series does not converge, either.

Note. Our statement of the limit comparison test is not symmetric, unlike the much weaker form of the test given in [9]. Therefore, we must be careful which series will be $\sum a_n$ and which will be $\sum b_n$. If we want to show that a series is convergent, this series must be $\sum a_n$, and if we want

[22.1] L'Hospital's rule is not really necessary here. Namely, with $f(x) = 3^x$ we have

$$\lim_{h\to 0}\frac{3^h-1}{h}\lim_{h\to 0}\frac{f(0+h)-f(0)}{h-0} = f'(0) = 3^0\ln 3 = \ln 3.$$

to show this series is divergent, this series must be $\sum b_n$ (and assume the series is convergent, so as to obtain a contradiction). This complication is unavoidable, unless one is satisfied with the much less sensitive test in [9]. This is not a real problem, however, since when one is close to knowing the answer, one may interchange a_n and b_n if necessary, and the calculation was not wasted.

Problem 22.2. Show that the series

$$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$$

is convergent.

Solution. The problem is quite simple, but the way it is stated makes it difficult to figure out what is going on. Indeed, we have[22.2]

$$(\ln n)^{\ln n} = e^{\ln n \cdot \ln \ln n} = n^{\ln \ln n} > n^2 \qquad \text{if} \quad n > e^{e^2} \approx 1618.178;$$

the first equation holds because $\ln n = e^{\ln \ln n}$; the second one, because $e^{\ln n} = n$, and the third one holds because because if $n > e^{e^2}$ then $\ln n > e^2$, and then $\ln \ln n \geq 2$. Thus

$$\frac{1}{(\ln n)^{\ln n}} = \frac{1}{n^{\ln \ln n}} < \frac{1}{n^2} \qquad \text{if} \quad n \geq 2000.$$

Using the comparison test Theorem 18.1 with $b_n = 1/n^2$ and $a_n = 1/(\ln n)^{\ln n}$, we can conclude that since $\sum_{n=2000}^{\infty} 1/n^2$ is convergent, the series $\sum_{n=2000}^{\infty} 1/(\ln n)^{\ln n}$ is also convegent. Since if a tail of a series is convergent then the whole series is also convergent, it follows that

$$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$$

is also convergent.

Problem 22.3. Show that the series

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

is convergent.

Solution. Writing $a_n = n!/n^n$, we will use the Limit Ratio Test. We have

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} = \frac{(n+1)!}{n!} \frac{n^n}{(n+1)^{n+1}} \\ &= (n+1) \cdot \frac{n^n}{(n+1)^{n+1}} = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n \\ &= \left(\frac{1}{1+1/n}\right)^n = \frac{1}{(1+1/n)^n} \to \frac{1}{e}\end{aligned}$$

as $n \to \infty$; we will explain the steps in this calculation. The second equality follows by multiplying the numerator and denominator by $n^n(n+1)^{n+1}$ of the fraction on its left. The third equality follows by noting that $(n+1)! = n! \cdot (n+1)$. The first equality in the last row follows by dividing the

[22.2]When writing $\ln(\ln n)$, the parentheses are totally superfluous, since there is no other way to read $\ln \ln n$.

numerator and the denominator of the fraction inside the parentheses by n. To obtain the right-hand side we noted that[22.3]

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e. \tag{22.1}$$

Noting that $e > 1$, so that $\lim_{n\to\infty}|a_{n+1}/a_n| < 1$, the Limit Ratio Test implies that the series in this problem is convergent.

Note. As for the limit relation mentioned just before, more generally, for all x we have the relation

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \tag{22.2}$$

can be taken as the definition of the function $\exp x = e^x$;[22.4] see [2, currently equation (2) on p. 2]. See also Subsubsection 11.7 for a further discussion of the limit in equation (22.2).

22.1 Reading

[9, §11.7, pp. 779–780].

22.2 Homework

[9, §11.7, p. 781], try all odd numbered problems (some are quite challenging).

23 Power series

A series of form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots \tag{23.1}$$

is called a power series. Here we take $x^0 = 1$ by convention even in case $x = 0$; this is necessary in order to simplify writing, since 0^0 is undefined. One can prove that the following.

Theorem 23.1 (Radius of convergence)**.** *Given a power series as in (23.1), there is an $R \ge 0$, either a real number or $R = +\infty$, such that this power series absolutely converges for $|x| < R$ and it diverges or $|x| > R$.*

The question of convergence when $|x| = R$ is not covered by a general theorem; the series may be absolutely convergent, conditionally convergent, or divergent; we will give examples later. The interval $(-R, R)$ is called the interval of convergence, and R, the radius of convergence. If one wants to use a more accurate language, any of the intervals $(-R, R]$, $[-R, R]$, or $[-R, R)$ can also be called an interval of convergence, according as to which is the case.

[22.3] We will explain below how to establish this well-known limit relation.
[22.4] The natural exponentiation function e^x is often written as $\exp x$ or $\exp x$. This is the approach taken. Writing $\exp x$ instead of e^x is especially convenient when x is replaced by a long expression.

23.1 Example for calculating the radius of convergence

Writing $n!$ for the factorial of n (or n factorial), that is[23.1]

$$n! = \prod_{k=1}^{n} k = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n \qquad (n \geq 0). \tag{23.2}$$

In case $n = 0$ this product is empty; accordingly, we have $0! = 1$. Consider the series

$$S_1(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

Writing a_n for the nth term of this series, that is, $a_n = x^n/n!$, we can use the Ratio Test to decide whether this series in convergent. Excluding the trivial case $x = 0$, when the ratio test is unusable, we have

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}/(n+1)!}{x^n/n!}\right| = \lim_{n\to\infty} \left|\frac{n!x}{(n+1)!}\right| = \lim_{n\to\infty} \left|\frac{x}{n+1}\right| = 0.$$

This shows that the series $S_1(x)$ is convergent for all x, to the series is convergent for all x.

