Exam 1, Mathematics 1206, Section TY11 18 copies

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1.*a*) Find the exact value of $\operatorname{arcsec}(-2/\sqrt{3})$ (no calculator should be used for this, since a calculator can only give approximate answers).

Solution. By definition, $y = \operatorname{arcsec} x$ if $x = \sec y$ and either $0 \le y < \pi/2$ or $\pi \le y < 3\pi/2$. Hence, we are looking for a y for which $\sec y = -2/\sqrt{3}$, i.e., $\cos y = 1/\sec y = -\sqrt{3}/2$. Now $\cos(\pi/6) = \sqrt{3}/2$. As $\cos(\pi - t) = -\cos t$, we have $-\sqrt{3}/2 = \cos(\pi - \pi/6) = \cos(5\pi/6)$. However, $5\pi/6$ is in the second quadrant, and we want a y in the first or third quadrant. As $\cos(-t) = \cos t$, we can get a number in the third quadrant: $\cos(-5\pi/6) = \cos(5\pi/6) = -\sqrt{3}/2$. Finally, as $\cos t = \cos(t + 2\pi)$, $y = (-5\pi/6) + 2\pi = 7\pi/6$ will satisfy all the requirements: $\cos(7\pi/6) = -\sqrt{3}/2$, i.e., $\sec(7\pi/6) = -2/\sqrt{3}$, and $\pi \le 7\pi/6 < 3\pi/2$ (this inequality is easy to verify by multiplying through with 6). Hence $\operatorname{arcsec}(-2/\sqrt{3}) = 7\pi/6$.

b) Find $\cos(\arctan x)$.

Solution. Writing $y = \arctan x$, we have $\tan y = x$, and $\sec y = \sqrt{\tan^2 y + 1} = \sqrt{x^2 + 1}$ (with the positive value of the square root, since $-\pi/2 < y < \pi/2$, i.e., y is in the first or the fourth quadrant, and the secant of those angles is nonnegative). Hence

$$\cos(\arctan x) = \cos y = \frac{1}{\sec y} = \frac{1}{\sqrt{x^2 + 1}}.$$

c) Calculate the derivative of $x^2 \arctan x$.

Solution. Using the product rule of differentiation, we have

$$\begin{aligned}(x^2 \arctan x)' &= (x^2)' \arctan x + x^2 (\arctan x)' \\ &= 2x \cdot \arctan x + x^2 \cdot \frac{1}{x^2 + 1} = 2x \arctan x + \frac{x^2}{x^2 + 1}.\end{aligned}$$

2. Set up the integrals to do the following calculations. *Do not calculate any of the integrals.*

a) To find the area between the curves $y = x^2 - 3x$ and $y = 3x$.

Solution. First we solve the equations $y = x^2 - 3x$ and $y = 3x$. Since the left-hand sides are equal, we can equate the right-hand sides: $x^2 - 3x = 3x$, i.e. $x^2 - 6x = 0$. We can factor the left-hand side: $x(x - 6) = 0$. The only way for a product to be zero is for at least one of the factors to be zero; that is $x = 0$ or $x = 6$. This shows that the two curves intersect at the abscissas $x = 0$ and $x = 6$. The curve $y = 3x$ is on top, and the curve $y = x^2 - 3x$ is on the bottom. Hence, the area is

$$A = \int_0^6 3x\,dx - \int_0^6 (x^2 - 3x)\,dx = \int_0^6 (6x - x^2)\,dx.$$

b) To find the arc length of the curve $y = \ln x$ between the abscissas $x = 1$ and $x = 2$.

Solution. The formula for the arc length L between abscissas a and b of the curve $y = f(x)$ is $L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx$. In the present case, $y' = f'(x) = 1/x$, so

$$L = \int_1^2 \sqrt{1 + \frac{1}{x^2}}\,dx.$$

c) To find the volume obtained by rotating the region between the curves $y = \sqrt{x} - 1$, $x = 2$, $x = 5$, and $y = 0$ about the x axis by using the method of slices (also called the method of disks, washers, or cross sections).

[1] All computer processing for this manuscript was done under Fedora Linux. $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX was used for typesetting.

Solution. The formula for the volume V of the solid obtained by rotating the region between abscissas a and b and under the curve $y = f(x)$ about the x axis is $V = \int_a^b \pi(f(x))^2\,dx$; this formula is obtained by using the method of slices. In the present case,

$$V = \int_2^5 \pi(\sqrt{x} - 1)^2\,dx.$$

d) To find the the volume obtained by rotating the region between the curves $y = e^{-x}$, $y = 0$, $x = 0$, and $x = 1$ about the y axis by using the method of cylindrical shells.

