Exam 2, Mathematics 1206, Section TY9 13 copies
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1. Calculate the following limits:

a) $\lim_{x\to+\infty} x(\ln(x+3)-\ln x)$, *b*) $\lim_{x\to+\infty}(xe^{1/x}-x)$.

Solution to Part *a*)**.** We have

$$\lim_{x\to\infty} x(\ln(x+3)-\ln x) = \lim_{x\to\infty}\frac{\ln(x+3)-\ln x}{1/x} = \lim_{x\to\infty}\frac{\frac{1}{x+3}-\frac{1}{x}}{-1/x^2} = \lim_{x\to\infty}\frac{\frac{-3}{(x+3)x}}{-1/x^2} = \lim_{x\to\infty}\frac{3x^2}{x^2+3x};$$

here the second equality was obtained by using l'Hospital's rule. The limit on the right-hand side can easily be established by dividing both the numerator and the denominator by x^2. Alternatively, one can use l'Hospital's rule twice:

$$\lim_{x\to\infty}\frac{3x^2}{x^2+3x} = \lim_{x\to\infty}\frac{6x}{2x+3} = \lim_{x\to\infty}\frac{6}{2} = 3.$$

Solution to Part *b*)**.** We have

$$\lim_{x\to+\infty}(xe^{1/x}-x) = \lim_{x\to+\infty}\frac{e^{1/x}-1}{1/x} = \lim_{x\to+\infty}\frac{-e^{1/x}/x^2}{-1/x^2}$$
$$= \lim_{x\to+\infty}\exp(1/x) = \exp\bigl(\lim_{x\to+\infty}(1/x)\bigr) = \exp(0) = 1;$$

we used the notation $\exp(t) = e^t$ above; the second equality follows by l'Hospital's rule, and the fourth equality holds in view of the following:

Lemma. *Let f and g be functions and L a real number such that $\lim g(x) = L$, and assume f is continuous at L. Then*

$$\lim f(g(x)) = f(L).$$

Here lim may mean any of $\lim_{x\to a}$, $\lim_{x\searrow a}$, $\lim_{x\nearrow a}$ for a real number a, or $\lim_{x\to+\infty}$, $\lim_{x\to-\infty}$, or $\lim_{x\to\pm\infty}$. The last displayed equation may be more easy to remember if it is written as

$$\lim f(g(x)) = f(\lim g(x));$$

this equation may be expressed in words by saying that the order of taking a limit and applying a continuous function can be interchanged.

2. Decide whether the following improper integrals are convergent or divergent. Give clear reasons for your answer (no credit will be given for a correct answer unless the correct reason is also given). Do not calculate the integrals.

a) $\displaystyle\int_2^{+\infty}\frac{dx}{(x+1)(\ln x)^2}$, *b*) $\displaystyle\int_0^{+\infty}\frac{dx}{\sqrt{x^2+1}}$, *c*) $\displaystyle\int_1^{+\infty}\frac{x\ln x\,dx}{x^3+1}$.

Solution. The answer to each of these questions can be given almost instantaneously by using the following:

[1] All computer processing for this manuscript was done under Fedora Linux. $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX was used for typesetting.

Limit Comparison Theorem. *Let a be a real number, and let f and g be real-valued continuous*[2] *functions on $[a, +\infty)$ such that $f(x) \geq 0$ and $g(x) \geq 0$ for all $x \in [a, +\infty)$. Assume that the limit* $\lim_{x\to+\infty} g(x)/f(x)$ *exist.*[3] *Assume, further, that the integral $\int_a^{+\infty} f$ is convergent. Then the integral $\int_a^{+\infty} g$ is also convergent.*

Instead of this, one could also use the following, but at the price of some extra effort:

Comparison Theorem. *Let a be a real number, and let f and g be real-valued continuous*[4] *functions on $[a, +\infty)$ such that $0 \leq g(x) \leq f(x)$ for all $x \in [a, +\infty)$. Assume, further, that the integral $\int_a^{+\infty} f$ is convergent. Then the integral $\int_a^{+\infty} g$ is also convergent.*

We will use the Limit Comparison Theorem in the solution.

Solution to Part a**).** Take

$$g(x) = \frac{1}{(x+1)(\ln x)^2} \quad \text{and} \quad f(x) = \frac{1}{x(\ln x)^2}.$$

As

$$\lim_{x\to+\infty} \frac{g(x)}{f(x)} = \lim_{x\to+\infty} \frac{x}{x+1} = 1,$$

and the integral

$$\int_2^{+\infty} f(x)\, dx = \int_2^{+\infty} \frac{1}{x(\ln x)^2}\, dx$$

is convergent, it follows that the integral in question is convergent.