Next, consider the series

$$S_2(x) = \sum_{n=0}^{\infty} \frac{1}{n} x^n.$$

Writing a_n for the nth term of this series, that is, $a_n = x^n/n$, we can use the Ratio Test to decide whether this series in convergent:

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}/(n+1)}{x^n/n}\right| = \lim_{n\to\infty} \left|\frac{nx}{n+1}\right| = |x| \lim_{n\to\infty} \frac{n}{n+1} = |x|.$$

Thus, $L = |x|$. According to the Ratio Test, this series is convergent if $L < 1$, that is, $|x| < 1$, and divergent if $L > 1$, i.e., $|x| > 1$. Thus the, the radius of convergence is 1. Thus, the series is absolutely convergent if $-1 < x < 1$. As for the endpoints -1 and 1 of the interval of convergence, we have

$$S_2(-1) = \sum_{n=0}^{\infty} \frac{1}{n} (-1)^n.$$

This series is conditionally convergent. Furthermore

$$S_2(1) = \sum_{n=0}^{\infty} \frac{1}{n}.$$

This series is divergent. Thus the interval of convergence is $[-1, 1)$. At the left endpoint, $S_2(x)$ is conditionally convergent, and at the right endpoint it is divergent.

[23.1] The symbol $\prod_{n=a}^{b}$ works similarly to the symbol $\sum n = a^b$, except that in the former we have to multiply the items, and in the latter one we have to add them. Here the index n can only go up, starting at a, and ending up at b. In case $a > b$, there is nothing to multiply or add; there are called the empty product and the empty sum, respectively. The empty product is always taken to be 1, and the empty sum is taken to be 0. The reason for this is that 1 is a neutral element for multiplication (nothing happens when one multiplies by 1.), while 0 is neutral for addition.

As the next example, consider the series

$$S_3(x) = \sum_{n=0}^{\infty} n! x^n.$$

We could use the Ratio Test to calculate the radius of convergence as before. However, we want to use the Root Test, just so as to show that we have also use the Root Test; this will be significantly more difficult, but it allows us to show how to do certain more complicated calculations. We are going to calculate the limit L of the Root Test (this need not be the same as the L of the Ratio Test). With $a_n = n!x^n$, we have

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{n!|x^n|} = \lim_{n\to\infty} \sqrt[n]{n!}\,|x| = |x| \lim_{n\to\infty} \sqrt[n]{n!} \qquad (x \neq 0).$$

It will soon turn out that the limit on the right is $+\infty$. This also explain why we wrote $x \neq 0$ at the end of the line. Namely, the limit rules apply only when the individual limits exist.[23.2] Hence, the third equality is not justified in case $x = 0$. In case $x = 0$, we obviously have $L = 0$, yet the right-hand side is meaningless even if we calculate with extended real numbers.[23.3] The evaluate the limit on the right-hand side, observe that writing $\lfloor t \rfloor$ for the integer part of t,[23.4] for large enough n we have[23.5]

$$n! = \prod_{k=1}^{n} k = \Big(\prod_{k=1}^{\lfloor n/2 \rfloor - 1} k \Big) \prod_{\lfloor n/2 \rfloor}^{n} k \geq \Big(\prod_{k=1}^{\lfloor n/2 \rfloor - 1} 1 \Big) \prod_{\lfloor n/2 \rfloor}^{n} \lfloor n/2 \rfloor = \prod_{\lfloor n/2 \rfloor}^{n} \lfloor n/2 \rfloor;$$

here, on the right-hand side and the middle member[23.6] the first product was put in parentheses so as to end the scope of the product symbol $\prod$.[23.7] After the second equality, we split the range of the product on the left into two parts, and the inequality was obtained by lowering the factors in each of the products to their lowest value in each of the ranges. lowest value. Noting that the product on the right-hand side ha $n - \lfloor n/2 \rfloor + 1 \geq n/2$ factors, we have

$$n! \geq (\lfloor n/2 \rfloor)^{n/2}.$$

Hence

$$\sqrt[n]{n!} = (n!)^{1/n} \geq \left((\lfloor n/2 \rfloor)^{n/2} \right)^{1/n} = (\lfloor n/2 \rfloor)^{1/2} \to \infty \qquad \text{as} \quad n \to \infty.$$

Thus $L = \infty$ for all x except for $x = 0$. Thus, $S_3(x)$ diverges unless $x = 0$. This means that $R = 0$ in this case.

[23.2] For example, the product rule in Theorem 16.2 assumes that the limits A and B exist. Some of the times, these rules can be extended to the case when one of both of the limits are infinite (so they do not exist). For example, the third rule is valid even if A is infinite, unless $c = 0$. This expection is represented by the case $x = 0$ in the present case.

[23.3] The set of extended real numbers is the consist of the elements of $\mathbb{R}$, $-\infty$, and ∞. That is this set equals the set $\mathbb{R} \cup \{-\infty, \infty\}$. Certain operations can be easily extended so as to involve extended real numbers, such as $\infty + \infty = \infty$ and $r \cdot \infty = -\infty$ if $0 > r \in \mathbb{R}$, but $0 \cdot \infty$ is meaningless. Calculations with extended real numbers are helpful for stating the limit rules in Theorem 16.2 in case A or B are infinite.

[23.4] That is, $\lfloor t \rfloor$ the largest integer not exceeding t. In other words, it is the only integer n satisfying the inequalities $t - 1 < n \leq t$. Traditionally, the integer part of t was denoted as $[t]$, but computerized typesetting largely changed that.

[23.5] It is enough to require that $n \geq 2$, so that $\lfloor n/2 \rfloor \geq 1$

[23.6] When there is a series of expressions between equation or inequality signs, each of these expressions is called a member.

