Exam 3, Mathematics 1206, Section TY11 18 copies
11:00 am–1:00 pm, December 5, 2024, 1005-IH
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1. Decide whether each of the following series are convergent or divergent. Give reasons for your answers. No credit will be given for yes or no answers without proper explanations.

a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{4n}{3+9n}\right)^n$, *b)* $\displaystyle\sum_{n=1}^{\infty}\tan(1/n)$, *c)* $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n+n\sin^2 n}$,

d) $\displaystyle\sum_{n=1}^{\infty}\frac{n^2 2^n}{n!}$, *e)* $\displaystyle\sum_{n=1}^{\infty}\frac{2^n n!}{n^n}$.

Solution to Part *a*)**.** Write

$$a_n=\left(\frac{4n}{3+9n}\right)^n,$$

Using the root test to check for the convergence of $\sum_{n=1}^{\infty}$, note that

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\frac{4n}{3+9n}=\frac{4}{9}<1,$$

showing that this series is convergent.

Solution to Part *b*)**.** This problem can be solved by using the Limit Comparison Test. We restate the Comparison Test and the Limit Comparison Test for easy reference.

Comparison Test. *Let* $\sum_{n=1}^{\infty}a_n$ *and* $\sum_{n=1}^{\infty}b_n$ *be two series such that* $0\le a_n\le b_n$ *for every* $n\ge 1$. *Assume that the series* $\sum_{n=1}^{\infty}b_n$ *is convergent. Then the series* $\sum_{n=1}^{\infty}a_n$ *is also convergent.*

Limit Comparison Test. *Let* $\sum_{n=1}^{\infty}a_n$ *and* $\sum_{n=1}^{\infty}b_n$ *be two series such that* $a_n\ge 0$ *and* $b_n>0$ *for every* $n\ge 1$. *Assume that the series* $\sum_{n=1}^{\infty}b_n$ *is convergent. Assume, further, that the limit*

$$\lim_{n\to\infty}\frac{a_n}{b_n}$$

exists.[2] *Then the series* $\sum_{n=1}^{\infty}a_n$ *is also convergent.*

To show that

$$\sum_{n=1}^{\infty}\tan(1/n),$$

is divergent, assume that it is convergent, and use the Limit Comparison Test with the choice of $a_n=1/n$ and $b_n=\tan(1/n)$. We have

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{1/n}{\tan 1/n}=\lim_{x\to 0}\frac{x}{\tan x}=\lim_{x\to 0}\frac{\cos x}{\frac{\sin x}{x}}=\frac{\lim_{x\to 0}\cos x}{\lim_{x\to 0}\frac{\sin x}{x}}=\frac{1}{1}=1.$$

Thus it follows that the series $\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}1/n$ is convergent. However, this is not true; consequently, the assumption that $\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\tan(1/n)$ is convergent must be incorrect; i.e., the latter series is divergent, as claimed.

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[1] All computer processing for this manuscript was done under Fedora Linux. $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX was used for typesetting.

[2] Recall that when we say that a limit exists, it must be finite. That is, if a limit is $+\infty$, $-\infty$, or $\pm\infty$, the limit does not exist.

Solution to Part *c*)**.** The series is divergent. Assume that it is convergent, and use the comparison test with $a_n = 1/n$ and

$$b_n = \frac{2}{n + n\sin^2 n}$$

According to the assumption, $\sum_{n=1}^{\infty} b_n$ is convergent. We have

$$a_n = \frac{1}{n} = \frac{2}{n+n} \le \frac{2}{n + n\sin^2 n} = b_n,$$

so the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 1/n$ is also convergent, according to the Comparison Test. This is, however, not the case, showing that the assumption that the series in question is convergent is wrong. Thus, the series is divergent, as claimed.

Solution to Part *d*)**.** We use the Ratio Test. Let

$$a_n = \frac{n^2 2^n}{n!}.$$

We have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{a_{n+1}}{a_n} = \frac{(n+1)^2 2^{n+1}/(n+1)!}{n^2 2^n/n!} = \left(\frac{n+1}{n}\right)^2 \cdot \frac{2}{n+1} \to 0$$

as $n \to \infty$. Thus the series in question converges.

