Exam 3, Mathematics 1206, Section TY9 13 copies
9:05 am–11:40 pm, November 30, 2023, 1005-IH
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1. Decide whether each of the following series is convergent. Give reasons for your answers. No credit will be given for yes or no answers without proper explanations.

a) $\displaystyle\sum_{n=1}^{\infty}\frac{n\ln n}{n^2+1}$ *b)* $\displaystyle\sum_{n=2}^{\infty}\frac{n^2-1}{n^4+2}$ *c)* $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt[3]{n-1}}{\sqrt{n^3+1}}$

Solution. For each of these problems we can use the following:

Limit Comparison Theorem. *Let $a_n \ge 0$ and $b_n \ge 0$ for all $n \ge 1$, and assume that the limit $\lim_{n\to\infty} a_n/b_n$ exists. Assume, further, that the series $\sum_{n=1}^{\infty} b_n$ converges. Then $\sum_{n=1}^{\infty} a_n$ also converges.*

Note that when we say when a limit is infinite, the limit is also said to not exist. So, by saying above that the limit $\lim_{n\to\infty} a_n/b_n$ exists we also mean that this limit is finite. Further, the theorem remains true even if we omit the assumption that $a_n \ge 0$ for all $n \ge 1$; this assumption is only included since it was convenient to make it at the time the theorem was first discussed. On the other hand, the assumption that $b_n \ge 0$ for all $n \ge 1$ cannot be omitted.

Solution to part *a)***.** The series is divergent. To see this, assume that it is convergent, and use the Limit Comparison Theorem with

$$a_n = \frac{1}{n} \quad\text{and}\quad b_n = \frac{n\ln n}{n^2+1}.$$

Then we have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{n\ln n}{n^2+1}} = \lim_{n\to\infty}\frac{n^2+1}{n^2\ln n} = 0.$$

Hence the assumption that the series $\sum_{n=1}^{\infty}\frac{n\ln n}{n^2+1}$ is convergent implies that the series $\sum_{n=1}^{\infty}\frac{1}{n}$ also converges. However, we know that the latter series diverges. Hence the assumption that the former series converges cannot be correct; that is, the series $\sum_{n=1}^{\infty}\frac{n\ln n}{n^2+1}$ diverges.

Solution to part *b)***.** The series converges. To use the Limit Comparison Theorem, take

$$a_n = \frac{n^2-1}{n^4+2} \quad\text{and}\quad b_n = \frac{1}{n^2}.$$

We have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{n^2-1}{n^4+2}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{n^2(n^2-1)}{n^4+2} = 1.$$

Since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, it follows that $\sum_{n=1}^{\infty}\frac{n^2-1}{n^4+2}$ also converges.

Solution to part *c)***.** The series is convergent. To use the Limit Comparison Theorem, take

$$a_n = \frac{\sqrt[3]{n-1}}{\sqrt{n^3+1}} \quad\text{and}\quad b_n = \frac{\sqrt[3]{n}}{\sqrt{n^3}} = \frac{n^{1/3}}{n^{3/2}} = n^{1/3-3/2} = n^{-7/6}.$$

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[1] All computer processing for this manuscript was done under Fedora Linux. $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX was used for typesetting.

The series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} n^{-7/6}$ is convergent. Furthermore,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt[3]{n-1}/\sqrt[3]{n}}{\sqrt{n^3+1}/\sqrt{n^3}} = \frac{\lim_{n\to\infty} \sqrt[3]{n-1}/\sqrt[3]{n}}{\lim_{n\to\infty} \sqrt{n^3+1}/\sqrt{n^3}} = \frac{1}{1} = 1.$$

Hence it follows that $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n-1}}{\sqrt{n^3+1}}$ is also convergent.

2. Decide whether each of the following series is absolutely convergent, conditionally convergent, or divergent. Give reasons for your answer. No credit will be given for a correct answer without a correct explanation.

$$a)\quad \sum_{n=2}^{\infty} (-1)^{n-1} \frac{\ln n}{n}, \qquad b)\quad \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^2+\ln n}, \qquad c)\quad \sum_{n=2}^{\infty} (-1)^{n-1} \frac{n-1}{n^2(\ln n)^2}.$$

Solution to part *a*)**.** The series is conditionally convergent. Indeed,

$$\frac{d}{dx}\frac{\ln x}{x} = \frac{\frac{1}{x}\cdot x - (\ln x)\cdot 1}{x^2} = \frac{1-\ln x}{x^2} < 0$$

for $x > e$, showing that $\frac{\ln n}{n}$ is decreasing for $n \ge 3\,(> e)$. Clearly, $\lim_{n\to\infty} \frac{\ln n}{n} = 0$ showing that the series is a convergent alternating series. The limit comparison test shows that the series is not absolutely convergent. Indeed, taking

$$b_n = \frac{\ln n}{n} \quad \text{and} \quad a_n = \frac{1}{n},$$

we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{\ln n}{n}} = \lim_{n\to\infty} \frac{1}{\ln n} = 0.$$

Hence, assuming that $\sum_{n=2}^{\infty} \frac{\ln n}{n}$ is convergent, it follows that $\sum_{n=2}^{\infty} \frac{1}{n}$ is also convergent. The latter series, however, is divergent; hence the former series must also be divergent.