[23.7] One hard and fast rule is the scope (or effect) of a $\prod$ symbol ends at the end of the term, i.e., at the next $+$ or $-$ sign after the symbol $\prod$ that is not protected (i.e., inside) parentheses. Whether a second $\prod$ symbol ends the scope of the first one is not at all clear. Note that using k as the product variable in both products is harmless, since k is meaningless outside the scope of each of the products, and so it can be reused.

23.2 Limit superior: the real truth about the radius of convergence

The *limit superior*, or *upper limit*, of a sequence $\{a_n\}$, using a common but really bad notation[23.8] is defined as the largest extended real number (that is, a real number, ∞, or $-\infty$) that is the limit of a convergent subsequence of $\{a_n\}$, or of a subsequence with limit $+\infty$ and $-\infty$.[23.9] This is not the most practical, but perhaps the most easily understood definition of limit superior. The usual notation is $\limsup_{n\to\infty} a_n$. One can show that in this extended sense (that is, allowing ∞ and $-\infty$ as values), every sequence has a limit superior; see the section on Supremum and Limits in [1, currently §4 on p. 8–13]. Given the power series

$$\sum_{n=0}^{\infty} c_n x^n,$$

its radius of convergence R can be calculated by the formula

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|c_n|}; \tag{23.3}$$

here we have $R = 0$ when the formula gives $1/R = +\infty$, and $R = +\infty$ when it gives $1/R = 0$.

23.2.1 An example

We will not prove this formula,[23.10] but we will give an example that shows its importance. Consider the series

$$S_4(x) = \sum_{n=0}^{\infty} \frac{1 + (-1)^n}{2} x^n.$$

Writing c_n for the coefficient of x^n here, for $n \ge 0$ we have

$$c_n = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Thus taking the subsequence of $\{c_n\}$ formed be the terms for even n, it is easy to see that equation (23.3) gives $R = 1$. We can easily find the radius of convergence by using the Ratio Test or the Root test for the series $S_4(x)$ if we write it in the form

$$S_4(x) = \sum_{n=0}^{\infty} x^{2n}.$$

Using the Root Test is slightly easier in ths case, but the reader is encouraged to get the same result by using the Ratio Test instead. Writing a_n for the nth term of this series, that is, $a_n = x^{2n}$, we have

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \sqrt[n]{|x^{2n}|} = x^2,$$

[23.8] The symbol $\{x\}$ is commonly used to denote the *singleton* of x, the set whose only element is x. The symbol used here for a sequence has a conflict; on the other hand, context matters, so it is difficult to misunderstand this notation used in a proper context.

[23.9] A sequence with limit $+\infty$ or $-\infty$ is divergent, so it would be strange to say "limit of a convergent subsequence" in case the limit is infinite. Perhaps it is equally strange to say that the limit of a sequence is infinite and also to say that a sequence with an infinite limit has no limit, but this is commonly done.

[23.10] The proof is based by using the Root Test with Conditions (21.7) and (21.8) istead of the way we stated the Limit Root Test in Theorem 21.3.

Now, according to the Root Test, the series $S_4(x)$ is convergnet if $L < 1$ and it is divergent if $L > 1$. Thus, the series is convergent if $|x| < 1$, and it is divergent if $|x| > 1$. This shows that the we have $R = 1$ in this case.

This example might misleadingly imply that one does not really need formula (23.3) is not realy needed, since one can easily find a way to use the Ratio Test or the Root Test instead. This is by no means the case. Consider, for example, the series

$$S_5(x) = \sum_{n=0}^{\infty} \frac{2 + (-1)^n}{3} x^n.$$

Writing c_n for the coefficient of x^n here, for $n \ge 0$ we have

$$c_n = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 1/3 & \text{if } n \text{ is odd.} \end{cases}$$

In this case, formula (23.3) gives the radius of convergence $R = 1$; in fact, even the limit in this formula exists, so one can use lim instead of lim sup. On the other hand, a direct use of the ratio test is not suitable to determine the radius of convergence. On can find the radius of convergence by the ratio test if one splits the series as the sum of terms for even n and the sum of terms for odd n.

23.3 Power series not centered at the origin

A more general form of power series than the one given in (23.1) is

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots. \tag{23.4}$$

This is called a power series centered at a; if $a = 0$ as in (23.1), then we say that the power series is centered at the origin. Every question about a power series centered at a can be translated into a similar question about a power series centered at the origin by making a substitution $t = x - a$. For example, for a series as given here there is an extended real number $R \ge 0$ such that this series is convergent on the interval $(a-R, a+R)$, and perhaps at the endpoints, and is divergent if $x < a-R$ or $x > a+R$. In this case, R is called the radius of convergence, and $(a-R, a+R)$ is called the interval of convergence.

23.4 Reading

[9, §11.8, pp. 781–785].

23.5 Homework

[9, §11.8, p. 786], 3–27, odd numbers.

24 Representing functions with power series

So far, the only function we know how to represent by a power series is

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad (|x| < 1) \tag{24.1}$$

(see equation (17.3)). Various algebraic manipulations and the following theorem allows us to represent many more functions.

Theorem 24.1. *Assume*

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \qquad (-R < x < R)$$

for some $R > 0$ ($R \in \mathbb{R}$ or $R = +\infty$). Then f is differentiable in $(-R, R)$,

$$f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} \qquad (-R < x < R),$$

and

$$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1} \qquad (-R < x < R).$$

The radii[24.1] *convergence of all three series are the same.*

Each term on the second series is the derivative of he corresponding term in the first series; the summation in the second series starts with $n = 1$ because the term for $n = 0$ in the first series is constant, and so its derivative is 0. Similarly, in the third series, each term is the definite integral of the corresponding term (with x replaced by t) in the first series. This theorem is often described as saying that a power series can be integrated and differentiated termwise inside its interval of interval of convergence. Note that the theorem does not say anything about the endpoints of the interval of convergence. For example, assuming v is an endpoint of the radius of convergence, and the first series converges at v, and $f(v)$ is the sum of this series at v, it does not follow that f is differentiable at v, or even that f is continuos at v.[24.2]

A proof of this theorem is beyond the scope of this article. The mathematical tools needed for a proof will only be discussed a more advanced course.

