1. a) Find the directional derivative of \( f(x, y) = x^2 + y^2 \) at the point (2, -1) in the direction of the vector \( \mathbf{u} = \mathbf{v}/|\mathbf{v}| \), where \( \mathbf{v} = (3, 4) \).

**Solution.** We have

\[
\nabla f(x, y) = 2xi + 3y^2j \quad \text{and so} \quad \nabla f(2, -1) = 4i + 3j.
\]

We have \( \mathbf{u} = \mathbf{v}/|\mathbf{v}| = (3i + 4j)/5 \), and

\[
D_u f(2, -1) = \mathbf{u} \cdot \nabla f(2, -1) = \frac{3i + 4j}{5} \cdot (4i + 3j) = \frac{24}{5}.
\]

b) Find the unit vector in whose direction the directional derivative of the function \( f(x, y) \) in part a) is the largest at the point (2, -1).

**Solution.** The directional derivative is largest in the direction of the gradient vector. The unit vector in this direction is

\[
\frac{\nabla f(2, -1)}{|\nabla f(2, -1)|} = (4/5)i + (3/5)j.
\]

2. a) Find the places of local maxima and local minima of \( f(x, y) = x^2 + y^2 - 6x - 8y \).

**Solution.** To find the critical points, we need to solve the system of the equations \( f_x(x, y) = 0, f_y(x, y) = 0 \), i.e., of the equations \( 2x - 6 = 0, 2y - 8 = 0 \). We obtain the solution \( (x, y) = (3, 4) \). This is indeed a place of local extremum according to the second derivative test, since \( f_{xx}(x, y) = 2, f_{yy}(x, y) = 2, f_{xy}(x, y) = 0 \), and so \( f_{xx}(3, 4)f_{yy}(3, 4) - (f_{xy}(3, 4))^2 = 4 > 0 \). As \( f_{xx}(3, 4) > 0 \), this point is a local minimum.

b) Find the places and values of the absolute maximum and absolute minimum of the function \( f(x, y) \) in part a) on the circle \( x^2 + y^2 = 100 \) using the method of Lagrange multipliers.

**Solution.** Writing \( g(x, y) = x^2 + y^2 - 100 \), we need find the extrema of \( f(x, y) \) under the constraint \( g(x, y) = 0 \). Using the method of Lagrange multipliers and writing \( F(x, y, \lambda) = f(x, y) + \lambda g(x, y) \), we need to solve the system of equations \( F_x(x, y, \lambda) = 0, F_y(x, y, \lambda) = 0, g(x, y) = 0 \) for \( x, y \), and \( \lambda \) to obtain the possible places \( (x, y) \) of extrema. That is, we need to solve the equations \( 2x(1+\lambda) - 6 = 0, 2y(1+\lambda) - 8 = 0 \), and \( x^2 + y^2 = 100 \). Expressing \( x \) and \( y \) in terms of \( \lambda \) from the first two equations, and substituting the result into the third equation, we obtain

\[
\left( \frac{3}{\lambda + 1} \right)^2 + \left( \frac{4}{\lambda + 1} \right)^2 = 100, \quad \text{i.e.,} \quad \left( \frac{5}{\lambda + 1} \right)^2 = 100,
\]

gives \( \lambda + 1 = \pm 1/2 \). Substituting this into the first two equations, we obtain the points \( (x, y) = (6, 8) \) or \( (-6, -8) \). These are the only possible places of local extrema. Since the set \( \{(x, y) : x^2 + y^2 = 100 \} \) is closed and bounded and the function \( f(x, y) \) is continuous, \( f(x, y) \) must have an absolute maximum and an absolute minimum on this set. Given that \( f(6, 8) = 0 \) and \( f(-6, -8) = 200 \), the absolute minimum on this set is assumed at \( (6, 8) \), and the absolute maximum, at \( (-6, -8) \).

c) Find the places and values of the absolute maximum and of the absolute minimum of \( f(x, y) \) given in part a) in the region \( \{(x, y) : x^2 + y^2 \leq 100 \} \).

**Solution.** The place of local minimum \( (3, 4) \) of \( f \) is in the set \( \{(x, y) : x^2 + y^2 < 100 \} \) (that is inside, and not on the boundary, of the set \( \{(x, y) : x^2 + y^2 \leq 100 \} \). As \( f(3, 4) = -25 < 0 = f(6, 8) \), the absolute minimum on the set \( \{(x, y) : x^2 + y^2 \leq 100 \} \) is assumed at \( (3, 4) \), and the value of the absolute minimum is \( -25 \), while the absolute maximum is assumed on the boundary \( \{(x, y) : x^2 + y^2 = 100 \} \) of this set, at the point \( (-6, -8) \), and the value of the absolute maximum is \( 200 \).

