

1. Let X be the random variable with density function

$$f_X(x) = \begin{cases} 4x^3 & \text{if } 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

a) Find the expectation of X .

Solution. We have

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^1 4x^4 dx = \frac{4}{5} \cdot x^5 \Big|_{x=0}^{x=1} = \frac{4}{5}.$$

b) Find the variance of X .

Solution. We have $\text{Var}(X) = E(X^2) - (E(X))^2$. Here

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x) dx = \int_0^1 4x^5 dx = \frac{4}{6} \cdot x^6 \Big|_{x=0}^{x=1} = \frac{2}{3} \cdot x^6 \Big|_{x=0}^{x=1} = \frac{2}{3}.$$

Hence

$$\text{Var}(X) = E(X^2) - (E(X))^2 = \frac{2}{3} - \frac{16}{25} = \frac{50}{75} - \frac{48}{75} = \frac{2}{75} = .02666\dots$$

(a repeating decimal).

2. 1. Let X be an variable having distribution $N(180, 25)$ (normal distribution with expectation 180 and standard deviation 5). Find the probability that a) $X \geq 177$, b) $X \leq 186$, and c) $177 \leq X \leq 186$.

Solution to Part a). The random variable

$$Y = \frac{X - 180}{5}$$

is a standard normal variable. Thus we have

$$P(X \geq 177) = P\left(Y \geq \frac{177 - 180}{5}\right) = P(Y \geq -.6) = P(Y \leq .6) = \Phi(.6) \approx .7257;$$

here the third equation is justified by the symmetry about the origin of the standard normal distribution, and the approximate equality is obtained by using the table of the standard normal distribution.

Solution to Part b). Similarly as in Part a), we have

$$P(X \leq 186) = P\left(Y \leq \frac{186 - 180}{5}\right) = P(Y \leq 1.2) = \Phi(1.2) \approx .8849.$$

Solution to Part c). We have

$$\begin{aligned} P(177 \leq X \leq 186) &= P(X \leq 186 \text{ and } 177 \leq X) \\ &= P(X \leq 186 \text{ and not } X < 177) = P(X \leq 186) - P(X < 177); \end{aligned}$$

¹All computer processing for this manuscript was done under Debian Linux. $\mathcal{A}\mathcal{M}\mathcal{S}\text{-}\mathcal{T}\mathcal{E}\mathcal{X}$ was used for typesetting. The Perl programming language was used in creating the $\mathcal{A}\mathcal{M}\mathcal{S}\text{-}\mathcal{T}\mathcal{E}\mathcal{X}$ source file.

it is important to note that the last equality holds because if $X \leq 186$ then also $X < 177$. That is, if A is the event $X \leq 186$ and B is the event $X < 177$, then $B \subset A$, and so $A = B \cup (A \setminus B)$. Thus $P(A) = P(B) + P(A \setminus B)$; thus

$$P(A \setminus B) = P(A) - P(B).$$

Hence, taking the probability values obtained in Parts a) and b), we obtain that

$$\begin{aligned} P(177 \leq X \leq 186) &= P(X \leq 186) - P(X < 177) = P(X \leq 186) - (1 - P(X \geq 177)) \\ &\approx .8849 - (1 - .7257) \approx .8849 - 1 + .7257 = .6106. \end{aligned}$$

3. A pair of fair coins are tossed 1200 times. Let X denote the number of times both coins come up head.

a) Write a formula for the probability $P(X = k)$ ($0 \leq k \leq 1200$).

Solution. The probability that both coins come up head in any single toss is $1/4$. Therefore X has a binomial distribution $X \sim \text{Bin}(1200, 1/4)$. That is,

$$P(X = k) = \binom{1200}{k} \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{1200-k}.$$

b) Write the sum for the exact probability that $290 \leq X \leq 310$.

Solution. We have

$$P(290 \leq X \leq 310) = \sum_{k=290}^{310} P(X = k) = \sum_{k=290}^{310} \binom{1200}{k} \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{1200-k}.$$

c) Use a well-known approximation to calculate the approximate probability that $290 \leq X \leq 310$.

Solution. We use normal approximation with continuity correction. A $\text{Bin}(n, p)$ distribution has expectation np and standard deviation $\sqrt{np(1-p)}$. Accordingly, writing $D(X)$ for the standard deviation of X , we have

$$E(X) = 1200 \cdot \frac{1}{4} = 300 \quad \text{and} \quad D(X) = \sqrt{1200 \cdot \frac{1}{4} \cdot \frac{3}{4}} = 15.$$