24.1 Examples

Using equation (24.1) with $-x$ replacing x, we obtain[24.3]

$$\text{(24.2)} \qquad \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n \qquad (|x| < 1).$$

Integrating this, by Theorem 24.1 we obtain

$$\text{(24.3)} \quad \ln(1+x) = \int_0^x \frac{1}{1+t}\,dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \qquad (|x| < 1).$$

Here, after the second equality, we made the substitution $k = n + 1$, in which case $n = k - 1$. The expression after the last equation is the same as the expression before, except that we replaced k

[24.1] Radii is the plural of radius. Note that it is not stated in the theorem that this radius of convergence is R. We could add this – however, adding this would not change what the statement says in any way.

[24.2] Note that if a function is differentiable at a point, it must be continuous at that point.

[24.3] Since the series equation (24.1) converges if $|x| < 1$, the series we obtain will converge if $|-x| < 1$, which means exacly the same as $|x| < 1$.

with n (since we prefer to use n as the summation variable, for the sake of uniformity). Substituting $x = 1$ (this is not allowed by Theorem 24.1), we obtain (correctly, as we will explain) the following.

$$\ln 2 = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \pm \dots .$$

This equation is correct, but it needs to be proved: $x = 1$ represent an endpoint of the interval of convergence of the series (24.2), and Theorem 24.1) has nothing to say about what happens at an endpoint of the interval of convergence. This series is an interesting curiosity, but it is not of any practical use for calculationg $\ln 2$, because it converges much too slowly.

Differentiating equation (24.1), we obtain

$$\text{(24.4)} \qquad \frac{1}{(1-x)^2} = \frac{d}{dx} \frac{1}{1-x} = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{k=0}^{\infty} (k+1) x^k = \sum_{n=0}^{\infty} (n+1) x^n \qquad (|x| < 1)$$

Here, after the third equation, we made the substitution $k = n - 1$. when $n = k + 1$. After the last equation we wrote again n instead of k.

Writing x^2 instead of x in equation (24.2), we obtain

$$\text{(24.5)} \qquad \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n (x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} \qquad (|x| < 1).$$

Again, since the series in (24.2) converges for $|x| < 1$, the present series will converge exactly if $|x^2| < 1$, which means the same as $|x| < 1$, which means exactly the same as $|x| < 1$. This is what we wrote. Integrating this, we obtain

$$\arctan x = \int_0^x \frac{1}{1+t^2}\,dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad (|x| < 1).$$

Substituting $x = 1$ (again, not allowed by Theorem 24.1), and noting that $\arctan 1 = \pi/4$, we obtain

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \pm \dots .$$

Again, this is correct but it needs to be proved, since we obtained it by an invalid application of Theorem 24.1). The proof is beyond the level of the present course.

24.2 Estimating $\ln(1 + x)$ near $x = 0$.

Assuming $|x| < 1$ and Using (24.3), we have

$$\left|\ln(1+x) - x\right| = \left|\sum_{n=2}^{\infty} (-1)^{n-1} \frac{x^n}{n}\right| \le \frac{|x|^2}{2} \sum_{n=0}^{\infty} |x|^n \le \frac{|x|^2}{2(1-|x|)}.$$

Assuming $|x| \le 1/2$, the denominator on the right-hand side is at least 1; hence we obtain

$$\text{(24.6)} \qquad \left|\ln(1+x) - x\right| \le |x|^2 \qquad (|x| \le 1/2).$$

24.3 Reading

[9, §11.9, pp. 787–792].

24.4 Homework

[9, §11.9, p. 793], 3, 9, 11, 13, 17, 21, 25, 29, 31, 33.

25 Taylor and Maclaurin series

Assume we have

$$(25.1)\quad f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots \qquad (a-R < x < a+R)$$

for some $R > 0$.[25.1] Substituting $x = a$, all terms on the right-hand side will be zero unless $n = 0$, so we have

$$f(a) = c_0.$$

According to Theorem 24.1, the series on the right-hand side above can be differentiated termwise, certainly at the point $x = a$. Differentiating once, we obtain

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4(x-a)^3 + \dots .$$

Substituting $x = a$, we obtain

$$f'(a) = c_1.$$

Differentiating again, we have

$$\begin{aligned} f''(x) &= \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2} \\ &= 2\cdot 1c_2 + 3\cdot 2c_3(x-a) + 4\cdot 3(x-a)^2 + 5\cdot 4(x-a)^3 + \dots . \end{aligned}$$

Substituting $x = a$, we obtain

$$f''(a) = 2\cdot 1c_2.$$

Differentiating yet again, we arrive at

$$\begin{aligned} f'''(x) &= \sum_{n=3}^{\infty} n(n-1)(n-2)c_n(x-a)^{n-3} \\ &= 3\cdot 2\cdot 1c_3 + 4\cdot 3\cdot 2c_4(x-a) + 5\cdot 4\cdot 3c_5(x-a)^2 + \dots . \end{aligned}$$

Substituting $x = a$, we obtain

$$f'''(a) = 3\cdot 2\cdot 1c_2.$$

[25.1] We are not implying that the series on the right has R as its radius of convergence, but we are saying that the series has a positive radius of convergence.

