Exam 1, Mathematics 1206, Section TY9 13 copies
9:05 am–11:40 pm, September 28, 2023, 1005-IH
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1.*a*) Find the exact value of $\operatorname{arcsec}(-2)$ (no calculator should be used for this, since a calculator can only give approximate answers).

Solution. By definition, $y = \operatorname{arcsec} x$ if $x = \sec y$ and either $0 \le y < \pi/2$ or $\pi \le y < 3\pi/2$. Hence, we are looking for a y for which $\sec y = -2$, i.e., $\cos y = 1/\sec y = -1/2$. Now $\cos \pi/3 = 1/2$. As $\cos(\pi - t) = -\cos t$, we have $-1/2 = \cos(\pi - \pi/3) = \cos 2\pi/3$. However, $2\pi/3$ is in the second quadrant, and we want a y in the first or third quadrant. As $\cos(-t) = \cos t$, we can get a number in the third quadrant: $\cos(-2\pi/3) = \cos(2\pi/3) = -1/2$. Finally, as $\cos t = \cos(t + 2\pi)$, $y = (-2\pi/3) + 2\pi = 4\pi/3$ will satisfy all the requirements: $\cos 4\pi/3 = -1/2$, i.e. $\sec 4\pi/3 = -2$, and $\pi \le 4\pi/3 < 3\pi/2$ (this inequality is easy to verify by multiplying through with 6). Hence $\operatorname{arcsec}(-2) = 4\pi/3$.

b) Find $\tan(\arcsin x)$.

Solution. Writing $y = \arcsin x$, we have $\sin y = x$, and $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$ (with the positive value of the square root, since $-\pi/2 \le y \le \pi/2$, i.e., y is in the first or fourth quadrant, and the cosine of those angles is nonnegative). Hence

$$\tan(\arcsin x) = \tan y = \frac{\sin y}{\cos y} = \frac{x}{\sqrt{1 - x^2}}.$$

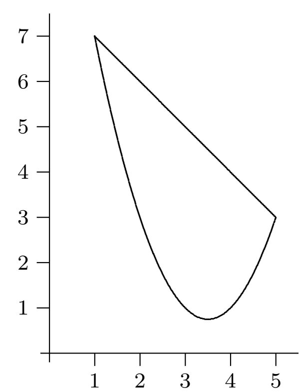
c) Calculate the derivative of $x \arctan x$.

Solution. Using the product rule of differentiation, we have

$$\begin{aligned}(x \arctan x)' &= (x)' \arctan x + x(\arctan x)' \\ &= 1 \cdot \arctan x + x \cdot \frac{1}{x^2 + 1} = \arctan x + \frac{x}{x^2 + 1}.\end{aligned}$$

2. Set up the integrals to do the following calculations. *Do not calculate any of the integrals.* Always integrate along the x-axis. For each of the parts, consider the region bounded by the curves $y = x^2 - 7x + 13$ and $y = 8 - x$.

a) To find the area of the above region.



Solution. First we solve the system equations $y = x^2 - 7x + 13$ and $y = 8 - x$. Since the left-hand sides are equal, we can equate the right-hand sides: $x^2 - 7x + 13 = 8 - x$, i.e., $x^2 - 6x + 5 = 0$. We can factor the left-hand side: $(x - 1)(x - 5) = 0$. The only way for a product to be zero is for at least one of the factors to be zero; that is $x = 1$ or $x = 5$. This shows that the two curves intersect at the abscissas $x = 1$ and $x = 5$. The line segment $y = 8 - x$ is on top, and the parabola $y = x^2 - 7x + 13$ is on the bottom (see the graph on the right). Hence, the area is

$$A = \int_1^5 (8 - x)\,dx - \int_1^5 (x^2 - 7x + 13)\,dx = \int_1^5 (-x^2 + 6x - 5)\,dx.$$

b) To find the volume obtained by rotating the above region about the x-axis.

Solution. The formula for the volume V of the solid obtained by rotating the region between abscissas a and b and under the curve $y = f(x)$ about the x axis is $V = \int_a^b \pi(f(x))^2\,dx$; this formula is obtained by using the method of slices. In the present case,

$$V = \int_1^5 \pi(8 - x)^2\,dx - \int_1^5 \pi(x^2 - 7x + 13)^2\,dx = \int_1^5 \pi(-x^4 + 14x^3 - 74x^2 + 166x - 105)\,dx.$$

[1]All computer processing for this manuscript was done under Fedora Linux. $\mathcal{AMS}$-TEX was used for typesetting.

c) To find the the volume obtained by rotating the above region about the y axis.

