Exam 3, Mathematics 2501, Section TR3 21 copies
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1. Consider a random sample of a normal distribution with standard deviation 2 and unknown expectation μ (i.e., of the distribution $\mathcal{N}(\mu, 4)$):

$$44.1, \qquad 48.4, \qquad 41.5, \qquad 42.0.$$

The hypothesis

$$H_0 : \mu = 46.0$$

is to be tested against the alternative hypothesis

$$H_1 : \mu > 46.0$$

(one-sided test).

a) Find the p-value of this test.

Solution. While the problem can be solved as stated, in a more reasonable statement, the hypothesis

$$H_0 : \mu = 42.0$$

should have been tested against the alternative hypothesis

$$H_1 : \mu > 42.0$$

(one-sided test).

Given independent identically distributed random variables $X_1, X_2, \dots, X_n$, with $\mathcal{N}(\mu, \sigma^2)$ distribution, and given observed values $x_1, x_2, \dots, x_n$, let

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i,$$

and

$$\bar{x}_n = \frac{1}{n}\sum_{i=1}^{n} x_i,$$

Assume σ is known, but μ is not. Given μ_0, the the p-value of the test $H_0 : \mu = \mu_0$ against the alternative hypothesis $H_1 : \mu > \mu_0$ is $p = P(\bar{X}_n > \bar{x}_n)$, calculated under the assumption that $\mu = \mu_0$. Under this assumption, $\bar{X}_n$ has distribution $\mathcal{N}(\mu_0, \sigma^2/n)$. So the variable $Y = (\bar{X}_n - \mu_0)/(\sigma/\sqrt{n})$ standard normal distribution. Thus, the p-value of the test is

$$\begin{aligned} p = \mathrm{P}(\bar{X}_n > \bar{x}_n) &= \mathrm{P}\left(\frac{\bar{X}_n - \mu_0}{\sigma/\sqrt{n}} > \frac{\bar{x}_n - \mu_0}{\sigma/\sqrt{n}}\right) = \mathrm{P}\left(Y > \frac{\bar{x}_n - \mu_0}{\sigma/\sqrt{n}}\right) \\ &= 1 - \mathrm{P}\left(Y \le \frac{\bar{x}_n - \mu_0}{\sigma/\sqrt{n}}\right) = 1 - \Phi\left(\frac{\bar{x}_n - \mu_0}{\sigma/\sqrt{n}}\right); \end{aligned}$$

the last equation holds since Y has standard normal distribution.

In the present case, $n = 4$, $\sigma = 2$, $x_1 = 44.1$, $x_2 = 48.4$, $x_3 = 41.5$, and $x_4 = 42.0$. and so $\bar{x}_4 = 44.0$; in addition, $\mu_0 = 46$. Therefore, the measured value of the standardized variable is

$$y = \frac{\bar{x}_n - \mu_0}{\sigma/\sqrt{n}} = \frac{44 - 46}{2/\sqrt{4}} = -2.$$

[1]All computer processing for this manuscript was done under Debian Linux. $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX was used for typesetting. The Perl programming language was used in creating the $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX source file.

We have

$$\Phi(2) = .977,25,$$

and so, for the p-value we have

$$p = 1 - \Phi(-2) = \Phi(2) \approx 0.97725 = 97\%.$$

Actually, as $\bar{x}_n < \mu_0$, and can say without any calculations that the hypothesis H_0 against H_1 cannot be rejected at any level of significance.

b) Test H_0 at the level 5% of significance.

Solution. Since the p-value we calculated in Part *a*) is larger than 5%, the test is result is not significant at level 5% of significance.

c) Test H_0 at the level 1% of significance.

Solution. Since the p-value we calculated in Part *a*) is greater than 1%, the test is result is not significant at level 1% of significance.

2. Consider a random sample of a normal distribution with unknown standard deviation and unknown expectation μ (i.e., of the distribution $\mathcal{N}(\mu, \sigma^2)$ with unknown σ):

$$44.1, \qquad 48.4, \qquad 41.5, \qquad 42.0.$$

The hypothesis

$$H_0 : \mu = 46.0$$

is to be tested against the alternative hypothesis

$$H_1 : \mu > 46.0$$

a) Test H_0 at the level 5% of significance.

Solution to Part *a*). While the problem can be solved as stated, in a more reasonable statement, the hypothesis

$$H_0 : \mu = 42.0$$

should have been tested against the alternative hypothesis

$$H_1 : \mu > 42.0$$

(one-sided test).

We use the test described in Subsection 29.2 on p. 76 in the notes. We have $n = 4$, $x_1 = 44.1$, $x_2 = 48.4$, $x_3 = 41.5$, and $x_4 = 42.0$, We then calculate $\bar{x}_n$ as

$$\bar{x}_4 = \frac{x_1 + x_2 + x_3 + x_4}{4} = 44.0,$$

and s as

$$s = \sqrt{\frac{(x_1 - \bar{x}_4)^2 + (x_2 - \bar{x}_4)^2 + (x_3 - \bar{x}_4)^2 + (x_4 - \bar{x}_4)^2}{3}} = \sqrt{\frac{29.62}{3}} \approx \sqrt{9.813,333,333} \approx 3.142,186,076.$$

As for the test variable t, with $\mu_0 = 46.0$ we have

$$t = \frac{\bar{x}_4 - \mu_0}{s/\sqrt{n}} = -1.272,999,085.$$

The critical region for this test is the interval

$$(t_\alpha(n-1), \infty).$$

With α being .05 (the significance level for one-sided test), we have $t_{0.05}(3) = 2.353,363\ldots$, this interval is

$$(2.353,363, +\infty).$$

Since t is not in this interval, the null hypothesis is not rejected at the significance level $.05 = 5\%$.

