Exam 3, Mathematics 2501, Section TR3 30 copies
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1. The numbers

$$81.1, \qquad 84.7, \qquad 77.8, \qquad 76.4$$

represent the temperature measured at a certain location and at a certain time. This is regarded as a random sample from a distribution $\mathcal{N}(m,\sigma^2)$, where m is the true temperature, and σ^2 measures the precision of the method.

a) Find an unbiased estimate for m.

Solution. Given a random sample $x_1, x_2, \dots, x_n$ (i.e., observations of the identically distributed independent random variables $X_1, X_2, \dots, X_n$), an unbiased estimate for the mean is

$$\bar{x} \overset{def}{=} \frac{1}{n}\sum_{k=1}^{n} x_n.$$

In the present case, $n = 4$ and $x_1 = 81.1$, $x_2 = 84.7$, $x_3 = 77.8$, and $x_4 = 76.4$. Hence $\bar{x} = 80.0$ is an unbiased estimate for m.

b) Find an unbiased estimate for σ^2 if it is known that $m = 79.0$.

Solution. We have $\sigma^2 = \mathrm{Var}(X_k) = \mathrm{E}\big((X_k - m)^2\big)$ (for all k with $1 \le k \le n$, since the variables X_k are identically distributed). Given that m is known, an unbiased estimate for σ^2 us the same as an unbiased estimate for the mean (expectation) $\mathrm{E}\big((X_k - m)^2\big)$. That is, the estimate for σ^2 we are looking for is

$$\frac{1}{n}\sum_{k=1}^{n}(x_k - m)^2.$$

With the present data, $m = 79.0$ and x_k and n as above, we have 11.275 as the estimate for σ^2 we are looking for.

c) Find an unbiased estimate for σ^2 if m is unknown.

Solution. If m is not known, an unbiased estimate for σ^2 is

$$s^2 = \frac{1}{n-1}\sum_{k=1}^{n}(x_k - \bar{x})^2,$$

where $\bar{x}$ is as above. In the present case, this gives $s^2 \approx 13.70$.

2. The numbers

$$81.1, \qquad 84.7, \qquad 77.8, \qquad 76.4$$

represent the temperature measured at a certain location and at a certain time. This is regarded as a random sample from a distribution $\mathcal{N}(m, 16)$, where m is the true temperature, and $\sigma^2 = 16$ measures the precision of the method; the value of m is not known. Find a 90% confidence interval for m.

Solution. The level $1-\alpha$ confidence interval for the mean of a normal distribution given that the standard deviation is known is

$$\left(\bar{x} - \lambda_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{x} + \lambda_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right),$$

[1]All computer processing for this manuscript was done under Debian Linux. $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX was used for typesetting. The Perl programming language was used in creating the $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX source file.

where n is the size of the sample, $\bar{x} = (1/n)\sum_{k=1}^{n} x_k$; further, given a standard normal variable Y (i.e., a random variable with distribution $\mathcal{N}(0,1)$), and given y with $0 < y < 1$, the quantity λ_y is defined to be such that

$$P(Y > \lambda_y) = y.$$

In other words, with Φ being the distribution function of the standard normal variable, we have

$$\Phi(\lambda_y) = 1 - y.$$

This confidence interval is discussed on p. 231 in the textbook.

In the present case, $\alpha = .10$, so $\lambda_{\alpha/2} = \lambda_{.05} = 1.6449$, as given in the tables on pp. 325 in the textbook. Further, $n = 4$, $x_1 = 81.1$, $x_2 = 84.7$, $x_3 = 77.8$, and $x_4 = 76.4$, and so $\bar{x} = 80.0$. We are also given that $\sigma = 4$. Hence, the 90% confidence interval for m is

$$(76.71, 83.29).$$

3. The numbers

$$81.1, \qquad 84.7, \qquad 77.8, \qquad 76.4$$

represent the temperature measured at a certain location and at a certain time. This is regarded as a random sample from a distribution $\mathcal{N}(m, \sigma^2)$, where m is the true temperature, and σ^2 measures the precision of the method; the values of m and σ^2 are not known. Find a 90% confidence interval for m.

Solution. The level $1 - \alpha$ confidence interval for the mean of a normal distribution given that the standard deviation is not known is

$$\left(\bar{x} - t_{\alpha/2}(n-1)\frac{s}{\sqrt{n}}, \bar{x} + t_{\alpha/2}(n-1)\frac{s}{\sqrt{n}}\right),$$

where n is the size of the sample, $\bar{x} = (1/n)\sum_{k=1}^{n} x_k$,

$$s = \sqrt{\frac{1}{n-1}\sum_{k=1}^{n}(x_i - \bar{x})^2}\,;$$

further, given a random variable Y with a Student-t distribution with degrees of freedom f, and given y with $0 < y < 1$, the quantity $t_y(f)$ is defined such that

$$P\big(Y > t_y(f)\big) = y.$$

This confidence interval is discussed on p. 232 in the textbook.