In general, differentiating (25.1 k times ($k \geq 0$), we arrive at

$$f^{(k)}(x) = \sum_{n=k}^{\infty} \Big(\prod_{i=0}^{k-1} (n-i) \Big) c_n (x-a)^{n-k} \qquad (a - R < x < a + R)$$

Substituting $x = a$, all terms on the right-hand side are will be zero except if $n = k$. W)e obtain

$$f^{(k)}(a) = \Big(\prod_{i=0}^{k-1} (k-i) \Big) c_k = k! c_k.$$

Since this equation must be true for any $k \geq 0$, we can conclude (using n instead of k) that $c_n = f^{(n)}(a)/n!$. That is, if equation (25.1) is valid, then we must have

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \qquad (a - R < x < a + R). \tag{25.2}$$

Above, we started with a representation of a function by a power series, and showed that the series must have the form given by equation (25.2). We want to reverse this procedure: starting with an arbitrary function $f(x)$, will then equation (25.2) be true? In this general form, the answer is no, since in order for this equation to hold, at the least it must make sense, so f must be differentiable at a infinitely many times,[25.2] or more concisely, f must be infinitely differentiable at a. But even this is not sufficient.

25.1 A function not represented by its Maclaurin series

The series on the right-hand side of equation (25.2) is called the Taylor series of f. In case $a = 0$, one often call this series the Maclaurin series of f. Since equation (25.2) cannot be guaranteed to hold in general, one often writes

$$f(x) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n, \tag{25.3}$$

and regardless whether there is equality, or, indeed, whether the series on the right converges, one still calls the series on the right-hand side the Taylor series of f or, in case $a = 0$, the Maclaurin series of f.

An example for a function that is not represented by its Maclaurin series is the function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Normally, calculating the derivatives of f at 0 would be a quite painful exercise, since the differentiation rules cannot be used at $x = 0$, except for the following

Theorem 25.1. *Let (a, b) be an interval, and let $c \in (a, b)$. Assume that f is continuous on (a, b). Assume, further that f is differentiable on (a, b), except possibly at the point c. Assume, further, that $\lim_{x \to c} f'(x) = L$. Then f is differentiable also at c and $f'(c) = L$.*

[25.2] Common parlance for saying, more correctly, that f is differentiable n times for arbitrary positive integer n.

This theorem is immensely helpful in the present situation, yet we believe that, unfortunately, it is not well known. We will not give the proof here, but see [7, Problem 10, p. 7] for a proof.

While calculating the $f^{(n)}(x)$ for the function $f(x) = e^{-1/x^2}$ $(x \neq 0)$ is still time consuming, it is quite easy to see that $f^{(n)}(x)$ is the sum of terms of form

$$C\frac{1}{x^k}e^{-1/x^2}$$

for various integral[25.3] values of C and $k > 0$. Since the limit of all such terms is zero when $x \to 0$, it follows from Theorem 25.1, that $f^{(n)}(0) = 0$.[25.4] That is, the Maclaurin series of f is identically 0, yet f is not identically 0.

25.2 The remainder term of the Taylor series

The remainder term of the Taylor formula is defined as

$$R_n(x, a) \stackrel{def}{=} f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

It expresses the difference between $f(x)$ and a partial sum of its Taylor series. To show that a function is equal to its Taylor series, one needs to show that the that this remainder term tends to zero in some interval. Fortunately, this can be accomplished in many cases, because a number of useful expression for this remainder term are known. The best known one is the Lagrange remainder term:

Theorem 25.2. *Let $n \geq 0$ be an integer. Let U be an open interval in $\mathbb{R}$ and let $f : U \to \mathbb{R}$ be a function that is $n+1$ times differentiable. For any $x, a \in U$ with $x \neq a$, there is a $\xi \in (x, a)$ (if $x < a$) or $\xi \in (a, x)$ (if $x > a$) such that*

$$R_n(x, a) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}. \tag{25.4}$$

This formula is valid also in case $x = a$ if one takes $\xi = a$.

The case $x = a$ needs a special comment since the intervals (x, a) and (a, x) are empty, and no ξ can be found that belongs to interval; but the formula correctly gives $R_n(x, a) = 0$ in this case, independently of the choice of ξ[25.5] The value of ξ in this formula is not known; so the remainder term cannot be calculated, only estimated. For a discussion of the Lagrange remainder term, and many other forms of the remainder term of the Taylor series, see the section on the remainder term in Taylor's formula, [1, currently §23, pp. 47–50].

25.3 The Maclaurin series of the natural exponential function

For $f(x) = e^x$, for all $n \geq 0$ we have $f^{(n)}(x) = e^x$, and taking so $f^{(n)}(0) = 1$ for all n. Hence, according to equation (25.3) we have

$$e^x \sim \sum_{n=0}^{\infty} \frac{1}{n!}x^n, \tag{25.5}$$

[25.3]Integral is an adjectival form of integer. That is, we simply mean that C and k are integers to illustrate this use of the word "integral." Of course, C being an integer is of no importance, so it is not necessary to show that this is so; on the other hand, this is quite obvious.

[25.4]In a detailed proof, one needs to use induction on n, so that when one discusses $f^{(n)}$ for $n > 0$, one already knows that $f^{(n-1)}$ is continuous at 0.

[25.5]So, when applying this formula, there is no need to pay special attention to the case $t = a$.

It will be shown that here equation holds for all x in place of $\sim$. First note that the series on the right-hand side is convergent for all x. In fact, this series was the shown after equation (23.2) in Subsection 23.1 An important consequence of this convergence is that

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0. \tag{25.6}$$

We will need this to estimate the the remainder term in formula (25.4) for e^x:

$$R_n(x,0) = e^\xi \frac{x^{n+1}}{(n+1)!}.$$

First note that $e^\xi > 0$ for all ξ. Given that ξ is between 0 and x, we have $e^\xi \le 1$ for $x \le 0$, and $e^\xi < e^x$ for $x > 0$. Hence $e^\xi \le \max(1, e^x)$. Thus

$$|R_n(x,0)| \le \frac{|x|^{n+1}}{(n+1)!} \max(1, e^x).$$

Thus, (25.6) implies that $\lim_{n\to\infty} R_n(x,0) = 0$ for every x in case of the function e^x. Thus, we have equality in (25.5); i.e., for all real x we have