Solution. The formula for the volume V of the solid obtained by rotating the region between abscissas a and b and under the curve $y = f(x)$ about the y axis is $\int_a^b 2\pi x f(x)\,dx$; this formula is obtained by using the method of cylindrical shells. In the present case,

$$\int_0^1 2\pi x e^{-x}\,dx.$$

3. *a*) Calculate the integral

$$\int \frac{4x+12}{x^2+4x+8}\,dx.$$

Solution. We split up the integral into a sum of the integrals

$$I_1 = \int \frac{4x+8}{x^2+4x+8}\,dx \quad \text{and} \quad I_2 = \int \frac{4}{x^2+4x+8}\,dx.$$

The first integral is easy to calculate, since the numerator is twice the derivative of the denominator. Thus, with the substitution $t = x^2 + 4x + 8$ we have $dx = (2x + 4)\,dx$, and so

$$I_1 = \int \frac{2\,dt}{t} = 2\ln|t| + C_1 = 2\ln(x^2+4x+8) + C_1,$$

where C_1 is an arbitrary constant; we dropped the absolute value on the right-hand side since $x^2 + 4x + 8 = (x + 2)^2 + 4$ is always positive.

As for the second integral, we have

$$I_2 = \int \frac{4}{x^2+4x+8}\,dx = \int \frac{4}{(x+2)^2+4}\,dx = \int \frac{1}{\left(\frac{x+2}{2}\right)^2+1}\,dx;$$

to obtain the second equality, both the numerator and the denominator were divided by 4. Next, we use the substitution $t = (x + 2)/2$, in which case $dt = (1/2)\,dx$. We obtain

$$I_2 = \int \frac{2\,dt}{t^2+1} = 2\arctan t + C_2 = 2\arctan\frac{x+2}{2} + C_2,$$

where C_2 is an arbitrary constant. Hence

$$\int \frac{4x+12}{x^2+4x+8}\,dx = I_1 + I_2 = 2\ln(x^2+4x+8) + 2\arctan\frac{x+2}{2} + C,$$

where $C = C_1 + C_2$ is an arbitrary constant.

b) Calculate the integral

$$\int \frac{1}{x^2\sqrt{x^2+1}}\,dx.$$

Solution. We will use the substitution $t = \arctan x$. In this case we have $x = \tan t$, $dx = \sec^2 t\,dt$ and $\sqrt{x^2+1} = \sqrt{\tan^2 t + 1} = \sqrt{\sec^2 t} = \sec t$; note that the right-hand side is $+\sec t$ and not $\pm\sec t$, since we take the nonnegative value of the square root, and, further, the choice of $t = \arctan x$ means, by the definition of arctan that t is in the interval $(-\pi/2, \pi/2)$, and $\sec t$ is positive in this interval. Thus,

$$I = \int \frac{1}{x^2\sqrt{x^2+1}}\,dx = \int \frac{1}{\tan^2 t \sec t}\sec^2 t\,dt = \int \frac{\sec t}{\tan^2 t}\,dt = \int \frac{1/\cos t}{\sin^2 t/\cos^2 t}\,dt = \int \frac{\cos t\,dt}{\sin^2 t}.$$

To calculate this integral, we need to substitute $u = \sin t$, when $du = \cos t\,dt$. That is,

$$I = \int \frac{du}{u^2} = -\frac{1}{u} + C = -\frac{1}{\sin t} + C = -\frac{1}{\tan t \cos t} + C = -\frac{\sec t}{\tan t} + C = -\frac{\sqrt{x^2+1}}{x} + C.$$

4. Calculate the following integrals:

a) $\displaystyle\int e^{3x}\sin x\,dx.$ *b)* $\displaystyle\int x\arctan x\,dx.$

Solution to Part *a*)**.** Through two integrations by parts, and equation is obtained for the integral. When integrating by parts, we should both times differentiate the exponential function and integrate the trigonometric function, or else both times integrate the exponential function and differentiate the trigonometric function. (If we use a mixed approach, and in one of the integrations by parts one differentiates the exponential function and in the other one, we integrate the exponential function, we obtain a useless trivial identity.) In the present situation, it seems better to differentiate the exponential function, since this avoids having to deal with fractions:

$$I = \int e^{3x}\sin x\,dx = e^{3x}(-\cos x) - \int 3e^{3x}(-\cos x)\,dx = -e^{3x}\cos x + 3\int e^{3x}\cos x\,dx$$

Performing another integration by parts, the right-hand side equals

$$-e^{3x}\cos x + 3\left(e^{3x}\sin x - \int 3e^{3x}\sin x\,dx\right) = -e^{3x}\cos x + 3e^{3x}\sin x - 9\int e^{3x}\sin x\,dx.$$

The integral of on the right-hand side is the same as the integral we wanted to evaluate to begin with. Since that integral was denoted by I, we can write I also for the integral on the right-hand side. Well, not quite. This is because the indefinite integral of a function is only determined up to an additive constant. That is, even if the indefinite integral of a function is I, another occurrence of the indefinite integral of the same function might be $I + C'$ for an arbitrary constant C'. Hence we will write $I + C'$ for the integral on the right-hand side.