Solution to Part b**).** Instead of considering the integral on the interval $[0, +\infty)$, we can consider it on the interval $[1, +\infty)$; this will have no effect on whether the integral is divergent or convergent. Take

$$f(x) = \frac{1}{\sqrt{x^2+1}} \quad \text{and} \quad g(x) = \frac{1}{x},$$

and assume that $\int_1^{+\infty} f$ is convergent. As

$$\lim_{x\to+\infty} \frac{g(x)}{f(x)} = \lim_{x\to+\infty} \frac{\sqrt{x^2+1}}{x} = 1,$$

it follows that $\int_1^{+\infty} g(x)\, dx = \int_1^{+\infty} x^{-1}\, dx$ is convergent. However, it is known that this integral is divergent; this is a contradiction, showing that the assumption that $\int_1^{+\infty} f$ is wrong. I.e., this integral is divergent.

Solution to Part c**).** Take

$$g(x) = \frac{x \ln x}{x^3+1} \quad \text{and} \quad f(x) = x^{-3/2}.$$

Then

$$\lim_{x\to+\infty} \frac{g(x)}{f(x)} = \lim_{x\to+\infty} \frac{\frac{x\ln x}{x^3+1}}{x^{-3/2}} = \lim_{x\to+\infty} \frac{x^{5/2}\ln x}{x^3+1} = \lim_{x\to+\infty} \frac{x^3}{x^3+1} \cdot \lim_{x\to+\infty} \frac{\ln x}{x^{1/2}} = 1 \cdot 0 = 0.$$

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[2]Continuity here is assumed only for the sake of simplicity. A weaker, but sufficient assumption would be that the integrals of f and g exist on any interval $[a, b]$ with $a < b < +\infty$.

[3]So, in particular, the limit is not $+\infty$. For the existence of the limit, it is essential that $g(x) \neq 0$ when x is large enough, but it is not required that $f(x) \neq 0$ for all $x \in [a, +\infty)$.

[4]Continuity here is assumed only for the sake of simplicity, similarly as in the Limit Comparison Theorem above.

Since $\int_1^{+\infty} f(x)\,dx = \int_1^{+\infty} x^{-3/2}\,dx$ is convergent, it follows that $\int_1^{+\infty} g$ is also convergent.

3. *a*) Decide whether the improper integral $\displaystyle\int_0^{1/2} \frac{dx}{x(\ln x)^2}$ is convergent. If it is convergent, calculate the integral.

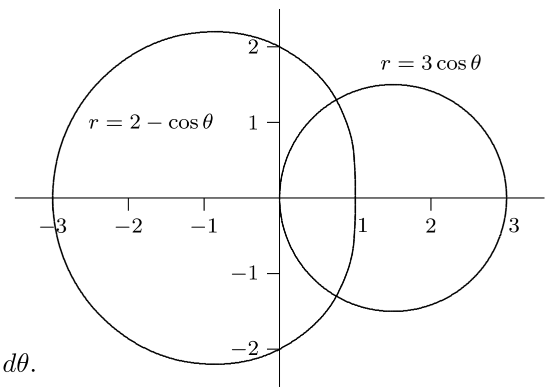
Solution. We have

$$\int_0^{1/2} \frac{dx}{x(\ln x)^2} = \lim_{\epsilon\searrow 0} \int_\epsilon^{1/2} \frac{1}{(\ln x)^2}\cdot\frac{dx}{x} = \lim_{\epsilon\searrow 0} \int_{\ln\epsilon}^{\ln(1/2)} \frac{1}{t^2}\,dt = \lim_{A\to-\infty} \int_A^{-\ln 2} \frac{1}{t^2}\,dt;$$

Here the second equality follows by substituting $t = \ln x$, when $dt = dx/x$,[5] and the third equality follows since we have $\lim_{\epsilon\searrow 0} \ln\epsilon = -\infty$. The expression on the right-hand side equals

$$-\lim_{A\to-\infty} \frac{1}{t}\bigg|_A^{-\ln 2} = -\lim_{A\to-\infty}\left(\frac{1}{-\ln 2} - \frac{1}{A}\right) = \frac{1}{\ln 2}.$$

b) Write the integral that finds the area that is the inside the curve $r = 3\cos\theta$ (given in polar coordinates) and outside the curve $r = 2 - \cos\theta$. *Do not evaluate the integral!*