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n. \tag{25.7}$$

25.4 The Maclaurin series of sine and cosine

For $f(x) = \sin x$ we have $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, $f^{(4)}(x)$, and from this point everything repeats. Given that $\sin 0 = 0$ and $\cos 0 = 1$. Thus, $f^{(k)}(0)$ for $k = 0, 1, 2, 3, \ldots$ give the sequence

$$0,\quad 1,\quad 0,\quad -1,\quad 0,\quad 1,\quad 0,\quad -1,\quad 0,\quad 1,\quad 0,\quad -1,\quad 0,\quad 1,\quad \ldots$$

Thus, we can write the Taylor series of $\sin x$ as

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

Thus,

$$\sin x = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + R_{2n+2}(x,0);$$

as for the subscript of the remainder term, it is the same as the exponent of highest power of x in the partial sum. It appears that this highest exponent is $2k+1$, after all x^{2k+1} appears in the last term of the sum above. However, x^{2k+2} also occurs, with zero coefficient; after all, every even power of x occurs with 0 coefficient. For $f(x) = \sin x$, this remainder term has form

$$R_{2n+2}(x,0) = f^{(2n+3)}(\xi) \frac{x^{2n+3}}{(2n+3)!},$$

where $f^{(2n+3)}(\xi) = \pm \sin \xi$ or $\pm \cos \xi$; thus, its absolute value value is ≤ 1. Therefore, equation (25.6) implies that $\lim_{n\to\infty} R_{2n+2}(x,0) = 0$. Hence

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots$$

This is the Taylor series of $\sin x$ at $x = 0$, convergent for all real x. Differentiating this series (cf. Theorem 24.1), we obtain a series expressing $\cos x$.

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{2!} - \dots.$$

According to what we said on account of equation (25.2), this is the Taylor series of $\cos x$ at $x = 0$.

25.5 The binomial series

25.5.1 Binomial coefficients

For $k \ge 0$ an integer and α a real, define the binomial coefficient $\binom{\alpha}{k}$ as

$$\binom{\alpha}{k} \stackrel{def}{=} \prod_{j=0}^{k-1} \frac{\alpha - j}{k - j} = \frac{\prod_{j=0}^{k-1}(\alpha - j)}{k!}. \tag{25.8}$$

For $k = 0$ this gives $\binom{\alpha}{k} = 1$, as the empty product is defined to be 1. For $k < 0$ it would also give 1. However, for $k < 0$, if one defines $\binom{\alpha}{k}$ at all, one should write $\binom{\alpha}{k} \stackrel{def}{=} 0$ so as to preserve known identities involving binomial coefficients. A consequence of formula (25.8) is that

$$\binom{\alpha}{k+1} = \binom{\alpha}{k} \frac{\alpha - k}{k + 1}.$$

This formula is useful if one wants to calculate $\binom{\alpha}{k}$ for $k = 0, 1, 2, 3, \dots$. If α is a positive integer, the definition given in (25.8) is that agrees with the usual definition of binomial coefficients.

25.5.2 Asymptotics for the binomial coefficient $\binom{\alpha}{n}$ as $n \to \infty$

According to (25.8), for $k \ge 2(|\alpha| + 1)$ we have

$$\begin{aligned}\binom{\alpha}{k} &= \prod_{j=1}^{k} \frac{\alpha + 1 - j}{j} = (-1)^k \prod_{j=1}^{k} \frac{j - (\alpha + 1)}{j} \\ &= (-1)^k \left(\prod_{j:1 \le j < 2(|\alpha|+1)} \left(1 - \frac{\alpha + 1}{j}\right)\right) \prod_{j:2(|\alpha|+1) \le j \le k} \left(1 - \frac{\alpha + 1}{j}\right).\end{aligned} \tag{25.9}$$

The first product on the right-hand side depends only on α, and not on k. We can estimate the logarithm of the second product with the aid of (24.6), since for every value of j in the second product we have

$$\left|\frac{\alpha + 1}{j}\right| \le \frac{|\alpha| + 1}{j} \le \frac{1}{2}.$$

Hence we have

$$\left|\ln\left(1 - \frac{|\alpha| + 1}{j}\right) - \frac{\alpha + 1}{j}\right| \le \frac{(\alpha + 1)^2}{j^2} \qquad \text{if} \quad (j \ge 2(|\alpha| + 1) \tag{25.10}$$

according to (24.6). Thus, we have

$$\begin{aligned}
&\left|\ln \prod_{j:2(|\alpha|+1)\le j\le k}\left(1-\frac{\alpha+1}{j}\right)-\sum_{j:2(|\alpha|+1)\le j\le k}\frac{\alpha+1}{j}\right| \\
&\qquad\le \sum_{j:2(|\alpha|+1)\le j\le k}\left|\ln\left(1-\frac{\alpha+1}{j}\right)-\frac{\alpha+1}{j}\right|\le \sum_{j:2(|\alpha|+1)\le j\le k}\frac{(\alpha+1)^2}{j^2}
\end{aligned} \tag{25.11}$$

for the logarithm of the second product on the right-hand side of (25.9).

Writing

$$\begin{aligned}
f(k,\alpha) &\overset{def}{=} \ln \prod_{j:2(|\alpha|+1)\le j\le k}\left(1-\frac{\alpha+1}{j}\right)-(\alpha+1)\sum_{j=1}^{k}\frac{1}{j} \\
&= -(\alpha+1)\sum_{j:1\le j<2(|\alpha|+1)}\frac{1}{j} \\
&\quad+\left(\ln \prod_{j:2(|\alpha|+1)\le j\le k}\left(1-\frac{\alpha+1}{j}\right)-(\alpha+1)\sum_{j:2(|\alpha|+1)\le j\le k}\frac{1}{j}\right)
\end{aligned} \tag{25.12}$$

Noting that the sum on the right-hand side of (25.11) is convergent if the upper limit of the summation is extended to infinity, it follows that

$$L \overset{def}{=} \lim_{k\to\infty} f(k,\alpha) \tag{25.13}$$

exists.