Solution to Part *e*)**.** We again use the Ratio Test. Write

$$a_n = \frac{2^n n!}{n^n}.$$

We have

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{a_{n+1}}{a_n} = \frac{2^{n+1}(n+1)!/(n+1)^{n+1}}{2^n n!/n^n} = 2(n+1)\cdot\frac{n^n}{(n+1)^{n+1}} = 2\cdot\frac{n^n}{(n+1)^n} \\ &= \frac{2}{\frac{(n+1)^n}{n^n}} = \frac{2}{\left(\frac{n+1}{n}\right)^n} = \frac{2}{\left(1+\frac{1}{n}\right)^n} \to \frac{2}{e}\end{aligned}$$

as $n \to \infty$. The right-hand side is less than 1 (since $e > 2$; in fact $e \approx 2.718{,}281{,}828$). Hence the series in question is convergent.

2. How many terms of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n! 2^n}$$

need to be added so as to find the sum of this series with an error < 0.01? Give reasons for your answer, and show all your calculations needed to give the answer, but *do not* evaluate the sum.

Solution. The series is an alternating series, and its the error (i.e., the difference between its limit and a given partial sum)[3] is between 0 and the first term we fail to add. The term for $n = 3$ is $-1/48$, and the term for $n = 4$ is $1/348$. That is, if one adds the first three terms of the series, i.e., if one takes

$$\sum_{n=1}^{3} \frac{(-1)^n}{n! 2^n} = -\frac{1}{2} + \frac{1}{8} - \frac{1}{48} \approx 0.395833,$$

the error is between 0 and $1/384$, and so its absolute value is definitely less than 0.01.

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[3] Often, when one talks about the error, one means what should properly be called the absolute value of the error. If one defines the error as we did here, rather than using absolute values, one can make a more precise statement about the error of the alternating series.

3. Find the interval of convergence of the following power series. For each endpoint of the interval of convergence, decide whether the series is absolutely convergent, conditionally convergent, or divergent.

$$a)\quad \sum_{n=1}^{\infty}\frac{1}{n2^n}(x+3)^n, \qquad b)\quad \sum_{n=1}^{\infty}\frac{(2x-12)^n}{8^n\sqrt{n}}, \qquad c)\quad \sum_{n=1}^{\infty} n!x^n.$$

Solution, general comments. Given a power series

$$\sum_{n=0}^{\infty} c_n(x-a)^n,$$

and using the ratio test, this series is convergent if the limit

$$\lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = \lim_{n\to\infty}\frac{|c_{n+1}|}{|c_n|}\,|x-a| = |x-a|\lim_{n\to\infty}\frac{|c_{n+1}|}{|c_n|}$$

is less than 1, and it is divergent if this limit is greater than 1. That is, writing

$$L = \lim_{n\to\infty}\frac{|c_{n+1}|}{|c_n|},$$

the series is convergent if $L|x-a|<1$, and it is divergent if $L|x-a|>1$. Thus, assuming $L\ne 0$, the series is convergent if $|x-a|<1/L$, and it divergent if $|x-a|>1/L$; in other words, the radius of convergence is $1/L$. If $L=0$ then the series is convergent for every x, i.e., $R=+\infty$, and if $L=+\infty$ then the series is never convergent unless $x=a$; so $R=0$ in this case. Therefore, we can write that

$$\frac{1}{R} = \lim_{n\to\infty}\frac{|c_{n+1}|}{|c_n|},$$

with the understanding that if the limit is 0 then $R=+\infty$ and if this limit is $+\infty$ then $R=0$.