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1 All computer processing for this manuscript was done under Debian Linux. A\LaTeX was used for typesetting.
3.a) In the iterated integral
\[ \int_0^{\pi/4} \int_{x^{1/3}}^{\pi/4} \sin(y^4) \, dy \, dx, \]
first (i) reverse the order of integration, and then (ii) evaluate the integral. (Note: it is not possible to evaluate the integral without first reversing the order of integration.)

Solution. The iterated integral in question equals the double integral \( \iint_S \sin(y^4) \, dA \) where \( S \) is the region \( S = \{ (x, y) : 0 \leq x \leq \pi^{3/4} \& \ x^{1/3} \leq y \leq \pi^{1/4} \} \). Writing \( S' = \{ (x, y) : 0 \leq y \leq \pi^{1/4} \& \ 0 \leq x \leq y^3 \} \), we will show that the region \( S' \) is the same as \( S \).

To see that \( S = S' \), we need to show that every point that belongs to \( S \) also belongs to \( S' \), and every point that belongs to \( S' \) also belongs to \( S \). To this end, assume that \( (x, y) \in S \). Then \( 0 \leq x \leq \pi^{3/4} \) and \( x^{1/3} \leq y \), and so \( 0 \leq y \) follows. Since we also have \( y \leq \pi^{1/4} \) according to the definition of \( S \). That is, \( 0 \leq y \leq \pi^{1/4} \), and so the first inequality in the definition of \( S \) is satisfied. We also have \( x^{1/3} \leq y \) by the definition of \( S \), the inequality of \( x \leq y^3 \) follows. Since we have \( 0 \leq x \) according to the first inequality in the definition of \( S \), we have \( 0 \leq x \leq y^3 \), and so the second inequality in the definition of \( S' \) is also satisfied. Hence, for any point \( (x, y) \in S \), both inequalities in the definition of \( S' \) are satisfied, i.e., we also have \( (x, y) \in S' \).

In a similar way, assuming that \( (x, y) \in S' \), we can show that \( (x, y) \in S \), and so \( S = S' \) follows. Using the description of the region given by the definition of \( S' \), we can rewrite the double integral as an iterated integral that will be easy to evaluate
\[ \iint_{S'} \sin(y^4) \, dA = \int_0^{\pi/4} \int_0^{y^3} \sin(y^4) \, dx \, dy = \int_0^{\pi/4} y^3 \sin(y^4) \, dy, \]
where carrying out the integration inside was a simple matter since the integrand did not depend on \( x \). To evaluate the integral on the right-hand side, use the substitution \( t = y^4 \). We have \( dt = 4y^3 \, dy \), and \( t = 0 \) when \( y = 0 \) and \( t = \pi \) when \( y = \pi^{1/4} \). Hence the integral on the right-hand side equals
\[ \frac{1}{4} \int_0^\pi \sin t \, dt = -\frac{1}{4} \cos t \bigg|_{t=0}^{t=\pi} = -\frac{1}{4} (-1) - 1 = \frac{2}{4} = 1/2. \]

b) Let \( D \) be the region \( D = \{ (x, y) : x \geq 0 \& \ x^2 + (y - 2)^2 \leq 4 \} \). Write the corresponding region \( D' \) in polar coordinates.

Solution. Geometrically, the region \( D \) represents the closed disk (the inside and the boundary or circumference of the circle) with center \((0, 2)\) and radius 2. We could easily argue geometrically to get a description of this region in polar coordinates; we will argue without invoking geometric intuition.

When converting to polar coordinates, we want to avoid multiple representation of points.\(^2\) To this end, we will assume that \( r \geq 0 \) and \( 0 \leq \theta < 2\pi \). The inequality \( x^2 + (y-2)^2 \leq 4 \) can be written as \( x^2 + y^2 - 4y + 4 \leq 4 \), i.e., \( x^2 + y^2 \leq 4y \). Since \( y = r \sin \theta \) and \( r^2 = x^2 + y^2 \), this inequality can be written as \( r^2 \leq 4 \sin \theta \), i.e., \( r \leq 4 \sin \theta \).\(^3\) The inequality \( x \geq 0 \) in the definition of \( x \) can be written as \( 0 \leq \theta \leq \pi/2 \) (first quadrant) or \( 3\pi/2 \leq \theta < 2\pi \) (fourth quadrant), given that we assume that \( 0 \leq \theta < 2\pi \). Since the inequality \( r \leq 4 \sin \theta \) is not satisfied in the fourth quadrant (since we assumed \( r \geq 0 \)), we can ignore the fourth quadrant. That is, the region \( D \) can be represented in polar coordinates as \( D' = \{ (r, \theta) : 0 \leq \theta \leq \pi/2 \& \ 0 \leq r \leq 4 \sin \theta \} \).

c) With the region \( D \) given in part b), evaluate the double integral \( \iint_D x \, dA \) by first converting it to polar coordinates.