Actually, as $t < 0$, and can say without any calculations that the hypothesis H_0 against H_1 cannot be rejected at any level of significance.

b) Explain why it is not feasible to find the *p*-value of this test without a computer.

Solution to Part *b*). The *p*-value of this test is, with $\bar{T}$ denoting a variable with *t*-distribution of degree of freedom 3, is

$$\mathrm{P}(T > -1.272,999) = .853,644;$$

however, one needs a computer to calculate this, so for a statistician using tables only, this calculation cannot be done. Since this *p*-value is larger than .05, the null hypothesis is not rejected at the significance level of .05.

3. In a random sample of 400 persons eating lunch at a department store cafeteria, 260 persons had dessert. Construct a 90% confidence interval for the true proportion of persons usually eating dessert in this cafeteria.

Solution. With n the number of persons questioned, and x the number of persons qualifying (i.e., eating dessert in the present case), writing $\hat{\theta} = x/n$, the level $1-\alpha$ confidence interval for the true proportion of persons qualifying is

$$\left(\hat{\theta} - \lambda_{\alpha/2}\sqrt{\frac{\hat{\theta}(1-\hat{\theta})}{n}}, \hat{\theta} + \lambda_{\alpha/2}\sqrt{\frac{\hat{\theta}(1-\hat{\theta})}{n}}\right),$$

where λ_y is defined as follows: given a standard normal variable Y (i.e., a random variable with distribution $\mathcal{N}(0,1)$), and given y with $0 < y < 1$, the quantity λ_y is defined to be such that

$$P(Y > \lambda_y) = y.$$

In other words, with Φ being the distribution function of the standard normal variable, we have

$$\Phi(\lambda_y) = 1 - y.$$

This confidence interval is discussed on p. 245 in the textbook (in the notation used there $p^* = \hat{\theta}$ and $q^* = 1 - \hat{\theta}$).

In the present case, $n = 400$, $x = 260$, so $\hat{\theta} = .65$. Further, $\alpha = .10$, and $\lambda_{\alpha/2} = \lambda_{.05} = 1.6449$ given on p. 325 of the textbook. Thus, the confidence interval in question is

$$(.611, .689).$$

4. *a*) Let P and Q be events, and let I_P, and I_Q be their indicator variables. Describe the event whose probability equals the expectation

$$\mathrm{E}\big(1 - (1 - I_P)(1 - I_Q)\big)$$

Solution. The event is

$$\big(P^* \cap Q^*\big)^* = P \cup Q.$$

b) Given events P and Q, use indicator variables to derive the formula for $\mathrm{P}(P \cup Q)$.

Solution. We have

$$\begin{aligned}\mathrm{P}(P \cup Q) &= \mathrm{E}\big(1 - (1 - I_P)(1 - I_Q)\big) = \mathrm{E}\big(1 - (1 - I_P - I_Q + I_P I_Q)\big) \\ &= \mathrm{E}(I_P + I_Q - I_P I_Q) = \mathrm{E}(I_P) + \mathrm{E}(I_Q) - \mathrm{E}(I_P I_Q) \\ &= \mathrm{P}(P) + \mathrm{P}(Q) - \mathrm{P}(P \cap Q).\end{aligned}$$

5. Let $n > 0$ be an integer,and $X \sim \mathrm{Bin}(n, x)$ be a binomial variable. Given $\delta > 0$, show that

$$\sum_{\substack{k:0 \le k \le n \\ |k - xn| \ge n\delta}} \binom{n}{k} x^k (1 - x)^{n-k} \le \frac{1}{n\delta^2}.$$

Solution. Given that X is a binomial variable $\mathrm{Bin}(n, x)$, we have $m \stackrel{def}{=} \mathrm{E}(X) = nx$ and $\sigma^2 \stackrel{def}{=} \mathrm{V}(X) = nx(1 - x)$. According to Chebyshev's inequality (see class notes, equation (21.1) on p. 56) for any $\epsilon > 0$ we have we have

$$\mathrm{P}(|X - m| \ge \epsilon) \le \frac{\sigma^2}{\epsilon^2}.$$

With m and σ as above, and with $\epsilon = n\delta$, this becomes

$$\mathrm{P}(|X - nx| \ge n\delta) \le \frac{nx(1 - x)}{n^2\delta^2} = \frac{x(1 - x)}{n\delta^2} < \frac{1}{n\delta^2};$$

the last inequality holds since $0 \le x \le 1$.

Noting that

$$\mathrm{P}(X = k) = \binom{n}{k} x^k (1 - x)^{n-k},$$

we have

$$\sum_{\substack{k:0 \le k \le n \\ |k - xn| \ge \delta}} \binom{n}{k} x^k (1 - x)^{n-k} = \mathrm{P}(|X - nx| \ge n\delta).$$

As we just showed, the right-hand side here is

$$\le \frac{1}{n\delta^2},$$

establishing the inequality in the question.