Solution. The graphs of the two curves are given in the diagram. Two find the points of intersection, we solve the simultaneous equations $r = 3\cos\theta$ and $r = 2 - \cos\theta$. Equating the right-hand sides gives $3\cos\theta = 2 - \cos\theta$, that is $\cos\theta = 1/2$. This gives $\theta = \pm\pi/3 + 2k\pi$ ($k = 0, \pm1, \pm2, \pm3, \dots$).[6] We of can of course take $k = 0$ here, since taking a different values of k do not represent different points. Using the formula $A = \frac{1}{2}\int_a^b r^2\,d\theta$, the area inside the circle $r = 3\cos\theta$ between $\theta = -\pi/3$ and $\theta = \pi/3$ is

$$\int_{-\pi/3}^{\pi/3} \frac{1}{2}(3\cos\theta)^2\,d\theta.$$

The area inside the curve $r = 2 - \cos\theta$ between $\theta = -\pi/3$ and $\theta = \pi/3$ is

$$\int_{-\pi/3}^{\pi/3} \frac{1}{2}(2 - \cos\theta)^2\,d\theta.$$

The area we seek is the first area minus the second area, that is

$$\frac{1}{2}\int_{-\pi/3}^{\pi/3} \Big((3\cos\theta)^2 - (2-\cos\theta)^2\Big)\,d\theta = \int_0^{\pi/3} \Big((3\cos\theta)^2 - (2-\cos\theta)^2\Big)\,d\theta.$$

[5] One may be inclined to make this substitution into the integral on the left. It needs to be remembered, however, that we have no substitution rule for improper integrals. While such a rule could be formulated, one would need to carefully consider the conditions under which such a rule is valid, and it does not seem to be worth while to engage in such considerations, since we will need to use the substitution rule involving improper integrals only a few times. It seems much simpler to first convert the improper integral to a limit of Riemann integrals, and then use the substitution rule for integrals.

[6] The points of intersection that these values of θ give are $(1.5, -\pi/3 + 2k\pi)$ and $(1.5, \pi/3 + 2k\pi)$. Replacing (r,θ) with $(r, \theta + 2k\pi)$ in the second equation will of course not produce any changes. Replacing (r,θ) with $(-r, \theta + (2k+1)\pi)$ in the second equation will result in the pair of equations $r = 3\cos\theta$ and $-r = 2 + \cos\theta$ (we did not make any change in the first equations). The have the solutions $(-1.5, \pm 2\pi/3 + 2k\pi)$ for (r,θ), but these coordinates represent the same points as above.

The equation here holds because we have

$$\int_{-\pi/3}^{0}\Big((3\cos\theta)^2-(2-\cos\theta)^2\Big)\,d\theta=\int_{0}^{\pi/3}\Big((3\cos\theta)^2-(2-\cos\theta)^2\Big)\,d\theta;$$

this can be seen formally by making the substitution $t=-\theta$ in the integral on the left, and then interchanging the limits, and finally changing the t back to θ. Informally, it can be seen by noticing that the region is symmetric about the x axis, so the area below the x axis is the same as the area above the x axis.

4. Decide whether or not each of the following sequences is convergent. Give reasons for your answers. If the given sequence is convergent, find its limit.

a) $a_n=\dfrac{n-2}{n+3}$, *b)* $a_n=\dfrac{3^n-n}{3^n+n^2}$, *c)* $a_n=\sin n\pi$,

d) $a_n=\dfrac{\ln n}{n}$, *e)* $a_n=\dfrac{\sin n}{n^2}$, *f)* $a_n=(-1)^n\dfrac{n-1}{n+1}$.

Solution to Part *a)***.** We have

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{n-2}{n+3}=\lim_{n\to\infty}\frac{1-2/n}{1+3/n}=\frac{\lim_{n\to\infty}(1-2/n)}{\lim_{n\to\infty}(1+3/n)}=\frac{1}{1}=1.$$

Solution to Part *b)***.** First note that

(1) $$\lim_{n\to\infty}3^{-n}n=\lim_{n\to\infty}3^{-n}n^2=0.$$

Indeed, we have

(2) $$\lim_{x\to+\infty}e^{-x}x=\lim_{x\to+\infty}\frac{x}{e^x}=\lim_{x\to+\infty}\frac{1}{e^x},$$

where to obtain the last equality we used l'Hospital's rule. Hence, for any positive α and β we have

$$\begin{aligned}\lim_{x^\alpha\to+\infty}e^{-\beta x}x^\alpha&=\lim_{x\to+\infty}\left(e^{-\beta x/\alpha}x\right)^\alpha=\left(\lim_{x\to+\infty}e^{-\beta x/\alpha}x\right)^\alpha\\&=\left(\frac{\alpha}{\beta}\right)^\alpha\left(\lim_{x\to+\infty}e^{-\beta x/\alpha}\,\frac{\beta x}{\alpha}\right)^\alpha=\left(\frac{\alpha}{\beta}\right)^\alpha\left(\lim_{t\to+\infty}e^{-t}t)\right)^\alpha=0;\end{aligned}$$

here the penultimate[7] equality was obtained by making the substitution $t=\beta x/\alpha$ (since if $x\to\infty$ then clearly $t=\beta x/\alpha\to\infty$), and the last equality hosts according to (2). According to (1) we have

$$\lim_{x\to+\infty}3^{-x}x=\lim_{x\to+\infty}e^{-x\ln 3}x=0$$

and

$$\lim_{x\to+\infty}3^{-x}x^2=\lim_{x\to+\infty}e^{-x\ln 3}x^2=0.$$

[7] The one before the last one.

