Exam 2, Mathematics 2501, Section TR2 24 copies
2:15–3:25 pm, March 26, 2026, IH-1105
Instructor: Attila Máté[1]

1. Let X be the random variable with density function

$$f_X(x) = \begin{cases} 4x^{-5} & \text{if } x \ge 1, \\ 0 & \text{otherwise.} \end{cases}$$

a) Find the expectation of X.

Solution. We have

$$\begin{aligned} \mathrm{E}(X) &= \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_1^{\infty} 4x^{-4}\,dx = \lim_{A\to\infty} \int_1^A 4x^{-4}\,dx = \lim_{A\to\infty} \left(-\frac{4}{3}\cdot x^{-3}\right)\Big|_{x=1}^{x=A} \\ &= \lim_{A\to\infty} \left(-\frac{4}{3}\cdot A^{-3} + \frac{4}{3}\right) = \frac{4}{3}. \end{aligned}$$

b) Find the variance of X.

Solution. We have $\mathrm{Var}(X) = \mathrm{E}\left(X^2\right) - \big(\mathrm{E}(X)\big)^2$. Here

$$\begin{aligned} \mathrm{E}\left(X^2\right) &= \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_1^{\infty} 4x^{-3}\,dx = \lim_{A\to\infty} \int_1^A 4x^{-3}\,dx = \lim_{A\to\infty} \left(-2\cdot x^{-2}\right)\Big|_{x=1}^{x=A} \\ &= \lim_{A\to\infty} \left(-2\cdot A^{-2} + 2\right) = 2. \end{aligned}$$

Hence

$$\mathrm{Var}(X) = \mathrm{E}\left(X^2\right) - \big(\mathrm{E}(X)\big)^2 = 2 - \frac{16}{9} = \frac{2}{9}.$$

c) Find the expectation of X^3.

Solution. We have

$$\begin{aligned} \mathrm{E}(X^3) &= \int_{-\infty}^{\infty} x^3 f_X(x)\,dx = \int_1^{\infty} 4x^{-2}\,dx = \lim_{A\to\infty} \int_1^A 4x^{-2}\,dx = \lim_{A\to\infty} \left(-4\cdot x^{-1}\right)\Big|_{x=1}^{x=A} \\ &= \lim_{A\to\infty} \left(-4\cdot A^{-3} + 4\right) = 4. \end{aligned}$$

2. In a factory, certain items are produced. Each item turns out to be defective with a probability of .001, independently of what happens to other items. After production, without checking the items, they are packed in boxes of 1600 each.

a) Assume a large number items are produced. Write an exact formula for the probability that a box contains exactly k defective items ($0 \le k \le 1600$).

Solution. Since the probability of finding a defective item is .001, packing a box corresponds to repeating an experiment 1600 times, with the experiment being "successful" if a defective item is picked. If we write X for the random variable that k defective are found, the X has a binomial distribution $X \sim \mathrm{Bin}(1600, .001)$. Hence

$$P(X = k) = \binom{1600}{k} \cdot .001^k \cdot .999^{1600-k}.$$

[1]All computer processing for this manuscript was done under Debian Linux. $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX was used for typesetting. The Perl programming language was used in creating the $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX source file.

b) Using an appropriate approximation, calculate the probability that a box contains 4 defective items.

Solution. The expectation of the random variable X described in the solution for part *a*) is $1600 \cdot .001 = 1.6$. The distribution of X can be approximated by the Poisson distribution Po(1.6). Thus,

$$P(X = 4) \approx \frac{(1.6)^4}{4!} e^{-1.6} \approx .05513.$$

A more precise value of the probability, without using the Poisson approximation, using the Maxima program `exam2_probl3_4.pdf` (online, at this website), we get a more precise answer:

$$P(X = 4) = \binom{1600}{4} \cdot .001^4 \cdot .999^{1600-4} = 0.0551008253904361.$$

3. A lifetime X of a certain type of light bulb is described by the exponential distribution with density function $f_X(x) = \theta^{-1} e^{-x/\theta}$ $(x \geq 0)$; then θ is the mean lifetime. A new bulb is switched on, and it is found that its lifetime is 2000 hours. Construct a two-sided confidence interval for θ with confidence level 90%. Hint: the distribution function of X is

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(t)\,dt = \int_0^x \frac{1}{\theta} e^{-t/\theta}\,dt = 1 - e^{-x/\theta}.$$

Solution. We want to choose a_1 and a_2 such that

$$P(a_1 < X < a_2) = .9.$$

To accomplish this, we will cut off equal probabilities at each end of the distribution of X; that is, a_1 and a_2 will be so chosen that

$$.05 = \mathrm{P}(X \leq a_1) = F_X(a_1) = 1 - e^{-a_1/\theta}$$

and

$$.05 = \mathrm{P}(X \geq a_2) = 1 - \mathrm{P}(X < a_2) = 1 - \mathrm{P}(X \leq a_2) = 1 - F_X(a_2) = e^{-a_2/\theta}.$$