Solution. The formula for the volume V of the solid obtained by rotating the region between abscissas a and b and under the curve $y = f(x)$ about the y axis is $\int_a^b 2\pi x f(x)\,dx$; this formula is obtained by using the method of cylindrical shells. In the present case,

$$V = \int_1^5 2\pi x(8 - x)\,dx - \int_1^5 2\pi x(x^2 - 7x + 13)\,dx = \int_1^5 2\pi(-x^3 + 6x^2 - 5x)\,dx.$$

3. Calculate the integrals

$$a)\quad \int_0^{\pi/2} x^2 \cos x\,dx, \qquad\qquad b)\quad \int_0^{\pi/4} \tan^5 x \sec^2 x\,dx.$$

Solution. As for *a*), we need to use integration by parts twice:

$$\int_0^{\pi/2} x^2 \cos x\,dx = \int_0^{\pi/2} x^2 (\sin x)'\,dx = x^2 \sin x\Big|_0^{\pi/2} - \int_0^{\pi/2} 2x \sin x\,dx = \frac{\pi^2}{4} + \int_0^{\pi/2} 2x(\cos x)'\,dx$$
$$= \frac{\pi^2}{4} + 2x\cos x\Big|_0^{\pi/2} - \int_0^{\pi/2} 2\cos x\,dx = \frac{\pi^2}{4} - 2\sin x\Big|_0^{\pi/2} = \frac{\pi^2}{4} - 2.$$

As for *b*), we need to use the substitution $t = \tan x$. Then $dt = \sec^2 x$; if $x = 0$ then $t = 0$, and if $x = \pi/4$ then $t = 1$. Hence

$$\int_0^{\pi/4} \tan^5 x \sec^2 x\,dx = \int_0^1 t^5\,dt = \frac{t^6}{6}\Big|_0^1 = \frac{1}{6}.$$

4. Calculate the integral

$$\int \frac{1}{x^2\sqrt{x^2 - 1}}\,dx.$$

Solution. We will use the substitution $t = \operatorname{arcsec} x$. In this case we have $x = \sec t$, $dx = \sec t \tan t\,dt$ and $\sqrt{x^2 - 1} = \sqrt{\sec^2 t - 1} = \sqrt{\tan^2 t} = \tan t$; note that the right-hand side is $+\tan t$ and not $\pm\tan t$, since we take the nonnegative value of the square root, and, further, the choice of $t = \operatorname{arcsec} x$ means, by the definition of arcsec that t is either in the interval $[0, \pi/2)$ or in the one $[\pi, 3\pi/2)$, and $\tan t$ is nonnegative in either of these intervals. Thus,

$$I = \int \frac{1}{x^2\sqrt{x^2 - 1}}\,dx = \int \frac{1}{\sec^2 t \tan t} \sec t \tan t\,dt = \int \frac{1}{\sec t}\,dt = \int \cos t\,dt = \sin t + C.$$

To complete the solution of the problem, we need to express $\sin t$ in terms of x. We have

$$\sin t = \frac{\frac{\sin t}{\cos t}}{\frac{1}{\cos t}} = \frac{\tan t}{\sec t} = \frac{\sqrt{x^2 - 1}}{x}.$$

Hence

$$I = \frac{\sqrt{x^2 - 1}}{x} + C.$$

5. Calculate the integral

$$\int_0^1 \frac{2x^4 + 5x^2 + 1}{(x + 1)(x^2 + 1)^2}\,dx.$$

Solution. We need to use partial fraction decomposition. Note that the denominator of the integrand is already factored, since x^2+1, having no real zeros,[2] cannot be factored as a product of linear factors as long as we stay with real numbers. Hence we can readily write the form of the partial fraction decomposition as

$$\frac{2x^4+5x^2+1}{(x+1)(x^2+1)^2} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}.$$

Multiplying both sides by $(x+1)(x^2+1)^2$, we obtain

$$2x^4+5x^2+1 = A(x^2+1)^2 + (Bx+C)(x+1)(x^2+1) + (Dx+E)(x+1).$$

Note that while the former equation does not make sense for $x=1$ or $x=i$ $(=\sqrt{-1}\,)$, the latter equation holds even in these cases, since if two polynomials agree at infinitely many points, they must agree at every point (since, by rearranging the equation, we will get a polynomial equation with infinitely many solutions, and a polynomial that is not identically zero can only have finitely many zeros).