Solution. We have \( x = r \cos \theta \); further, we have \( dA = r \, dr \, d\theta \) in polar coordinates. Hence, with the sets \( D \) and \( D' \) described in part b) we have \( \iint_D x \, dA = \iint_{D'} r \cos \theta \, dA \), where the integral on the left is in Cartesian

\(^2\)If \( (r, \theta) \) are the coordinates of a point in polar coordinates, the coordinates \( (r, \theta + 2k\pi) \) and \( (-r, \theta + (2k+1)\pi) \) describe the same point for any integer (positive, negative, or zero) \( k \).

\(^3\)Note that the inequality \( r^2 \leq 4s \sin \theta \) allows \( r = 0 \) for any value of \( \theta \), while the inequality \( r \leq 4s \sin \theta \) allows \( r = 0 \) only for certain values of \( \theta \). While this may lose some pairs \((0, \theta)\) satisfying the former inequality. On the other hand, \((0, \theta)\) represents the origin of the coordinate system for any value of \( \theta \), so no points in the place represented by the region \( D \) will be lost.
coordinates, and the integral on the right is in polar coordinates, though this is not indicated by the notation. Given the description of the region \( D' \), the integral on the right can be written as an iterated integral

\[
\int\int\int_{D'} r^2 \cos \theta \, dr \, d\theta = \int\int_{D'} r^2 \cos \theta \, dr \, d\theta = \int_0^{\pi/2} d\theta \int_0^4 r^2 \sin \theta \, d\theta
\]

\[
= \int_0^{\pi/2} \left[ \frac{1}{3} r^3 \cos \theta \right]_{r=0}^{r=4} \, d\theta = \int_0^{\pi/2} \frac{64}{3} \cos \theta \, d\theta.
\]

To calculate the integral on the right-hand side, use the substitution \( t = \sin \theta \). Then \( dt = \cos \theta \, d\theta \); for \( \theta = 0 \) we have \( t = 0 \) and for \( \theta = \pi/2 \) we have \( t = 1 \). Hence the right-hand side equals

\[
\frac{64}{3} \int_0^1 t^3 \, dt = \frac{64}{3} \cdot \frac{1}{4} = \frac{16}{3}.
\]

4. a) Let \( E \) be the solid that is in the first octant and lies under the paraboloid \( z = 9 - x^2 - y^2 \). Write the triple integral \( \iiint_E (5\sqrt{x^2 + y^2 - 4z}) \, dV \) as an iterated integral in Cartesian coordinates. Do not evaluate the integral in Cartesian coordinates.

**Solution.** The solid \( E \) lies above the part \( D \) of the disk \( x^2 + y^2 \leq 9 \) in the first quadrant of the \( xy \)-plane. We have \( D = \{(x, y) : 0 \leq x \leq 3 \quad \& \quad 0 \leq y \leq \sqrt{9 - x^2}\} \), and so \( E = \{(x, y, z) : 0 \leq x \leq 3 \quad \& \quad 0 \leq y \leq \sqrt{9 - x^2} \quad \& \quad 0 \leq z \leq 9 - x^2 - y^2\} \). Hence

\[
\int\int\int_E (5\sqrt{x^2 + y^2 - 4z}) \, dV = \int_0^3 dx \int_0^{\sqrt{9-x^2}} dy \int_0^{9-x^2-y^2} (5\sqrt{x^2 + y^2 - 4z}) \, dz.
\]

b) Write the triple integral in part a) as an iterated integral in cylindrical coordinates. Do not evaluate the integral.

