Exam 2, Mathematics 1206, Section TY11 18 copies
11:00 am–1:00 pm, October 31, 2024, 1005-IH
Instructor: Attila Máté[1]

1. Calculate the following limits:

$$a)\quad \lim_{x\to 0}(1-2x)^{1/x}, \qquad\qquad b)\quad \lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right).$$

Solution to Problem *a*)**.** Writing $\exp t$ for e^t for reasons of typographical convenience, we have

$$\lim_{x\to 0}(1-2x)^{1/x}=\lim_{x\to 0}\exp\left(\frac{1}{x}\ln(1-2x)\right)=\exp\left(\lim_{x\to 0}\frac{\ln(1-2x)}{x}\right)=\exp\left(\lim_{x\to 0}\frac{\frac{-2}{1-2x}}{1}\right),$$

where the interchange of the limit and the exponential function used to obtain the second equality is justified in view of the continuity of the function e^x, and the third equality was obtained with the aid of l'Hospital's rule. The limit on the right-hand side inside the exponential function is easily evaluated to be -2; thus the limit on the left-hand side is $e^{-2}=1/e^2$.

Solution to Problem *b*)**.** Writing the difference of two fractions as a single fraction with a common denominator, and then using l'Hospital's Rule twice, we obtain:

$$\begin{aligned}\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)&=\lim_{x\to 1}\frac{x-1-\ln x}{(x-1)\ln x}=\lim_{x\to 1}\frac{1-\frac{1}{x}}{\ln x+(x-1)\cdot\frac{1}{x}}.\\ &=\lim_{x\to 1}\frac{1-\frac{1}{x}}{\ln x+1-\frac{1}{x}}=\lim_{x\to 1}\frac{\frac{1}{x^2}}{\frac{1}{x}+\frac{1}{x^2}}=\frac{1}{1+1}=\frac{1}{2};\end{aligned}$$

here the second and fourth equalities were obtained by using l'Hospital's rule, and the fifth equality was obtained as the limit of a function that is continuous at $x=1$, and so its limit is equal to its value at $x=1$.

2. *a*) Write the integral that finds the length of the curve $r=3-2\sin\theta$ in polar coordinates. *Do not calculate the integral.*

Solution. The formula for the arc length of a curve $r=f(\theta)$ between the values $\theta=a$ and $\theta=b$ in polar coordinates is

$$\int_a^b\sqrt{(f(\theta))^2+(f'(\theta))^2}\,d\theta.$$

In the present case, we have $f(\theta)=3-2\sin\theta$ and $f'(\theta)=-2\cos\theta$. Hence we have

$$\begin{aligned}(f(\theta))^2+(f'(\theta))^2&=(3-2\sin\theta)^2+(2\cos\theta)^2\\ &=9-12\sin\theta+4(\sin^2\theta+\cos^2\theta)=9-12\sin\theta+4=13-12\sin\theta.\end{aligned}$$

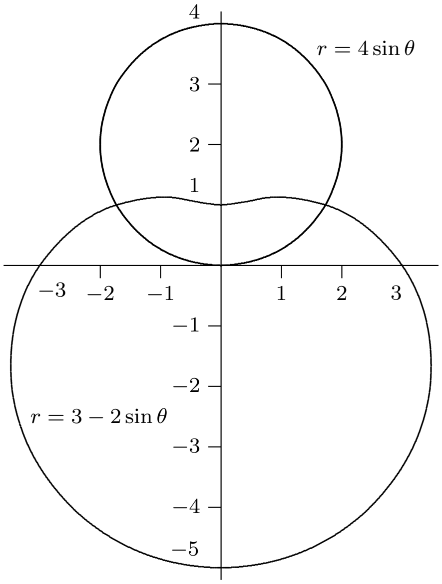
Hence the arc length in question is

$$\int_0^{2\pi}\sqrt{13-12\sin\theta}\,d\theta.$$

---

[1] All computer processing for this manuscript was done under Fedora Linux. $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX was used for typesetting.