In the present case, $\alpha = .10$, $n = 4$, $f = n - 1 = 3$, so $t_{\alpha/2}(n-1) = t_{.05}(3) = 2.35$, as given in the table on p. 326 in the textbook; a more precise value is $t_{\alpha/2}(n-1) = t_{.05}(3) = 2.35336$. Further, $x_1 = 81.1$, $x_2 = 84.7$, $x_3 = 77.8$, and $x_4 = 76.4$, and so $\bar{x} = 80.0$ and $s = \sqrt{13.70} \approx 3.70135$. Hence, the 90% confidence interval for m in this case is

$$(75.64, 84.36).$$

4. The numbers

$$81.1, \qquad 84.7, \qquad 77.8, \qquad 76.4$$

represent the temperature measured at a certain location and at a certain time. This is regarded as a random sample from a distribution $\mathcal{N}(m, \sigma^2)$, where m is the true temperature, and σ^2 measures the precision of the method; the values of m and σ^2 are not known.

a) Find a 90% confidence interval for σ^2.

Solution. The level $1-\alpha$ confidence interval for the standard deviation of a normal distribution given that the mean and standard deviation is not known is

$$\left(\frac{1}{\chi^2_{\alpha/2}(n-1)}\sum_{k=1}^{n}(x_k-\bar{x})^2, \frac{1}{\chi^2_{1-\alpha/2}(n-1)}\sum_{k=1}^{n}(x_k-\bar{x})^2\right),$$

where n is the size of the sample and $\bar{x} = (1/n)\sum_{k=1}^{n} x_k$; further, given a random variable Y with a χ^2 distribution with degrees of freedom f, and given y with $0 < y < 1$, the quantity $\chi^2_y(f)$ is defined such that

$$P\big(Y > \chi^2_y(f)\big) = y.$$

This confidence interval is discussed on p. 235 in the textbook.

In the present case, $\alpha = .10$, $n = 4$, $f = n-1 = 3$, so $\chi^2_{\alpha/2}(n-1) = \chi^2_{.05}(3) = 7.81$ and $\chi^2_{1-\alpha/2}(n-1) = \chi^2_{.95}(3) = .35$ as given in the table on p. 327 in the textbook; more precise values are $\chi^2_{\alpha/2}(n-1) = \chi^2_{.05}(3) = 7.814727$ and $\chi^2_{1-\alpha/2}(n-1) = \chi^2_{.95}(3) = .351846$. Further, $x_1 = 81.1$, $x_2 = 84.7$, $x_3 = 77.8$, and $x_4 = 76.4$, and so $\bar{x} = 80.0$ and $\sum_{k-1}^{n}(x_k-\bar{x})^2 = 41.1$. Hence, the 90% confidence interval for σ^2 in this case is

$$(5.259, 116.81).$$

b) Find a 90% confidence interval for σ.

Solution. The confidence interval for σ is obtained by taking the square roots of the endpoints of the confidence interval for σ^2. In the present case, it is

$$(2.29, 10.81).$$

5. In a random sample of 500 persons eating lunch at a department store cafeteria, 204 persons had dessert. Construct a 90% confidence interval for the true proportion of persons usually eating dessert in this cafeteria.

Solution. With n the number of persons questioned, and x the number of persons qualifying (i.e., eating dessert in the present case), writing $\Theta = x/n$, the level $1-\alpha$ confidence interval for the true proportion of persons qualifying is

$$\left(\Theta - \lambda_{\alpha/2}\sqrt{\frac{\Theta(1-\Theta)}{n}}, \Theta + \lambda_{\alpha/2}\sqrt{\frac{\Theta(1-\Theta)}{n}}\right),$$

where λ_y is defined as follows: given a standard normal variable Y (i.e., a random variable with distribution $\mathcal{N}(0,1)$), and given y with $0 < y < 1$, the quantity λ_y is defined to be such that

$$P(Y > \lambda_y) = y.$$

In other words, with Φ being the distribution function of the standard normal variable, we have

$$\Phi(\lambda_y) = 1 - y.$$

This confidence interval is discussed on p. 245 in the textbook (in the notation used there $p^* = \Theta$ and $q^* = 1 - \Theta$).

In the present case, $n = 500$, $x = 204$, so $\Theta = .408$. Further, $\alpha = .10$, and $\lambda_{\alpha/2} = \lambda_{.05} = 1.6449$ given on p. 325 of the textbook. Thus, the confidence interval in question is

$$(.3718, .4442).$$