Exam 1, Mathematics 2501, Section TR3 21 copies
3:40–4:50 pm, February 26, 2026, IH-1105
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1. Let A and B be two events such that $\mathrm{P}(A)=.5$, $\mathrm{P}(B)=.4$, and $\mathrm{P}(A\cap B)=.2$.

a) Find $\mathrm{P}(A\cup B)$.

Solution. We have

$$\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)=.5+.4-.2=.7.$$

b) Find $\mathrm{P}(A\setminus B)$.

Solution. We have

$$(A\cap B)\cup(A\setminus B)=A.$$

Noting that $A\cap B$ and $A\setminus B$ are disjoint, we have

$$\mathrm{P}(A\cap B)+\mathrm{P}(A\setminus B)=\mathrm{P}(A).$$

Hence

$$\mathrm{P}(A\setminus B)=\mathrm{P}(A)-\mathrm{P}(A\cap B)=.5-.2=.3.$$

Note that the answer does not depend on the value of $\mathrm{P}(B)$.

2. *a*) From an urn containing 4 red balls and 6 green balls, 6 balls are taken without replacement. Determine the probability that 2 of the balls are red and 4 of them are green.

Solution. The number of ways one can pick 2 balls out of 4 red balls without replacement is $\binom{4}{2}$. The number of ways one can pick 4 balls out of 6 green balls without replacement is $\binom{6}{4}$. The number of ways one can pick 6 balls out of a total of 10 balls without replacement is $\binom{10}{6}$. Hence, the probability is

$$\binom{4}{2}\binom{6}{4}\Big/\binom{10}{6}=\binom{4}{2}\binom{6}{2}\Big/\binom{10}{4}=\frac{\frac{4\cdot3}{1\cdot2}\cdot\frac{6\cdot5}{1\cdot2}}{\frac{10\cdot9\cdot8\cdot7}{1\cdot2\cdot3\cdot4}}=\frac{3}{7}\approx .42857.$$

b) Give the probability if the same experiment is performed with replacement, and the same outcome is obtained.

Solution. The probability of picking 2 red balls with replacement $(4/10)^2=(2/5)^2$. The probability of picking 4 green balls with replacement is $(6/10)^4=(3/5)^4$. The probability of first picking 2 red balls with replacement and then 4 green balls of replacement is the product of these, that is $(2/5)^2\cdot(4/5)^4$. The number of ways 2 red balls and 4 green balls can be arranged in a sequence (the order in which they are being picked) is

$$\binom{6}{2}=\frac{6\cdot5}{1\cdot2}=15.$$

Hence the probability of picking 2 red balls and 4 green balls (in any order) is

$$\binom{6}{2}\cdot\left(\frac{2}{5}\right)^2\cdot\left(\frac{3}{5}\right)^4=15\cdot\frac{324}{15625}=\frac{972}{3125}\approx .31104.$$

3. *a*) In a factory, parts are manufactured by three machines, M_1, M_2, and M_3 in proportions $25:30:45$. The percentages 8%, 4%, and 6% of these parts are defective, respectively. Find the probability that a randomly chosen part is defective.

[1]All computer processing for this manuscript was done under Debian Linux. $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX was used for typesetting. The Perl programming language was used in creating the $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX source file.

Solution. Write A for the event that a part is defective, and M_i with $i = 1, 2, 3$ for the event that it was manufactured on machine i. We have

$$P(A) = \sum_{i=1}^{3} P(A|M_i)\,P(M_i) = .08 \cdot .25 + .04 \cdot .3 + .06 \cdot .45 = .059.$$

b) Find the probability that a defective part was manufactured on the third machine.

Solution. Using Bayes's theorem, we have

$$P(M_3|A) = \frac{P(A|M_3)\,P(M_3)}{\sum_{i=1}^{3} P(A|M_i)\,P(M_i)} = \frac{.06 \cdot .45}{.08 \cdot .25 + .04 \cdot .3 + .06 \cdot .45} = \frac{.027}{.059} \approx .457627;$$

note that the denominator in the third member is the same as the answer to part *a*).