Instead of the ratio test, we can also use the root test, showing that the above series is convergent if

$$\lim_{n\to\infty}\sqrt[n]{|c_n(x-a)^n|} = \lim_{n\to\infty}\sqrt[n]{|c_n|}\,|x-a| = |x-a|\lim_{n\to\infty}\sqrt[n]{|c_n|}$$

is less than 1, and it is divergent if this limit is greater than 1. That is, writing

$$L = \lim_{n\to\infty}\sqrt[n]{|c_n|},$$

the series is convergent if $L|x-a|<1$, and it is divergent if $L|x-a|>1$. Hence, assuming $L\ne 0$, the series is convergent if $|x-a|<1/L$, and it is divergent if $|x-a|>1/L$; in other words, the radius of convergence is $1/L$. If $L=0$ then the series is convergent for every x, i.e., $R=+\infty$, and if $L=+\infty$ then the series is never convergent unless $x=a$; so $R=0$ in this case. So we can write that

$$\frac{1}{R} = \lim_{n\to\infty}\sqrt[n]{|c_n|}$$

with the understanding that if the limit is 0 then $R=+\infty$ and if this limit is $+\infty$ then $R=0$. Either of the above formulas are applicable to find the radius of convergence, though there are cases when one or the other limit does not exist. The second formula can be generalized to a slightly more complicated formula that always correctly gives the radius of convergence, but that formula is not discussed in the present course.

Solution to Part *a*)**.** Writing

$$c_n = \frac{1}{n2^n},$$

for the radius of convergence R of the series $\sum_{n=1}^{\infty} c_n t^n$, we have

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty}\frac{\frac{1}{(n+1)2^{n+1}}}{\frac{1}{n2^n}} = \lim_{n\to\infty}\frac{n2^n}{(n+1)2^{n+1}} = \frac{1}{2}.$$

Thus the series

$$\sum_{n=1}^{\infty}\frac{1}{n2^n}t^n$$

has radius of convergence is 2; that is, it is absolutely convergent in the interval $(-2, 2)$. As for the endpoints of this interval, for $t = 2$, this series becomes

$$\sum_{n=1}^{\infty}\frac{1}{n2^n}2^n = \sum_{n=1}^{\infty}\frac{1}{n},$$

and this series (called the harmonic series) is divergent. For $t = -2$, it becomes

$$\sum_{n=1}^{\infty}\frac{1}{n2^n}(-2)^n = \sum_{n=1}^{\infty}(-1)^n\frac{1}{n},$$

which is a conditionally convergent alternating series. Thus, the above series is absolutely convergent when $-2 < t < 2$, conditionally convergent when $t = -2$, and divergent otherwise. With $t = x + 3$ (or $x = t - 3$), this implies that the series

$$\sum_{n=1}^{\infty}\frac{1}{n2^n}(x+3)^n$$

is absolutely convergent when $-2 < x + 3 < 2$, i.e., when $-5 < x < -1$, and conditionally convergent when $x + 3 = -2$, i.e, when $x = -5$, and divergent otherwise.

Solution to Part *b*)**.** Writing

$$c_n = \frac{1}{8^n\sqrt{n}},$$

for the radius of convergence R of the series $\sum_{n=1}^{\infty} c_n t^n$, we have

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty}\frac{\frac{1}{8^{n+1}\sqrt{n+1}}}{\frac{1}{8^n\sqrt{n}}} = \lim_{n\to\infty}\frac{8^n\sqrt{n}}{8^{n+1}\sqrt{n+1}} = \lim_{n\to\infty}\frac{1}{8}\sqrt{\frac{n}{n+1}} = \frac{1}{8}.$$

So the series

$$\sum_{n=1}^{\infty}\frac{1}{8^n\sqrt{n}}t^n$$

has radius of convergence 8. That is, it is absolutely convergent when $-8 < t < 8$. For $t = 8$ it becomes

$$\sum_{n=1}^{\infty}\frac{1}{8^n\sqrt{n}}8^n = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}},$$

and this series is divergent by the integral test. For $t = -8$, the series becomes

$$\sum_{n=1}^{\infty}\frac{1}{8^n\sqrt{n}}(-8)^n = \sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}},$$

which is a convergent alternating series. It is not absolutely convergent, since the series formed by the absolute values is the same as the divergent series considered just before (namely, the power series at $t = 8$). Thus, the series is conditionally convergent for $t = -8$. The question of convergence of the series

$$\sum_{n=1}^{\infty}\frac{(2x-12)^n}{8^n\sqrt{n}}$$

reduces to that of the above series with $t = 2x - 12$. Therefore, this series is absolutely convergent when $-8 < 2x - 12 < 8$, i.e., when $4 < 2x < 20$, that is, when $2 < x < 10$. It is conditionally convergent when $2x - 12 = -8$, i.e., when $x = 2$, and it is divergent otherwise.