Solution to part *b*)**.** The series is divergent. This is because

$$\lim_{n\to\infty} \frac{n^2}{n^2+\ln n} = \lim_{n\to\infty} \frac{1}{1+\frac{\ln n}{n}} = 1,$$

and in order for a series to converge its general term must tend to 0.

Solution to part *c*)**.** The series is absolutely convergent. Using the Limit Comparison Test, take

$$a_n = \frac{n-1}{n^2(\ln n)^2} \quad \text{and} \quad b_n = \frac{1}{n(\ln n)^2}.$$

We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n-1}{n^2(\ln n)^2}}{\frac{1}{n(\ln n)^2}} = \lim_{n\to\infty} \frac{n-1}{n} = 1.$$

Since the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ is easily seen to converge by the integral test, it follows that the series $\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{n-1}{n^2(\ln n)^2}$ is also convergent.

3. Find the interval of convergence of the following power series. For each endpoint of the interval of convergence, decide whether the series is absolutely convergent, conditionally convergent, or divergent.

$$a) \quad \sum_{n=1}^{\infty} \frac{1}{n3^n}(x+4)^n, \qquad b) \quad \sum_{n=1}^{\infty} \frac{(2x-7)^n}{5^n\sqrt{n}}, \qquad c) \quad \sum_{n=1}^{\infty} n!(x-1)^n.$$

Solution, general comments. Given a power series

$$\sum_{n=0}^{\infty} c_n(x-a)^n,$$

and using the ratio test, this series is convergent if the limit

$$\lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = \lim_{n\to\infty}\frac{|c_{n+1}|}{|c_n|}\,|x-a| = |x-a|\lim_{n\to\infty}\frac{|c_{n+1}|}{|c_n|}$$

is less than 1, and it is divergent if this limit is greater than 1. That is, writing

$$L = \lim_{n\to\infty}\frac{|c_{n+1}|}{|c_n|},$$

the series is convergent if $L|x-a| < 1$, and it is divergent if $L|x-a| > 1$. Thus, assuming $L \neq 0$, the series is convergent if $|x-a| < 1/L$, and it divergent if $|x-a| > 1/L$; in other words, the radius of convergence is $1/L$. If $L = 0$ then the series is convergent for every x, i.e., $R = +\infty$, and if $L = +\infty$ then the series is never convergent unless $x = a$; so $R = 0$ in this case. Therefore, we can write that

$$\frac{1}{R} = \lim_{n\to\infty}\frac{|c_{n+1}|}{|c_n|},$$

with the understanding that if the limit is 0 then $R = +\infty$ and if this limit is $+\infty$ then $R = 0$.

Instead of the ratio test, we can also use the root test, showing that the above series is convergent if

$$\lim_{n\to\infty}\sqrt[n]{|c_n(x-a)^n|} = \lim_{n\to\infty}\sqrt[n]{|c_n|}\,|x-a| = |x-a|\lim_{n\to\infty}\sqrt[n]{|c_n|}$$

is less than 1, and it is divergent if this limit is greater than 1. That is, writing

$$L = \lim_{n\to\infty}\sqrt[n]{|c_n|},$$

the series is convergent if $L|x-a| < 1$, and it is divergent if $L|x-a| > 1$. Hence, assuming $L \neq 0$, the series is convergent if $|x-a| < 1/L$, and it is divergent if $|x-a| > 1/L$; in other words, the radius of convergence is $1/L$. If $L = 0$ then the series is convergent for every x, i.e., $R = +\infty$, and if $L = +\infty$ then the series is never convergent unless $x = a$; so $R = 0$ in this case. So we can write that

$$\frac{1}{R} = \lim_{n\to\infty}\sqrt[n]{|c_n|}$$

with the understanding that if the limit is 0 then $R = +\infty$ and if this limit is $+\infty$ then $R = 0$. Either of the above formulas are applicable to find the radius of convergence, though there are cases when one or the other limit does not exist. The second formula can be generalized to a slightly more complicated formula that always correctly gives the radius of convergence, but that formula is not discussed in the present course.