Thus, the distribution of X can be approximated by a that of a normal variable $Y \sim \mathcal{N}(300, 15^2)$, i.e., by a normal variable Y having expectation 300 and standard deviation 15. Writing

$$Z = \frac{Y - 300}{15},$$

Y is a standard normal variable. Hence, using continuity correction,

$$\begin{aligned} P(290 \leq X \leq 310) &\approx P(289.5 \leq Y \leq 310.5) \\ &= P\left(\frac{289.5 - 300}{15} \leq \frac{Y - 300}{15} \leq \frac{310.5 - 300}{15}\right) = P\left(-.7 \leq \frac{Y - 300}{15} \leq .7\right) \\ &= P(-.7 \leq Z \leq .7) = P(Z \leq .7) - P(Z < -.7) = P(Z \leq .7) - P(Z > .7) \\ &= P(Z \leq .7) - (1 - P(Z \leq .7)) = 2P(Z \leq .7) - 1 = 2\Phi(.7) - 1; \end{aligned}$$

here, the Φ denotes the distribution function of the standard normal distribution. The first equality in the last line follows since the distribution of Z is symmetric about 0. We can use the normal distribution table to find that $\Phi(.7) \approx .7580$. Thus, we gave

$$P(290 \leq X \leq 310) \approx .5160.$$

Calculating the sum Part *b*) directly, with a short program written in the programming language Maxima, we get a more precise answer

$$P(290 \leq X \leq 310) \approx .5160583906382258,$$

showing that the result obtained by the normal approximation is almost correct to four decimal places; not quite, since rounding the more precise result to four decimal places, we obtain

$$P(290 \leq X \leq 310) \approx .5161.$$

4. A rare disease occurs in 0.03% of the population.

a) What is the probability that out of 10000 randomly chosen individuals, exactly 4 have the disease? Write the formula expressing the exact value of this probability, but do not evaluate.

Solution. If X denotes the number of persons having the disease out of 10000 randomly selected individuals, then X follows the binomial distribution $\text{Bin}(10000, .0003)$. According to this, the probability that exactly

$$P(X = 4) = \binom{10000}{4} .0003^4 .9997^{9996}.$$

On a computer, this can easily be evaluated, giving the result $P(X = 4) \approx .1680565656971817$.

b) Use a well-known approximation to approximately calculate the probability given in Part *a*).

Solution. For small k , the probability $P(X = k)$ can be approximated by the corresponding value of the Poisson distribution $\text{Po}(10000 \cdot .0003) = \text{Po}(3)$. According to this,

$$P(X = 4) \approx \frac{3^4}{4!} e^{-3} \approx 0.1680313557415408.$$

Comparing this with the value we calculated for the same probability using the binomial distribution shows that the approximation obtained here is correct up to three decimals.

5. *a*) On a certain night soon after sunset in March, the probability that an observer sees a meteor (falling star) in any given second is 6/3600 (note that an hour is 3600 seconds; one mentions seconds here, since one second is a very short time under the circumstances – so 6 is the hourly rate when one measures this rate in very short time intervals). What is the probability that this observer will see exactly k meteors in a given hour ($k \geq 0$)?

Solution. If X indicates the number of meteors seen by the given observer, then X follows the $\text{Po}(6)$ distribution; that is,

$$P(X = k) = \frac{6^k}{k!} e^{-6}.$$

b) What is the probability that the same observer will see 4 meteors in a given hour.

Solution. According to the formula given in Part *a*), we have

$$P(X = 4) = \frac{6^4}{4!} e^{-6} \approx .1338526175399833.$$

Note. The question arises as to how justified one is to consider one second a “very short time” in the problem. In other words, if one sees a meteor with probability approaching $\lambda \Delta t$ in a time interval of length Δt as $\Delta t \rightarrow 0$, what value of λ would give rise to a sighting of at least one meteor with probability 6/3600 in any given second. Here λ is given for a time unit of one hour; the corresponding value of λ for a time unit

of one second would be $\lambda/3600$. That is, if Y is the random variable denoting the number of meteors seen in a given time interval one second length, then Y follows the Poisson distribution $\text{Po}(\lambda/3600)$; i.e.,

$$P(Y = k) = \frac{(\lambda/3600)^k}{k!} e^{-\lambda/3600}.$$

Hence

$$P(Y \geq 1) = 1 - P(Y = 0) = 1 - e^{-\lambda/3600}.$$

Thus the equation $P(Y \geq 1) = 6/3600$ gives

$$\frac{6}{3600} = 1 - e^{-\lambda/3600}.$$

From here we obtain

$$\lambda = -3600 \ln \left(1 - \frac{6}{3600} \right) \approx 6.005005562509411.$$

What is going on here is the following. Recall that the expectation of a $\text{Po}(\lambda)$ is λ . We have

$$\frac{\lambda}{3600} \approx .001668057100697058.$$

This is the expected number of meteors seen in any given second. On the other hand, the probability that at least one meteor is seen in any given second is

$$\frac{6}{3600} = \frac{1}{600} \approx .0016666666666666666$$

(a repeating decimal); the former number is somewhat larger, since one might see more than one meteor in a given second.