b) Write the integral that finds the area inside the curve $r = 4\sin\theta$ and outside the curve $r = 3 - 2\sin\theta$. *Do not calculate the integral.*



Solution. The graphs of the two curves are given in the diagram. Two find the points of intersection, we solve the simultaneous equations $r = 4\sin\theta$ and $r = 3 - 2\sin\theta$. Equating the right-hand sides gives $4\sin\theta = 3 - 2\sin\theta$, that is $\sin\theta = 1/2$. This gives $\theta = \pi/6 + 2k\pi$ and $\theta = 5\pi/6 + 2k\pi$ $(k = 0, \pm1, \pm2, \pm3, \dots)$.[2] We of can of course take $k = 0$ here, since taking a different values of k do not represent different points. Using the formula $A = \frac{1}{2}\int_a^b r^2\,d\theta$, the area inside the circle $r = 3\cos\theta$ between $\theta = \pi/6$ and $\theta = 5\pi/6$ is

$$\int_{\pi/6}^{5\pi/6} \frac{1}{2}(4\sin\theta)^2\,d\theta.$$

The area inside the curve $r = 3 - 2\sin\theta$ between $\theta = \pi/6$ and $\theta = 5\pi/6$ is

$$\int_{\pi/6}^{5\pi/6} \frac{1}{2}(3 - 2\sin\theta)^2\,d\theta.$$

The area we seek is the first area minus the second area, that is

$$\frac{1}{2}\int_{\pi/6}^{5\pi/6} \Big((4\sin\theta)^2 - (3 - 2\sin\theta)^2\Big)\,d\theta = \int_{\pi/6}^{\pi/2} \Big((4\sin\theta)^2 - (3 - 2\sin\theta)^2\Big)\,d\theta.$$

The equation here holds because we have

$$\int_{\pi/6}^{\pi/2} \Big((4\sin\theta)^2 - (3 - 2\sin\theta)^2\Big)\,d\theta = \int_{\pi/2}^{5\pi/6} \Big((4\sin\theta)^2 - (3 - 2\sin\theta)^2\Big)\,d\theta;$$

this can be seen formally by making the substitution $t = \pi - \theta$ in the integral on the left, and then interchanging the limits, and finally changing the t back to θ. Informally, it can be seen by noticing that the region is symmetric about the y axis, so the area right of the y axis is the same as the area left of the y axis.

3. Decide whether the following improper integrals are convergent or divergent. Give clear reasons for your answer (no credit will be given for a correct answer unless the correct reason is also given). Do not calculate the integrals.

a) $\displaystyle\int_2^{+\infty} \frac{dx}{x^2\ln x}$, *b*) $\displaystyle\int_0^{+\infty} \frac{x\,dx}{x^2+1}$, *c*) $\displaystyle\int_1^{+\infty} \frac{\ln x\,dx}{x^2+1}$.

Solution. The answer to each of these questions can be given almost instantaneously by using the following:

Limit Comparison Theorem. *Let a be a real number, and let f and g be real-valued continuous*[3] *functions on $[a, +\infty)$ such that $f(x) \ge 0$ and $g(x) \ge 0$ for all $x \in [a, +\infty)$. Assume that the limit $\lim_{x\to+\infty} g(x)/f(x)$ exist.*[4] *Assume, further, that the integral $\int_a^{+\infty} f$ is convergent. Then the integral $\int_a^{+\infty} g$ is also convergent.*

Instead of this, one could also use the following, but at the price of some extra effort:

---

[2] The points of intersection that these values of θ give are $(2, \pi/6+2k\pi)$ and $(2, 5\pi/6+2k\pi)$. Replacing (r, θ) with $(r, \theta+2k\pi)$ in the first equation will of course not produce any changes. Replacing (r, θ) with $(-r, \theta + (2k+1)\pi)$ in the first equation will result in the equation $-r = 4\sin(\theta + \pi)$; since $\sin(\theta + \pi) = -\sin\theta$, this leads to an equation identical to the first equation. That is, no new points of intersection will be produced by making these substitutions.