Solution to Part *c*)**.** Writing $c_n = n!$, for the radius of convergence R of the series

$$\sum_{n=1}^{\infty} c_n x^n$$

we have

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} n = +\infty.$$

That is, the radius of convergence of the above series is zero. I.e., the series

$$\sum_{n=1}^{\infty} n! x^n$$

converges only for $x = 0$, and it diverges for any other value of x.

4. Find the power series representation (centered at 0, i.e., in the form $\sum_{n=0}^{\infty} c_n x^n$) of

a) $\dfrac{1}{1+4x^2}$, *b*) $\dfrac{x}{(1+x)^2}$.

Solution to Part *a*)**.** We using the geometric series

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n,$$

valid when $|t| < 1$ (for $|t| > 1$ the series on the right is divergent), we have

$$\frac{1}{1+4x^2} = \frac{1}{1-(-4x^2)} = \sum_{n=0}^{\infty} (-4x^2)^n = \sum_{n=0}^{\infty} (-4)^n x^{2n} = \sum_{n=0}^{\infty} (-1)^n 4^n x^{2n}.$$

This is valid when $|-4x^2| < 1$, i.e., when $|x| < 1/2$.

Solution to Part *b*)**.** We will again use the sum formula for the geometric series. For $|x| < 1$ we have

$$\begin{aligned}\frac{x}{(1+x)^2} &= -x\frac{d}{dx}\frac{1}{1+x} = -x\frac{d}{dx}\frac{1}{1-(-x)} = -x\frac{d}{dx}\sum_{n=0}^{\infty}(-x)^n = -x\frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n x^n \\ &= -x\sum_{n=1}^{\infty}(-1)^n n x^{n-1} = -\sum_{n=1}^{\infty}(-1)^n n x^n = \sum_{n=1}^{\infty}(-1)^{n+1} n x^n = \sum_{n=1}^{\infty}(-1)^{n-1} n x^n.\end{aligned}$$

The last equation holds because $(-1)^{n+1} = (-1)^{n-1}$. The last expression (i.e., the last member of these equations) is not clearly preferable to the one before.

5. Find the first four terms of the Maclaurin series of $\dfrac{1}{\sqrt{1+x}}$ (that is, the last term you need to find will involve x^3). Make sure to write the coefficients as common fractions, *not as decimals*.

Solution. We have

$$\frac{1}{\sqrt{1+x}} = (1+x)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} x^n.$$

We need to evaluate the binomial coefficients $\binom{-1/2}{n}$ for $n = 0, 1, 2, 3$. We have

$$\binom{\alpha}{0} = 1, \quad \binom{\alpha}{1} = \alpha, \quad \text{and} \quad \binom{\alpha}{n} = \frac{\alpha - n + 1}{n}\binom{\alpha}{n-1}$$

for every real alpha and every integer $n \geq 1$. Thus

$$\binom{-1/2}{0} = 0, \quad \binom{-1/2}{1} = -1/2,$$

$$\binom{-1/2}{2} = \frac{-1/2 - 2 + 1}{2}\binom{-1/2}{1} = \frac{-3/2}{2} \cdot \frac{-1}{2} = \frac{3}{8},$$

$$\binom{-1/2}{3} = \frac{-1/2 - 3 + 1}{3}\binom{-1/2}{2} = \frac{-5/2}{3} \cdot \frac{3}{8} = -\frac{5}{16}.$$

Hence

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \dots .$$