The first of these two equations gives $e^{-a_1/\theta} = .95$, that is, $-a_1/\theta = \ln .95$, i.e.,

$$a_1 = -\theta \ln .95.$$

The second one of the above two equations gives $\ln .05 = -a_2/\theta$, i.e.,

$$a_2 = -\theta \ln .05.$$

Substitution the values of a_1 and a_2 into the starting equation, we have

$$\mathrm{P}(-\theta \ln .95 < X < -\theta \ln .05) = .9.$$

Dividing the first of these inequalities in the scope of $\mathrm{P}(\cdot)$ by $-\ln .95 > 0$ we obtain $\theta < X/(-\ln .95)$, and the second one by $-\ln .05 > 0$ we obtain $X/(-\ln .05) < \theta$; hence

$$\mathrm{P}\left(\frac{X}{-\ln .05} < \theta < \frac{X}{-\ln .95}\right) = .9.$$

Note that here the value of X random; θ is a fixed unknown constant. This gives

$$\left(\frac{X}{-\ln .05}, \frac{X}{-\ln .95}\right)$$

for the confidence interval for θ with confidence level 90% in terms of the random variable X. Substituting $X = 2000$, the actual life time of the light bulb in question, we obtain the confidence interval for θ, given the present observation:

$$(667.62, 38991.45).$$

4. The following four measurements of the distance between two points are taken:

$$42.1, \qquad 45.3, \qquad 37.9, \qquad 41.5.$$

This is regarded as a random sample from a distribution $\mathcal{N}(m, \sigma^2)$, where m is the true distance, and σ^2 measures the precision of the method.

a) Find an unbiased estimate for m.

Solution. Given a random sample $x_1, x_2, \dots, x_n$ (i.e., observations of the identically distributed independent random variables $X_1, X_2, \dots, X_n$), an unbiased estimate for the mean is

$$\bar{x} \overset{def}{=} \frac{1}{n}\sum_{k=1}^{n} x_k.$$

In the present case, $n = 4$ and $x_1 = 42.1$, $x_2 = 45.3$, $x_3 = 37.9$, and $x_4 = 41.5$. Hence $\bar{x} = 41.7$ is an unbiased estimate for m.

b) Find an unbiased estimate for σ^2 if it is known that $m = 40.0$.

Solution. We have $\sigma^2 = \mathrm{Var}(X_k) = \mathrm{E}\big((X_k - m)^2\big)$ (for all k with $1 \le k \le n$, since the variables X_k are identically distributed). Given that m is known, an unbiased estimate for σ^2 us the same as an unbiased estimate for the mean (expectation) $\mathrm{E}\big((X_k - m)^2\big)$. That is, the estimate for σ^2 we are looking for is

$$\frac{1}{n}\sum_{k=1}^{n}(x_k - m)^2.$$

With the present data, $m = 40.0$ and x_k and n as above, we have approximately 9.7899 as the estimate for σ^2 we are looking for.

c) Find an unbiased estimate for σ^2 if m is unknown.

Solution. If m is not known, an unbiased estimate for σ^2 is

$$s^2 = \frac{1}{n-1}\sum_{k=1}^{n}(x_k - \bar{x})^2,$$

where $\bar{x}$ is as above. In the present case, this gives $s^2 \approx 9.199$.

5. The following four measurements of the distance between two points are taken:

$$42.1, \qquad 45.3, \qquad 37.9, \qquad 41.5.$$

This is regarded as a random sample from a distribution $\mathcal{N}(m, 9)$, where m is the true distance, and $\sigma^2 = 9$ measures the precision of the method; the value of m is not known. Find a 95% confidence interval for m.

Solution. The level $1 - \alpha$ confidence interval for the mean of a normal distribution given that the standard deviation is known is

$$\left(\bar{x} - \lambda_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{x} + \lambda_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right),$$

where n is the size of the sample, $\bar{x} = (1/n)\sum_{k=1}^{n} x_k$; further, given a standard normal variable Y (i.e., a random variable with distribution $\mathcal{N}(0,1)$), and given y with $0 < y < 1$, the quantity λ_y is defined to be such that

$$P(Y > \lambda_y) = y.$$

In other words, with Φ being the distribution function of the standard normal variable, we have

$$\Phi(\lambda_y) = 1 - y.$$

This confidence interval is discussed on p. 231 in the textbook.

In the present case, $\alpha = .05$, so $\lambda_{\alpha/2} = \lambda_{.025} = 1.9600$, as given in the tables on pp. 324–325 in the textbook (the table on p. 324 only gives the value 1.96; the more precise value is given in the table on p. 325). Further, $n = 4$, $x_1 = 42.1$, $x_2 = 45.3$, $x_3 = 37.9$, and $x_4 = 41.5$, and so $\bar{x} = 41.7$. We are also given that $\sigma = 3$. Hence, the 95% confidence interval for m is

$$(38.76, 44.64).$$