

1. Let A and B be two events such that $P(A) = .6$, $P(B) = .7$, and $P(A \cap B) = .4$.

a) Find $P(A \cup B)$.

Solution. We have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = .6 + .7 - .4 = .9.$$

b) Find $P(A \setminus B)$.

Solution. We have

$$(A \cap B) \cup (A \setminus B) = A.$$

Noting that $A \cap B$ and $A \setminus B$ are disjoint, we have

$$P(A \cap B) + P(A \setminus B) = P(A).$$

Hence

$$P(A \setminus B) = P(A) - P(A \cap B) = .6 - .4 = .2.$$

Note that the answer does not depend on the value of $P(B)$.

2.a) From an urn containing 8 red balls and 6 green balls, six balls are taken without replacement. Determine the probability that 4 of the balls are red and 2 of them are green.

Solution. The number of ways one can pick 4 balls out of 8 red balls without replacement is $\binom{8}{4}$. The number of ways one can pick 2 balls out of 6 green balls without replacement is $\binom{6}{2}$. The number of ways one can pick 6 balls out of a total of 14 balls without replacement is $\binom{14}{6}$. Hence, the probability is

$$\binom{8}{4} \binom{6}{2} / \binom{14}{6} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 1 \cdot 2} \cdot \frac{7 \cdot 6 \cdot 5 \cdot 5}{7 \cdot 13 \cdot 11 \cdot 3} = \frac{2 \cdot 5 \cdot 5}{13 \cdot 11} = \frac{50}{143} \approx .34965.$$

b) Give the probability if the same experiment is performed with replacement, and the same outcome is obtained.

Solution. The probability of picking four red balls with replacement $(8/14)^4 = (4/7)^4$. The probability of picking two green balls with replacement is $(6/14)^2 = (3/7)^2$. The probability of first picking four red balls with replacement and then two green balls of replacement is the product of these, that is $(4/7)^4 \cdot (3/7)^2$. The number of ways four red balls and two balls can be arranged in a sequence (the order in which they are being picked) is

$$\binom{6}{4} = \binom{6}{2} = \frac{6 \cdot 5}{1 \cdot 2} = 15.$$

Hence the probability of picking two red balls and three green balls (in any order) is

$$\binom{6}{2} \cdot \left(\frac{4}{7}\right)^4 \cdot \left(\frac{3}{7}\right)^2 = 15 \cdot \frac{2304}{117649} = \frac{34560}{117649} \approx .29376.$$

3.a) In a factory, parts are manufactured by three machines, M_1 , M_2 , and M_3 in proportions 20 : 10 : 70. The percentages 6%, 3%, and 5% of these parts are defective, respectively. Find the probability that a randomly chosen part is defective.

¹All computer processing for this manuscript was done under Debian Linux. $\mathcal{A}\mathcal{M}\mathcal{S}\text{-}\mathcal{T}\mathcal{E}\mathcal{X}$ was used for typesetting. The Perl programming language was used in creating the $\mathcal{A}\mathcal{M}\mathcal{S}\text{-}\mathcal{T}\mathcal{E}\mathcal{X}$ source file.

Solution. Write A for the event that a part is defective, and M_i with $i = 1, 2, 3$ for the event that it was manufactured on machine i . We have

$$P(A) = \sum_{i=1}^3 P(A|M_i)P(M_i) = .06 \cdot .2 + .03 \cdot .1 + .05 \cdot .7 = .05.$$

b) Find the probability that a defective part was manufactured on the third machine.

Solution. Using Bayes's theorem, we have

$$P(M_3|A) = \frac{P(A|M_3)P(M_3)}{\sum_{i=1}^3 P(A|M_i)P(M_i)} = \frac{.05 \cdot .7}{.06 \cdot .2 + .03 \cdot .1 + .05 \cdot .7} = \frac{.035}{.05} = .7;$$

note that the denominator in the third member is the same as the answer to part a).

4. A fair die is rolled on four times in a row. What is the probability that the number 5 or 6 is rolled at least once?

Solution. Let A_k be the event that 5 or 6 is rolled on the k th try ($k = 1, 2, 3, 4$). These events are independent, and each has probability $1/3$. The complements A_k^* each have probability $2/3$, and these are also independent. Hence, the probability of $\bigcap_{k=1}^4 A_k^*$ is $(2/3)^4$. The probability in the question is that this does not happen, that is,

$$\left(\bigcap_{k=1}^4 A_k^*\right)^* = 1 - (2/3)^4 = 1 - \frac{16}{81} = \frac{65}{81} \approx .802469.$$

5. a) Let X be a random variable that can assume only positive integer values, and assume its probability function is $P(X = n) = A/3^n$ for some constant A ($n \geq 1$). Find A .

Solution. We must have

$$1 = \sum_{n=1}^{\infty} P(X = n) = \sum_{n=1}^{\infty} A \cdot \left(\frac{1}{3}\right)^n = \sum_{k=0}^{\infty} A \cdot \left(\frac{1}{3}\right)^{k+1},$$

where, to obtain the last equation, we wrote $k = n - 1$. The right-hand side here equals

$$\frac{A}{3} \cdot \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k = \frac{A}{3} \cdot \frac{1}{1 - 1/3} = \frac{A}{2}.$$

Hence we have $A = 2$.

b) Let X be a continuous random variable that can assume values between 0 and 3, and assume its density function is $f_X(x) = B(x^2 + 1)$ with some constant B ($0 \leq x \leq 3$). Find B .

Solution. That is, we have

$$f_X(x) = \begin{cases} B(x^2 + 1) & \text{if } 0 \leq x \leq 3, \\ 0 & \text{otherwise.} \end{cases}$$

We must have

$$1 = \int_{-\infty}^{\infty} f_X(x) dx = \int_0^3 B \cdot (x^2 + 1) dx = B \cdot \left(\frac{x^3}{3} + x\right) \Big|_{x=0}^{x=3} = 12B.$$

Hence $B = 1/12$.