Solution to Part *a*)**.** Writing

$$c_n = \frac{1}{n3^n},$$

for the radius of convergence R of the series $\sum_{n=1}^{\infty} c_n t^n$, we have

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty}\frac{\frac{1}{(n+1)3^{n+1}}}{\frac{1}{n3^n}} = \lim_{n\to\infty}\frac{n3^n}{(n+1)3^{n+1}} = \frac{1}{3}.$$

Thus the series

$$\sum_{n=1}^{\infty}\frac{1}{n3^n}t^n$$

has radius of convergence is 3; that is, it is absolutely convergent in the interval $(-3, 3)$. As for the endpoints of this interval, for $t = 3$, this series becomes

$$\sum_{n=1}^{\infty}\frac{1}{n3^n}3^n = \sum_{n=1}^{\infty}\frac{1}{n},$$

and this series (called the harmonic series) is divergent. For $t = -3$, it becomes

$$\sum_{n=1}^{\infty}\frac{1}{n3^n}(-3)^n = \sum_{n=1}^{\infty}(-1)^n\frac{1}{n},$$

which is a conditionally convergent alternating series. Thus, the above series is absolutely convergent when $-3 < t < 3$, conditionally convergent when $t = -3$, and divergent otherwise. With $t = x + 4$ (or $x = t - 4$), this implies that the series

$$\sum_{n=1}^{\infty}\frac{1}{n3^n}(x+4)^n$$

is absolutely convergent when $-3 < x + 4 < 3$, i.e., when $-7 < x < -1$, and conditionally convergent when $x + 4 = -3$, i.e, when $x = -7$, and divergent otherwise.

Solution to Part *b*)**.** Writing

$$c_n = \frac{1}{5^n\sqrt{n}},$$

for the radius of convergence R of the series $\sum_{n=1}^{\infty} c_n t^n$, we have

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty}\frac{\frac{1}{5^{n+1}\sqrt{n+1}}}{\frac{1}{5^n\sqrt{n}}} = \lim_{n\to\infty}\frac{5^n\sqrt{n}}{5^{n+1}\sqrt{n+1}} = \lim_{n\to\infty}\frac{1}{5}\sqrt{\frac{n}{n+1}} = \frac{1}{5}.$$

So the series

$$\sum_{n=1}^{\infty}\frac{1}{5^n\sqrt{n}}t^n$$

has radius of convergence 5. That is, it is absolutely convergent when $-5 < t < 5$. For $t = 5$ it becomes

$$\sum_{n=1}^{\infty} \frac{1}{5^n \sqrt{n}} 5^n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}},$$

and this series is divergent by the integral test. For $t = -5$, the series becomes

$$\sum_{n=1}^{\infty} \frac{1}{5^n \sqrt{n}} (-5)^n = \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}},$$

which is a convergent alternating series. It is not absolutely convergent, since the series formed by the absolute values is the same as the divergent series considered just before (namely, the power series at $t = 5$). Thus, the series is conditionally convergent for $t = -5$. The question of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(2x-7)^n}{5^n \sqrt{n}}$$

reduces to that of the above series with $t = 2x - 7$. Therefore, this series is absolutely convergent when $-5 < 2x - 7 < 5$, i.e., when $2 < 2x < 12$, that is, when $1 < x < 6$. It is conditionally convergent when $2x - 7 = -5$, i.e., when $x = 1$, and it is divergent otherwise.

Solution to Part *c*)**.** Writing $c_n = n!$, for the radius of convergence R of the series

$$\sum_{n=1}^{\infty} c_n t^n$$

we have

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} n = +\infty.$$

That is, the radius of convergence of the above series is zero. I.e., the series

$$\sum_{n=1}^{\infty} n! t^n$$

converges only for $t = 0$, and it diverges for any every value of t. Writing $t = x - 1$ (i.e., $x = t + 1$), we can conclude that the series

$$\sum_{n=1}^{\infty} n!(x-1)^n$$

converges only when $x - 1 = 0$, i.e., when $x = 1$, and it diverges for every other value of x.

4. *a*) Find the interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{x^n}{5^n n^2}.$$

Be sure to decide whether the series is absolutely convergent, conditionally convergent, or divergent at each endpoint of the interval of convergence.

Solution. Writing $a_n = \frac{x^n}{5^n n^2}$, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{5^{n+1}(n+1)^2}}{\frac{x^n}{5^n n^2}}\right| = \lim_{n\to\infty}\frac{|x|n^2}{5(n+1)^2} = \frac{|x|}{5}.$$

According to the ratio test, the series is convergent when this limit is < 1, i.e., when $|x| < 5$, and it is divergent when this limit is > 1, i.e., when $|x| > 5$. For $x = 5$ the series becomes $\sum_{n=1}^{\infty}\frac{1}{n^2}$, which is convergent by the integral test. For $x = -5$ it becomes $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$, and this series is absolutely convergent, since the series formed by the absolute values of its terms is identical to the former series.