**Solution.** In cylindrical coordinates, the region \( E' \) corresponding to \( E \) can be written as \( E' = \{(r, \theta, z) : 0 \leq \theta \leq \pi/2 \quad \& \quad 0 \leq r \leq 3 \quad \& \quad 0 \leq z \leq 9 - r^2\} \). Noting that \( \sqrt{x^2 + y^2} = r \) and \( dV = r \, dr \, d\theta \, dz \), we have

\[
\int\int\int_E (5\sqrt{x^2 + y^2 - 4z}) \, dV = \iiint_{E'} (5r - 4z) \, r \, dr \, d\theta \, dz = \int_0^{\pi/2} d\theta \int_0^3 dr \int_0^{9-r^2} (5r^2 - 4rz) \, dz.
\]

The integral on the right-hand side can easily be evaluated. In fact, the right-hand side equals

\[
\int_0^{\pi/2} d\theta \int_0^3 (5r^2 - 4rz^2) \bigg|_{z=0}^{9-r^2} \, dr = \int_0^{\pi/2} d\theta \int_0^3 (5r^2(9-r^2) - 4r(9-r^2)^2) \, dr
\]

\[
= \int_0^{\pi/2} \left[ \frac{-5}{3} - 8r^2 + 24r^4 \right]_0^3 \, d\theta = -\frac{81\pi}{2}.
\]

5. a) Let \( E \) be the solid that lies inside the sphere \( x^2 + y^2 + z^2 \leq 1 \) and above the cone \( z = \sqrt{x^2 + y^2}/3 \). Describe the corresponding region \( E' \) in spherical coordinates, with the aid of inequalities, i.e., in the form of a set \( \{(\rho, \theta, \phi) : \ldots\} \). (Hint: \( \arctan(1/\sqrt{3}) = \pi/6 \), so the angle between the above cone and the \( xy \)-plane is \( \pi/6 \).)
Solution. We have \( E = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1 \ & \ z \geq \sqrt{(x^2 + y^2)/3} \}. \) The inequality \( x^2 + y^2 + z^2 \leq 1 \) can be written as \( 0 \leq \rho \leq 1 \) in spherical coordinates.\(^4\) The inequality \( z \geq \sqrt{(x^2 + y^2)/3} \) can be written as \( z \geq r/\sqrt{3} \) in cylindrical coordinates. As \( \tan \phi = r/z \), in spherical coordinates this inequality can be written as \( \tan \phi \leq \sqrt{3} \), i.e., as \( 0 \leq \phi \leq \pi/3 \).\(^5\) Hence the corresponding region in spherical coordinates can be described as

\[
E' = \{(r, \theta, \phi) : 0 \leq \rho \leq 1 \ & \ 0 \leq \theta \leq 2\pi \ & \ 0 \leq \phi \leq \pi/3 \ \& \ 0 \leq \rho \leq 1 \};
\]

in the second description of \( E' \) we moved the inequality involving \( \rho \) to the last place.\(^6\)

b) Given the solid \( E \) described in part a), write the triple integral \( \iiint_E (x^2 + y^2 + z^2) \, dV \) in spherical coordinates, and then evaluate the integral.

Solution. Note that \( dV = r^2 \sin \phi \, dr \, d\theta \, d\phi \) in spherical coordinates and \( \rho^2 = x^2 + y^2 + z^2 \). We are going to use the second description of the set \( E' \) in converting the triple integral written in spherical coordinates to an iterated integral:

\[
\iiint_E (x^2 + y^2 + z^2) \, dV = \iiint_{E'} \rho^2 \, dV = \iiint_{E'} \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi = \iiint_{E'} \rho^4 \sin \phi \, d\rho \, d\theta \, d\phi
\]

\[
= \int_0^{2\pi} d\theta \int_0^{\pi/3} d\phi \int_0^1 \rho^4 \sin \phi \, d\rho = \int_0^{2\pi} d\theta \int_0^{\pi/3} \frac{1}{5} \sin \phi \, d\phi = \int_0^{2\pi} \frac{1}{5} \left( -\cos \phi \right) \bigg|_{\phi=0}^{\pi/3} d\theta
\]

\[
= \int_0^{2\pi} \frac{1}{5} \left( -\frac{1}{2} + 1 \right) d\theta = \int_0^{2\pi} \frac{1}{10} d\theta = \frac{2\pi}{10} = \frac{\pi}{5}.
\]

---

\(^4\)We require \( \rho \geq 0 \), since allowing \( \rho < 0 \) would result in allowing multiple representations of points in the solid.

\(^5\)The inequality \( 0 \leq \phi \) is needed to avoid multiple representation of points. In any case, in spherical coordinates one usually assumes that \( 0 \leq \phi \leq \pi \), though in some special applications one might want to avoid making this assumption.

\(^6\)The inequalities in the description of \( E' \) can of course be written in any order, but the order of the inequalities suggest the order of evaluation of the triple integral with iterated integrals. In a description suitable for setting up iterated integral, an inequality describing a range of a variable can only involve variables whose range has been described earlier. Since the no other variable is involved in describing the range of \( \rho \), and \( \rho \) is not involved in describing the ranges of the other variables, the inequality involving \( \rho \) can be placed anywhere.