We have

$$\begin{aligned}
\ln \prod_{j:2(|\alpha|+1)\le j\le k}\left(1-\frac{\alpha+1}{j}\right) &= f(k,\alpha)+(\alpha+1)\sum_{j=1}^{k}\frac{1}{j} \\
&= L+(\alpha+1)(\gamma+\ln k)+\left(f(k,\alpha)-L+(\alpha+1)\Big(\sum_{j=1}^{k}\frac{1}{j}-\ln k-\gamma\Big)\right),
\end{aligned}$$

where γ denotes Euler's constant defined in (18.6). The limit as $k\to\infty$ of the expression in the big parentheses on the right-hand side is 0 according to (25.13) and (18.6).

Writing A for the first product on the right-hand side of (25.9), we obtain that

$$\lim_{k\to\infty}(-1)^k\binom{\alpha}{k}\Big/ k^{\alpha+1} = A\exp\big(L+(\alpha+1)\gamma\big). \tag{25.14}$$

25.5.3 The binomial series

We have

Theorem 25.3. *For any real α and every real x with $|x|<1$ we have*

$$(1+x)^\alpha=\sum_{n=0}^{\infty}\binom{\alpha}{n}x^n. \tag{25.15}$$

The radius of convergence of this series is 1, but, depending on the value of α, it may also converge at 1 or -1. Writing $f(x) = (1+x)^\alpha$, we find for its nth derivative

$$f^{(n)}(x) = (1+x)^{\alpha-n} \prod_{j=0}^{n-1} (\alpha - j) = (1+x)^{\alpha-n} n! \cdot \binom{\alpha}{n}.$$

Thus, according to Taylor's Formula the equation

$$(1+x)^\alpha = \sum_{k=0}^{n} \binom{\alpha}{k} x^k + R_n(x,0)$$

holds. While it is true that $\lim_{n\to\infty} R_n(x,0) = 0$, for x with $|x| < 1$, we are not in a position to show this. For this, the Lagrange form of the remainder term given in equation (25.4 is not suitable; we need to use another form, not discussed in these notes. The details are given in the section on The binomial series in [1, currently §24, pp. 50–53].

If α is a positive integer, then, for $k > \alpha$ in equation (25.8) the factor for $j = \alpha$ in the numerator is zero; thus, $\binom{\alpha}{n} = 0$. Thus, if $\alpha > 0$ is an integer, all terms in the binomial series (25.15) are zero for $n > \alpha$; thus, the sum is a finite sum in this case. In fact, the equation becomes the well-known binomial theorem.

25.6 Examples

Problem 25.1. Find the Taylor series of $\sqrt{x}$ centered at $x = 4$.

Solution. Formula (25.15) with t replacing x and $\alpha = 1/2$. becomes

$$(1+t)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} t^n \qquad (|t| < 1).$$

On the right-hand side, we need powers of $x - 4$. This can be accomplished by taking $t = (x-4)/4$. In this case $1 + t = x/4$. Making this substitution, the above equation becomes

$$\frac{1}{2}\sqrt{x} = \left(\frac{x}{4}\right)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} \left(\frac{x-4}{4}\right)^n = \sum_{n=0}^{\infty} \binom{1/2}{n} \frac{1}{4^n}(x-4)^n.$$

Multiplying this equation by 2, we obtain the series we wanted:

$$\sqrt{x} = \sum_{n=0}^{\infty} \binom{1/2}{n} \frac{1}{2^{2n-1}}(x-4)^n.$$

This converges for $|t| < 1$, i.e., for $|x-4| < 4$, that is, for $-4 < x - 4 < 4$. This can also be written as $0 < x < 8$. It can be shown that the series is absolutely convergent even at the endpoints of this interval. We can write out the first few terms of this series:

$$\sqrt{x} = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3 - \frac{5}{16384}(x-4)^4 + \dots.$$

It follows from (25.14) with $\alpha = 1/2$ that this series also converges absolutely at the endpoints of the interval of convergence, that is, for $x = 0$ and $x = 8$. Whether the series converges to $\sqrt{x}$ also at the endpoints does not follow from the fact that it converges. While it is true that the limit of the series at the endpoints of the interval of convergence is indeed $\sqrt{x}$, this needs a separate proof; we will not discuss the proof of this here.

Problem 25.2. Evaluate the integral

$$\int_0^1 e^{-x^2/2}\,dx$$

using Taylor series with four decimal precision.

Solution. The integral cannot be evaluated in closed form in terms of elementary functions. That is, the techniques of integration discussed in the first part of this course cannot handle this integral. So, to evaluate this integral, and may use one of the methods of numerical integration, or else, one can express the integral in terms of a Taylor series, and evaluate the sum of this series numerically. We will follow this latter approach here. According to equation (25.7), we have

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!};$$

for all real x (that is, the series on the right is convergent for all t, and the sum of the series equals e^t). Substituting $t = -x^2/2$, it follows that

$$e^{-x^2/2} = \sum_{n=0}^{\infty} \frac{(-x^2/2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^n n!}.$$

Using Theorem 24.1 to integrate this series,we obtain

$$\int_0^x e^{-t^2/2}\,dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2^n n!(2n+1)}$$