Of course, these substitutions should be made only in one of the two equations. We picked the first equation for making these substitutions, since it is simpler to show that no new points of intersection are obtained this way than it would have been to show this if we had picked the second equation.

[3] Continuity here is assumed only for the sake of simplicity. A weaker, but sufficient assumption would be that the integrals of f and g exist on any interval $[a, b]$ with $a < b < +\infty$.

[4] So, in particular, the limit is not $+\infty$. For the existence of the limit, it is essential that $g(x) \ne 0$ when x is large enough, but it is not required that $f(x) \ne 0$ for all $x \in [a, +\infty)$.

Comparison Theorem. *Let a be a real number, and let f and g be real-valued continuous*[5] *functions on $[a, +\infty)$ such that $0 \le g(x) \le f(x)$ for all $x \in [a, +\infty)$. Assume, further, that the integral $\int_a^{+\infty} f$ is convergent. Then the integral $\int_a^{+\infty} g$ is also convergent.*

We will use the Limit Comparison Theorem in the solution.

Solution to Part *a*)**.** Take

$$g(x) = \frac{1}{x^2 \ln x} \quad \text{and} \quad f(x) = \frac{1}{x^2}.$$

As $\lim_{x\to+\infty} g(x)/f(x) = 0$ and the integral $\int_2^{+\infty} f(x)\,dx = \int_2^{+\infty} x^{-2}\,dx$ is convergent, it follows that the integral in question is convergent.

Solution to Part *b*)**.** Instead of considering the integral on the interval $[0, +\infty)$, we can consider it on the interval $[1, +\infty)$; this will have no effect on whether the integral is divergent or convergent. Take

$$f(x) = \frac{x}{x^2+1} \quad \text{and} \quad g(x) = \frac{1}{x},$$

and assume that $\int_1^{+\infty} f$ is convergent. As $\lim_{x\to+\infty} g(x)/f(x) = 1$, it follows that $\int_1^{+\infty} g(x)\,dx = \int_1^{+\infty} x^{-1}\,dx$ is convergent. However, it is known that this integral is divergent; this is a contradiction, showing that the assumption that $\int_1^{+\infty} f$ is wrong. I.e., this integral is divergent.

Solution to Part *c*)**.** Take

$$g(x) = \frac{\ln x}{x^2+1} \quad \text{and} \quad f(x) = x^{-3/2}.$$

Then $\lim_{x\to+\infty} g(x)/f(x) = 0$. Since $\int_1^{+\infty} f(x)\,dx = \int_1^{+\infty} x^{-3/2}\,dx$ is convergent, it follows that $\int_1^{+\infty} g$ is also convergent.

4. Decide whether or not each of the following *sequences (not series)* is convergent. Give reasons for your answers. If the given sequence is convergent, find its limit.

a) $a_n = \dfrac{n-1}{n+1}$, *b*) $a_n = \dfrac{2^n-1}{2^n+1}$, *c*) $a_n = \cos n\pi$,

d) $a_n = \dfrac{\cos n\pi}{\sqrt{n}}$, *e*) $a_n = \dfrac{\cos n}{n^2}$, *f*) $a_n = (-1)^n \dfrac{n}{n+1}$.

Solution to Part *a*)**.** We have

$$\lim_{n\to\infty} \frac{n-1}{n+1} = \lim_{n\to\infty} \frac{1-1/n}{1+1/n} = \frac{\lim_{n\to\infty}(1-1/n)}{\lim_{n\to\infty}(1+1/n)} = \frac{1}{1} = 1.$$

Solution to Part *b*)**.** We have

$$\lim_{n\to\infty} \frac{2^n-1}{2^n+1} = \lim_{n\to\infty} \frac{1-2^{-n}}{1+2^{-n}} = 1;$$

we omitted here some simple steps, such as the application of the quotient rule for limits, similarly to the way it was done in Part *a*).

Solution to Part *c*)**.** For even n we have $a_n = 1$, and for odd n we have $a_n = -1$, and so the sequence is not convergent.

Solution to Part *d*)**.** Since $|\cos n\pi| = 1$, we have $|a_n| = 1/\sqrt{n}$, and so $a_n \to 0$.