Hence (1) holds *a fortiori*.[8] Using (1), it is easy to see that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3^n - n}{3^n + n^2} = \lim_{n\to\infty} \frac{1 - 3^{-n}n}{1 + 3^{-n}n^2} = 1;$$

we omitted here some simple steps, such as the application of the quotient rule for limits, similarly to the way it was done in Part *a*).

Solution to Part *c*)**.** For any positive integer n we have $a_n = \sin n\pi = 0$, hence $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sin n\pi = \lim_{n\to\infty} 0 = 0$. On the other hand, note that the limit $\lim_{x\to+\infty} \sin \pi x$ does not exist. The difference is that in the former limit n is assumed to be an integer, while in the latter limit, x is not assumed to be an integer. For any real c, the expression $\sin \pi x$ assumes every value in the interval $[-1, 1]$ when x changes in the interval $[c - 1/2, c + 1/2]$.

Solution to Part *d*)**.** We have

$$\lim_{x\to+\infty} \frac{\ln x}{x} = \lim_{x\to+\infty} \frac{1/x}{1} = 0,$$

where we used l'Hospital's rule to obtain the first equality. Hence, a fortiori,

$$a_n = \lim_{n\to+\infty} \frac{\ln n}{n} = 0.$$

Solution to Part *e*)**.** We have $|\sin n| \le 1$,[9] and so $|a_n| \le 1/n^2$, and so $a_n \to 0$.

Solution to Part *f*)**.** We have

$$\lim_{n\to\infty} \frac{n-1}{n+1} = \lim_{n\to\infty} \frac{1 - 1/n}{1 + 1/n} = 1.$$

As $(-1)^n$ equals to 1 for even n and to -1 for odd n, this means that for large n, a_n is close to 1 if n is even and is close to -1 if n is odd. Therefore the sequence $\{a_n\}_{n=1}^\infty$ is divergent.

5. *a*) Decide whether the sum $\displaystyle\sum_{n=1}^\infty \frac{8^{n+1}}{9^n}$ is convergent. If it is convergent, find its limit.

Solution. The series is a convergent geometric series. To find its sum, we will start with the substitution $k = n - 1$, when $n = k + 1$ and k runs from 0 to ∞. We have

$$\sum_{n=1}^\infty \frac{8^{n+1}}{9^n} = \sum_{k=0}^\infty \frac{8^{k+2}}{9^{k+1}} = \sum_{k=0}^\infty \frac{8^2}{9} \cdot \frac{8^k}{9^k} = \frac{8^2}{9} \sum_{k=0}^\infty \left(\frac{8}{9}\right)^k = \frac{8^2}{9} \cdot \frac{1}{1 - 8/9} = \frac{8^2}{9 - 8} = 64;$$

to obtain the fourth equality, we used the sum formula

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad (|x| < 1)$$

[8] Latin for "even more so." That is, for example if the expression $3^{-x}x$ is close to 0 when x is a large real number, the same is certainly true when $x = n$ is a large positive integer. Note that in deriving (1) we first had to derive (2) since we needed to use l'Hospital's rule.

[9] Actually, equality will never hold when n is an integer, since $\sin x = \pm 1$ only if x equals $(k + 1/2)\pi$ for some integer k, and $(k + 1/2)\pi$ is never an integer, since π is irrational.

of the infinite geometric series.

b) Decide whether the sum $\sum_{n=3}^{\infty} \frac{1}{n(n+2)}$ is convergent. If it is convergent, find its limit.

Solution. The sum is a convergent telescoping sum. Indeed, we have

$$\sum_{n=3}^{\infty} \frac{1}{n(n+2)} = \sum_{n=3}^{\infty} \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right) = \frac{1}{2}\sum_{n=3}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right)$$
$$\frac{1}{2}\left(\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\left(\frac{1}{6}-\frac{1}{8}\right)+\left(\frac{1}{7}-\frac{1}{9}\right)+\left(\frac{1}{8}-\frac{1}{10}\right)+\dots\right)$$
$$= \frac{1}{2}\left(\frac{1}{3}+\frac{1}{4}\right) = \frac{1}{2}\cdot\frac{7}{12} = \frac{7}{24};$$

note that in the sum in parentheses before the third equality, every fraction $1/n$ for $n \geq 5$ occurs once with a positive sign, and once with a negative sign – thus all these fractions are canceled. The only ones that remain are $1/3$ and $1/4$; this explains the third equality.