Since the right-hand side equals I, the integral to be evaluated, we then have the equation

$$I = -e^{3x}\cos x + 3e^{3x}\sin x - 9(I + C')$$

Solving this equation for I, we obtain

$$I = \frac{1}{10}\bigl(-e^{3x}\cos x + 3e^{3x}\sin x\bigr) - \frac{9C'}{10}$$

As C' is an arbitrary constant, we can write C instead of $9C'/10$, where C is an arbitrary constant. Thus

$$I = \frac{1}{10}\bigl(-e^{3x}\cos x + 3e^{3x}\sin x\bigr) + C.$$

Solution to Part *b*)**.**

Solution to Part *a*)**.** We use integration by parts, according to which

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$

Taking $f(x) = \arctan x$ and $g'(x) = x$, we have

$$f'(x) = \frac{1}{x^2+1} \quad \text{and} \quad g(x) = \frac{x^2}{2},$$

and so

$$\begin{aligned}\int x\arctan x\,dx &= \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2}{x^2+1}\,dx \\ &= \frac{x^2}{2}\arctan x - \frac{1}{2}\int \left(1 - \frac{1}{x^2+1}\right)dx \\ &= \frac{1}{2}\big((x^2+1)\arctan x - x\big).\end{aligned}$$

5. Calculate the integral

$$\int \frac{2x+1}{x^4+x^2}\,dx.$$

Solution. We need to use partial fraction decomposition:

$$\frac{2x+1}{x^4+x^2} = \frac{2x+1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}.$$

That is, multiplying both sided by $x^2(x^2+1)$,

$$2x+1 = A(x^2+1)x + B(x^2+1) + (Cx+D)x^2.$$

The substitution $x = 0$ gives $1 = B$. The substitution $x = i$ makes $x^2+1 = 0$, and so the following equation results:

$$2i+1 = (Ci+D)(-1),$$

or

$$2i+1 = -Ci - D.$$

The real parts on the two sides of the equation must be equal; this gives the equation $1 = -D$. Similarly, the imaginary parts must be equal; this gives the equation $2 = -C$. Note that the substitution $x = -i$ also makes $x^2+1 = 0$; however, this substitution would give the same equations that we obtained through the substitution $x = i$, so nothing would be gained by making the substitution $x = -i$.

So far, we have obtained $B = 1$, $C = -2$, and $D = -1$. The easiest way to determine the coefficient A is by comparing the coefficients of x^3 on both sides of the equation. This gives the equation $0 = A + C$, i.e., $A = 2$. Hence

$$\frac{2x+1}{x^4+x^2} = \frac{2}{x} + \frac{1}{x^2} - \frac{2x+1}{x^2+1}.$$

Thus,

$$\int \frac{2x+1}{x^4+x^2}\,dx = \int \frac{2\,dx}{x} + \int \frac{dx}{x^2} - \int \frac{2x\,dx}{x^2+1} - \int \frac{dx}{x^2+1} = 2\ln|x| - \frac{1}{x} - \ln(x^2+1) - \arctan x + C.$$

In calculating the third integral, one needs to use the substitution $t = x^2+1$; as x^2+1 is always positive, one can write $\ln(x^2+1)$ instead of $\ln|x^2+1|$.

Note. The integral can also be evaluated by using the substitution $t = \arctan x$, in which case $x = \tan t$, $dx = \sec^2 t\,dt$, and $x^2 + 1 = \sec^2 t$. We obtain that

$$\begin{aligned}\int \frac{2x+1}{x^4+x^2}\,dx &= \int \frac{2x+1}{x^2(x^2+1)}\,dx = \int \frac{2\tan t+1}{\tan^2 t\sec^2 t}\sec^2 t\,dt = \int \frac{2\tan t+1}{\tan^2 t}\,dt \\ &= \int (2\cot t + \cot^2 t)\,dt.\end{aligned}$$

This integral is easy to evaluate. In evaluating the integral of the first term, we use the substitution $u = \sin t$, when $du = \cos t\,dt$. We have

$$\begin{aligned}\int 2\cot t\,dt &= \int \frac{2\cos t\,dt}{\sin t} = \int \frac{2du}{u} = 2\ln|u| + C = 2\ln|\sin t| + C = 2\ln\left|\frac{\tan t}{\sec t}\right| + C \\ &= 2\ln|\tan t| - 2\ln|\sec t| + C = 2\ln|\tan t| - \ln\sec^2 t + C = 2\ln|x| - \ln(x^2+1) + C.\end{aligned}$$

As for the integral of the second term,

$$\begin{aligned}\int \cot^2 t\,dt &= \int (\cot^2 t + 1)t\,dt - \int dt = \int \csc^2 t\,dt - t = -\cot t - t + C = -\frac{1}{\tan t} - t + C \\ &= -\frac{1}{x} - \arctan x + C.\end{aligned}$$

Substituting the values of these two integrals into the above equation, we obtain the same value of the integral to be calculated as above.