b) Find the first four terms of the Taylor series of $1/x$ at $x = 1$. That is, find the terms involving 1, $x-1$, $(x-1)^2$, $(x-1)^3$ in the Taylor expansion

$$\frac{1}{x} \sim \sum_{n=0}^{\infty} c_n (x-1)^n.$$

First solution. Writing $f(x) = 1/x = x^{-1}$, we have $f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$, $f'''(x) = -2\cdot 3x^{-3}$, ..., $f^{(n)}(x) = (-1)^n n! x^{-n-1}$, ... Hence $f^{(n)}(1) = (-1)^n n!$. Thus, the Taylor expansion of $f(x)$ is

$$\frac{1}{x} \sim \sum_{n=1}^{\infty}\frac{(-1)^n n!}{n!}(x-1)^n = \sum_{n=1}^{\infty}(-1)^n(x-1)^n = 1-(x-1)+(x-1)^2-(x-1)^3+\dots .$$

Second solution. We have the geometric series

$$\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$$

for $|y| < 1$. With $x = 1 - y$, when $y = 1 - x$, this becomes

$$\frac{1}{x} = \sum_{n=0}^{\infty}(1-x)^n = \sum_{n=0}^{\infty}((-1)(x-1))^n = \sum_{n=0}^{\infty}(-1)^n(x-1)^n,$$

valid for x with $|1-x| < 1$, i.e., with $-1 < x < 1$. Since a power series representing a function must be the Taylor series of this function, we have obtained the Taylor series of of $1/x$ centered at $x = 1$. The advantage of this approach is that it shows that the series obtained is in fact equal to $1/x$ in its region of convergence, whereas to see this with the aid of the first solution, we would have to examine the remainder term of the Taylor series.

5. *a*) Use the Maclaurin series $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ to write the Maclaurin series of $\int_0^x e^{-t^2}\,dt$.

Solution. We have

$$\int_0^x e^{-t^2}\,dt = \int_0^x \sum_{n=0}^{\infty}\frac{(-t^2)^n}{n!}\,dt = \sum_{n=0}^{\infty}\int_0^x \frac{1}{n!}\big((-1)t^2\big)^n\,dt$$
$$= \sum_{n=0}^{\infty}\int_0^x \frac{1}{n!}(-1)^n t^{2n}\,dt = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!(2n+1)}.$$

b) Using the results of part *a*) of this problem, find $\int_0^{1/4} e^{-x^2}\,dx$ with four decimal accuracy.

Solution. By convention, four decimal accuracy means that the absolute value of the error is less than $\frac{1}{2}\cdot 10^{-4}$.[2] Writing out the first few terms of the expansion of $\int_0^x e^{-t^2}\,dt$ for $x = 1/4$, we have

$$\begin{aligned}\int_0^{1/4} e^{-x^2}\,dx &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)4^{2n+1}} \\ &= \frac{1}{0!\cdot 1\cdot 4^1} - \frac{1}{1!\cdot 3\cdot 4^3} + \frac{1}{2!\cdot 5\cdot 4^5} - \frac{1}{3!\cdot 7\cdot 4^7} + \ldots = \frac{1}{4} - \frac{1}{192} + \frac{1}{10240} - \frac{1}{688128} + \ldots\end{aligned}$$

The series whose sum we are trying to estimate the sum of is an alternating series. We can approximate the sum of a series by calculating a partial sum, but to know how good the approximation is, we need to estimate the error we get by truncating (i.e., taking only a partial sum rather than the whole sum) the series. In case of an alternating series this is the simple matter: the error of a partial is between 0 and the first term we fail to add. That is, the absolute value of the error is less than the absolute value of the first term we fail to add. In this case, we want the error to be less than $\frac{1}{2}\cdot 10^{-4} = \frac{1}{20000}$. The first term in the series whose absolute value is less than this is

$$-\frac{1}{3!\cdot 7\cdot 4^7} = -\frac{1}{688128}.$$

This term does not need to be added. Thus, we obtain and approximation if we add only the first three terms. That is

$$\begin{aligned}\int_0^{1/4} e^{-x^2}\,dx &\approx \sum_{n=0}^{2} \frac{(-1)^n}{n!(2n+1)4^{2n+1}}\,dt \\ &= \frac{1}{0!\cdot 1\cdot 4^1} - \frac{1}{1!\cdot 3\cdot 4^3} + \frac{1}{2!\cdot 5\cdot 4^5} = \frac{1}{4} - \frac{1}{192} + \frac{1}{10240} \approx 0.244,889,322\ldots \approx 0.244,9.\end{aligned}$$

A more accurate value of the integral is $0.244,888,117,156,17$.

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[2] It cannot possibly mean that the first four decimals correctly give the value if rounded to four decimal places. For example, .00015000001 rounds to .0002, while .00014999999 rounds to .0001, whereas $.00015000001 - .00014999999 = 2\cdot 10^{-11}$.