4. *a*) From an urn with 8 red balls and 4 green balls, one ball is drawn with replacement until a green ball is obtained. Let X be a random variable whose value is the number of tries for a green ball to be drawn. Find the distribution of X; that is, find the probability that $X = k$ ($k = 1, 2, 3, \dots$).

Solution. The probability of drawing a red ball is $8/12 = 2/3$, and the probability of drawing a green ball is $4/12 = 1/3$. For the outcome $X = k$ one needs to draw red balls the first $k - 1$ times and a green ball on the kth try. Since the drawings are independent events, the probability for this to occur is

$$P(X = k) = \left(\frac{2}{3}\right)^{k-1} \cdot \frac{1}{3}.$$

b) Find the probability $P(X \geq 4)$ for the random variable X described in part *a*) of this problem.

First solution. We events $(X = k)$ for different values of k are mutually exclusive, and we have

$$(X \geq 4) = \bigcup_{k=4}^{\infty} (X = k).$$

Hence,

$$P(X \geq 4) = \sum_{k=4}^{\infty} P(X = k) = \sum_{k=4}^{\infty} \left(\frac{2}{3}\right)^{k-1} \cdot \frac{1}{3} = \frac{1}{3} \sum_{k=4}^{\infty} \left(\frac{2}{3}\right)^{k-1}.$$

Writing $n = k - 4$ in the sum, the right-hand side equals

$$\frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n+3} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n} \cdot \left(\frac{2}{3}\right)^{3} = \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n} = \frac{2^3}{3^4} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n}.$$

The sum on the right-hand side is easily evaluated by the sum formula

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x} \qquad (|x| < 1).$$

We obtain

$$P(X \geq 4) = \frac{2^3}{3^4} \frac{1}{1 - \frac{2}{3}} = \frac{2^3}{3^4} \frac{1}{\frac{1}{3}} = \frac{2^3}{3^3} = \frac{8}{27} = .296296.$$

Second solution. Instead of evaluating the infinite sum above, one may more simply observe that

$$\begin{aligned} P(X \geq 4) &= 1 - P(X = 1) - P(X = 2) - P(X = 3) = 1 - \frac{1}{3} - \frac{2}{3} \cdot \frac{1}{3} - \frac{4}{9} \cdot \frac{1}{3} \\ &= 1 - \frac{1}{3} - \frac{2}{9} - \frac{4}{27} = \frac{27 - 9 - 6 - 4}{27} = \frac{8}{27} = .296296. \end{aligned}$$

5. Three hunters shoot at the same deer simultaneously. They hit or miss independently of each other. The first hunter hits with probability 1/5, the second one with probability 1/6, and the third one with probability 1/7. Calculate the probability that the deer will be hit at least once. *Hint:* Consider the complement of the event in question, that is, the event that no hunter will hit the deer.

Solution. Writing A_i $(i = 1, 2, 3)$ for the event that the ith hunter will hit the deer, the event that the deer will not be hit is

$$\Omega \setminus (A_1 \cup A_2 \cup A_3) = (\Omega \setminus A_1) \cap (\Omega \setminus A_2) \cap (\Omega \setminus A_3),$$

where Ω is the probability space. Since the events on the right hand side are independent, the probability that all three of them occur is the product of the probability of each of them, that is

$$\left(1 - \frac{1}{5}\right)\left(1 - \frac{1}{6}\right)\left(1 - \frac{1}{7}\right) = \frac{4}{5} \cdot \frac{5}{6} \cdot \frac{6}{7} = \frac{4}{7}.$$

This is the probability that the deer will not be hit. The event that the deer will be hit at least once is the complement of the event that the deer will not be hit, so its probability is

$$1 - \frac{4}{7} = \frac{3}{7}.$$