The coefficients A, B, C, D, and E are fairly easy to determine if one proceeds judiciously. In fact, substituting $x=-1$, we obtain $8=4A$, i.e., $A=2$. Then, comparing the coefficients of x^4 on both sides, we obtain the equation $2=A+B$, and so $B=0$. Substituting $x=i$, and noting that $i^2=-1$ and $i^4=1$, we obtain $-2=(Di+E)(i+1)$, i.e.,

$$-2 = (-D+E) + (D+E)i.$$

Since D and E are real, and for two complex numbers to agree, both the real and the imaginary parts must agree, we get the equations $-2- = -D+E$ and $0=D+E$. Hence $D=1$ and $E=-1$. We have yet to determine C. Substituting $x=0$ in the above equation, we obtain $1=A+C+E$. Since $A=2$ and $E=-1$, this implies that $C=0$. Thus,

$$\frac{2x^4+5x^2+1}{(x+1)(x^2+1)^2} = \frac{2}{x+1} + \frac{x-1}{(x^2+1)^2}.$$

Next, we need to determine the integrals of the fractions on the right-hand side. The integral of the first fraction is easy to determine by using the substitution $t=x+1$, $dt=dx$. We also need to substitute the limits: if $x=0$ then $t=1$ and if $x=1$ then $t=2$. Hence, writing I_1 for this integral, we have

$$I_1 = \int_0^1 \frac{2}{x+1}\,dx = \int_1^2 \frac{1}{t}\,dt = 2\ln t\,\Big|_{t=1}^{t=2} = 2\ln 2.$$

In calculating the integral of the second fraction, we need to split it up as the difference of two integrals $I_2 - I_3$, where

$$I_2 = \int_0^1 \frac{x}{(x^2+1)^2}\,dx \quad \text{and} \quad I_3 = \int_0^1 \frac{1}{(x^2+1)^2}\,dx.$$

The first integral is easy to determine by using the substitution x^2+1, when we have $dt = 2x\,dx$. Substituting the limits, for $x=0$ we have $t=1$ and for $x=1$ we have $t=2$, so

$$I_2 = \frac{1}{2}\int_0^1 \frac{2x}{(x^2+1)^2}\,dx = \frac{1}{2}\int_1^2 \frac{dt}{t^2} = -\frac{1}{2}\,t^{-1}\,\Big|_{t=1}^{t=2} = \frac{1}{4}.$$

In calculating I_3 we can use the trigonometric substitution $t=\arctan x$, i.e., $x=\tan t$. Then $dx = \sec^2 x\,dx$ and $1+x^2 = \sec^x$. As for the limits, when $x=0$, $t=\arctan 0 = 0$ and when $x=1$, $t=\arctan 1 = \pi/4$. Hence

$$\begin{aligned} I_3 &= \int_0^1 \frac{1}{(1+x^2)^2}\,dx = \int_0^{\pi/4} \frac{1}{(\sec^2 t)^2}\sec^2 t\,dt = \int_0^{\pi/4} \frac{1}{\sec^2 t}\,dt \\ &= \int_0^{\pi/4} \cos^2 t\,dt = \int_0^{\pi/4} \frac{1+\cos 2t}{2}\,dt. \end{aligned}$$

[2] An equation has solutions or roots, and a function has zeros. That is, the roots, or solutions, of the equation $f(x)=0$ are called the zeros of the function $f(x)$.

Using the substitution $u = 2t$, we have $du = 2\,dt$, we have $u = 0$ when $t = 0$ and $u = \pi/2$ when $t = \pi/4$, and so

$$I_3 = \frac{1}{4}\int_0^{\pi/2} (1 + \cos u)\,du = \frac{1}{4}\Big[u + \sin u\Big]_{u=0}^{u=\pi/2} = \frac{\pi}{8} + \frac{1}{4}.$$

Putting things together, we have

$$\int_0^1 \frac{2x^4 + 5x^2 + 1}{(x+1)(x^2+1)^2}\,dx = I_1 + I_2 - I_3 = 2\ln 2 - \frac{\pi}{8}.$$