Substituting $x = 1$, we have

$$\int_0^1 e^{-x^2/2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!(2n+1)} = \sum_{n=0}^{\infty} (-1)^n b_n, \tag{25.16}$$

where we used x again as the variable of integration, and we wrote

$$b_n = \frac{1}{2^n n!(2n+1)}.$$

We have an alternating series on the right-hand side; we need to evaluate this series with four decimal precision. This means, by convention, that we allow an error of absolute value $\le 5 \cdot 10^{-5}$.[25.6] Now, the error in calculating the sum of an alternating series by taking a partial sum is at most the first term that is not included in this partial sum, i.e., the first term omitted; see Subsection 20.1. We have $b_4 = 1/3456 > 5 \cdot 10^{-5}$ and $b_5 = 1/42240 \le 5 \cdot 10^{-5}$; so we can omit the term involving b_5. That is, for the series on the right-hand side of (25.16) we have

$$\sum_{n=0}^{4} (-1)^n b_n \ge \sum_{n=0}^{\infty} (-1)^n b_n \ge \sum_{n=0}^{4} (-1)^n b_n - b_5 \ge \sum_{n=0}^{4} (-1)^n b_n - 5 \cdot 10^{-5}. \tag{25.17}$$

[25.6] When one says that one wants to evaluate a decimal fraction c with four decimal precision, one would expect that that one approximates c with a decimal fraction such that the first four decimal digits of c and its approximation agree. However, this expectation is unreasonable. For example, if $c = 4.999,999,999$, and one takes $\bar{c} = 5.000,0$ as its decimal approximation, then $\bar{c} - c = 10^{-9}$, so one should consider $\bar{c}$ and approximation of c that is much better than a four-decimal approximation, yet none of the decimal digits of c and $\bar{c}$ agree. So, to say that $\bar{c}$ approximates up to four-decimal precision has no well defined meaning without having a prior agreement what this should mean. In general, by convention, a k decimal digit approximation means that a maximum error of 5 is acceptable at the $k + 1$ decimal digit; more precisely, an error of absolute value $\le 5 \cdot 10^{-k-1}$ is acceptable.

Thus, with 4-decimal approximation, we have

$$\int_0^1 e^{-x^2/2}\,dx \approx \sum_{n=0}^{4}(-1)^n b_n = \frac{103499}{120960} \approx .855646 \pm .00005 \approx .8556;$$

the $\pm$ indicates that the result is between $.855646 + .00005$ and $.855646 - .00005$, to express the permissible error.

Actually, we did not do the best possible; we just said that the obtained result might have an error of .00005. However, equation (25.17) says that the integral is between .855646 and $.855646 - .00005$; that is, an error is committed only on the downside, not on the upside. The actual value of the integral is approximately $.855, 624, 361, 702, 433, 8$.

Problem 25.3. Find the Maclaurin series of

$$f(x) = \frac{x^2 + 3x - 4}{(x+1)^2(2x-1)}$$

Solution. The easiest way to find the Maclauring series of $f(x)$ is to first find the partial fraction decomposition of $f(x)$, and then to find the Maclaurin series of each of the linear fractions of the decomposition. We have

$$\frac{x^2 + 3x - 4}{(x+1)^2(2x-1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{2x-1}$$

with appropriate coeffcients A, B, and C. To determine the coefficients, multiply both sides by the denominator on the left-hand side:

$$x^2 + 3x - 4 = A(x+1)(2x-1) + B(2x-1) + C(x+1)^2.$$

Note that this equation is true even when $x + 1 = 0$ or when $2x - 1 = 0$, even though the equation before was meaningless in these cases. This is because, assuming the equation is not true for all x, the difference of the two sides is a nonzero polynomial, and so it can only be zero in finitely many places (i.e., not more than the degee of the of the polynomial), whereas the equation is clearly true at all but these exceptional places. Substituting $x = 1/2$, we obtain

$$-\frac{9}{4} = \frac{9}{4}C,$$

which implies $C = -1$. Substituting $x = -1$, we obtain the equation

$$-6 = -3B.$$

Hence it follows that $B = 2$. Finally, differentiating the equation with the intention of substituting $x = -1$, we obtain

$$2x + 3 = A(2x-1) + 2B + (x+1)\cdot\text{something}.$$

Here "something" on the right-hand side represents a polynomial of x (actually a constant, but that is of no interest) that does not need to be determined, since we will substitute $x = -1$ anyway. The term results form applying the product rule of differentiation to the first term on the right-hand side and from differentiation the third term there. Substituting $x = -1$ and noting that $B = 2$, we obtain the equation

$$1 = -3A + 4,$$

whence it follows that $A = 1$.

Thus

$$f(x) = \frac{1}{x+1} + \frac{2}{(x+1)^2} - \frac{1}{2x-1} = \frac{1}{x+1} + \frac{2}{(x+1)^2} + \frac{1}{1-2x}.$$

Each fraction on the right-hand side can easily expanded into a Maclaurin series. The Taylor series of the first fraction is given in equation (24.2), that is,

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \qquad (|x| < 1).$$

As for the third fraction, replacing x by $-x$ in equation (24.4), we obtain

$$\frac{1}{(1+x)^2} = \frac{1}{\big(1-(-x)\big)^2} = \sum_{n=0}^{\infty} (n+1)(-x)^n = \sum_{n=0}^{\infty} (-1)^n (n+1) x^n \qquad (|x| < 1).$$

The Taylor series of the second fraction can be obtained from the geometric series (17.3):

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n$$

This is convergent if $|2x| < 1$, i.e., if $|x| < 1/2$. Putting all these together, we obtain

$$f(x) = \sum_{n=0}^{\infty} \Big((-1)^n \big(1 + 2(n+1)\big) + 2^n \Big) x^n = \sum_{n=0}^{\infty} \big((-1)^n (2n+3) + 2^n \big) x^n.$$

In order for this series to be convergent, all three of the series that were added must be convergent. That is, the series is convergent if and only if $|x| < 1/2$.

25.7 Reading

[9, §11.10, pp. 795–808].

25.8 Homework

[9, §11.10, p. 808], 4 (this is an even numbered problem, so the solution is not in the book; the radius of convergence is 3, but explain why, find out what happens at the endpoints, and write out the Taylor series), 5, 7, 9, 11, 13, 17, 23, 25, 35, 37.

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