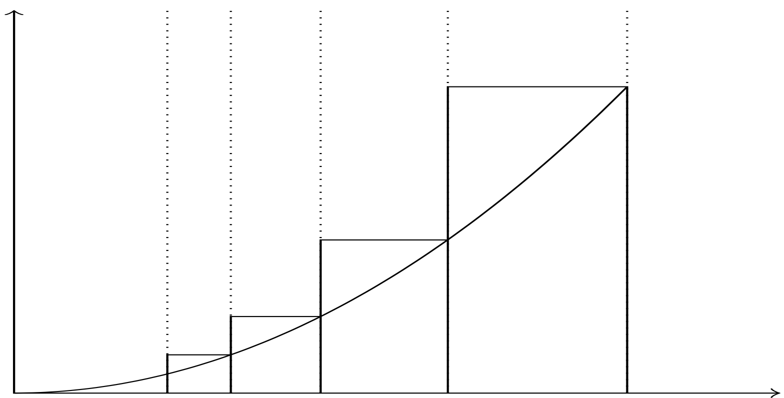
Fermat and Integration

Jeff Suzuki, Brooklyn College
(jsuzukibrooklyn.cuny.edu)

1 Preliminaries

Fermat's method of integration is based on partitioning an interval at points forming a geometric progression.



We'll need the finite geometric series summation formula:

$$1 + x + x^2 + \ldots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

Note that this is true for any $x \neq 1$, and we also have

$$\frac{1}{1 + x + x^2 + \ldots + x^n} = \frac{1 - x}{1 - x^{n+1}}$$

2 Paraboloids, Part One

Consider the region bound by $x = 1$, $x = b$ (where we assume $b > 1$), the x-axis, and the graph of $y = x^2$. Let n be given, and find r so that $r^n = b$. Let the interval $1 \leq x \leq b$ be partitioned at

$$x = 1, r, r^2, \ldots, r^n = b$$

1. Construct the upper rectangles and complete the following.

Rectangle	Height	Width	Area
1			
2			
3			
$\vdots$			
n			

2. Find the sum of the areas of the n upper rectangles.

3. Find the limit of this sum as $n \to \infty$.

3 Paraboloids, Part Two

Fermat also considered the case where n was rational (for example, the area bound by $y = \sqrt{x}$. Consider the region bound by $x = 1$, $x = b$ (where we assume $b > 1$), the x-axis, and the graph of $y = \sqrt{x}$. Let n be given, and find r so that $r^{2n} = b$. Let the interval $1 \leq x \leq b$ be partitioned at

$$x = 1, r^2, r^4, \ldots, r^{2n} = b$$

1. Construct the upper rectangles and complete the following.

Rectangle	Height	Width	Area
1			
2			
3			
$\vdots$			
n			

2. Find the sum of the areas of the n upper rectangles.

3. Find the limit of this sum as $n \to \infty$.

4 Hyperboloids

Now consider the area under $y = \dfrac{1}{x^3}$ and above the x-axis over the interval $1 \le x \le b$. As before, choose n and find r where $r^n = b$, then partition the interval at

$$x = 1, r, r^2, \ldots, r^n = b$$

1. Construct the upper rectangles and complete the following.

Rectangle	Height	Width	Area
1			
2			
3			
⋮			
n			

2. Find the sum of the areas of the n upper rectangles.

3. Find the limit as $n \to \infty$.

5 The Troublesome Hyperbola

What if $y = \frac{1}{x}$? As before, partition the x-axis at

$$x = 1, r, r^2, \ldots, r^n = b$$

1. Construct the upper rectangles and complete the following.

Rectangle	Height	Width	Area
1			
2			
3			
$\vdots$			
n			

2. Find the sum of all the areas.

6 The Natural Log

Now consider the region between $y = \frac{1}{x}$ and the x-axis over $1 \leq x \leq ab$, where $1 < b < a$.

1. Show that the region over $1 \leq x \leq b$ has the same area as the region over $a \leq x \leq ab$. (Hint: Compare the upper rectangles)

2. Let $L(t)$ be the area under $y = \frac{1}{x}$ over the interval $1 \leq x \leq t$. Show that $L(ab) = L(a) + L(b)$.

3. What type of function is $L(t)$?

7 Introducing e

1. By using upper and lower rectangles, show that

$$\frac{n}{r}(r-1) < \ln b < n(r-1)$$

2. Show that for $n > 1$,

$$\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n-1}\right)^n$$

3. Show the squeeze theorem applies.