---

[5] Continuity here is assumed only for the sake of simplicity, similarly as in the Limit Comparison Theorem above.

Solution to Part *e*)**.** We have $|\cos n| \le 1$,[6] and so $|a_n| \le 1/n^2$, and so $a_n \to 0$.

Solution to Part *f*)**.** We have

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1.$$

As $(-1)^n$ equals to 1 for even n and to -1 for odd n, this means that for large n, a_n is close to 1 if n is even and is close to -1 if n is odd. Therefore the sequence $\{a_n\}_{n=1}^{\infty}$ is divergent.

5. *a*) Decide whether the series

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n}+1}$$

is convergent.

Solution to part a). Before giving answers to the specific questions, we will state the

The Limit Comparison Test. *Let* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *be infinite series, and assume* $0 \le a_n \le b_n$ *and* $b_n \ne 0$ *hold for every integer* $n > 0$. *Assume that the limit* $\lim_{n\to\infty} \frac{a_n}{b_n}$ *exists.*[7] *Assume, further, that the series* $\sum_{n=1}^{\infty} b_n$ *is convergent. Then the series* $\sum_{n=1}^{\infty} a_n$ *is also convergent.*

Remark. Under the same conditions, if instead of assuming that $\sum_{n=1}^{\infty} b_n$ is convergent, we assume that $\sum_{n=1}^{\infty} a_n$ is divergent, then we can conclude that $\sum_{n=1}^{\infty} b_n$ is also divergent. In fact if the latter series were not divergent (i.e., if it were convergent) the Limit Comparison Test would imply that the former is series is convergent (i.e., it is not divergent).

The series is divergent. $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$ is divergent (changing the lower limit of summation from $n = 0$ to $n = 1$ does not affect questions of convergence). In fact, assuming that this series is convergent, by using the Limit Comparison Test we can conclude that the $\sum_{n=1}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{1}{n^{1/2}}$ is also convergent, as $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. This latter series is, however, known to be divergent (by the Integral Test), so the former series cannot be convergent.

b) Decide whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{2^{3n-5}}{3^{2n-3}}$ is convergent. If it is convergent, find its sum.

Solution to part b). The series is a geometric series, that is, a series of form $\sum_{n=0}^{\infty} x^n$. This series is divergent if $|x| \ge 1$, and it is convergent if $|x| < 1$. In the latter case we have

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

To compare the series in the problem to this latter, we first need to start the summation at 0. In order to do this, we will write $k = n - 2$, in which case the summation will go from $k = 0$ to $k = \infty$. We have $n = k + 2$; hence

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{2^{3n-5}}{3^{2n-3}} &= \sum_{k=0}^{\infty} \frac{2^{3(k+2)-5}}{3^{2(k+2)-3}} = \sum_{k=0}^{\infty} \frac{2^{3k+1}}{3^{2k+1}} = \sum_{k=0}^{\infty} \frac{2}{3} \cdot \frac{2^{3k}}{3^{2k}} = \sum_{k=0}^{\infty} \frac{2}{3} \cdot \frac{(2^3)^k}{(3^2)^k} = \frac{2}{3} \cdot \sum_{k=0}^{\infty} \frac{8^k}{9^k} \\ &= \frac{2}{3} \cdot \sum_{k=0}^{\infty} \left(\frac{8}{9}\right)^k = \frac{2}{3} \cdot \frac{1}{1-\frac{8}{9}} = \frac{2}{3} \cdot \frac{9}{9-8} = 6.\end{aligned}$$

Here, the sum on the second line is a convergent geometric series, since $|8/9| < 1$; the second equality in this line is obtained by using the sum formula of the geometric series mentioned above.

---

[6] Actually, equality will never hold when n is an integer, since $\cos x = \pm 1$ only if x is an integral multiple (i.e., integer multiple; "integral" here is an adjective form of "integer," and it does not refer to the mathematical concept of integration) of π, and π is an irrational number.

[7] So, in particular, this limit is not ∞. A limit that is infinite is does not exist; one only says that a limit exists when the limit is finite.