Blackboard exam 1, Mathematics Mathematics 3501, Section ETY6
Starts: 6:55 pm, Thurs, Mar 4; ends: 7:45 pm (late submission loses points).
Instructor: Attila Máté

1. *a*) Given the equation

$$x_1 + x_2 + x_3 = 16,$$

how many solutions (x_1, x_2, x_3) does it have, where $x_1 > 0$, $x_2 > 0$, $x_3 > 0$ are integers.

Solution. According to the discussion in Subsection 3.9, The number of positive solutions of an equation in the notes, this number is $\binom{n-1}{r-1}$ with $n = 16$ and $r = 3$, that is, it is

$$\binom{15}{2} = \frac{15 \cdot 14}{1 \cdot 2} = 15 \cdot 7 = 105.$$

b) How many integer solutions does the equation in Part *a* has if we require instead that $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$.

Solution. According the the discussion at the same place in the notes, the number of nonnegative solutions of the same equation is $\binom{n+r-1}{r-1}$ with the same values of n and r, that is

$$\binom{18}{2} = \frac{18 \cdot 17}{1 \cdot 2} = 9 \cdot 17 = 153.$$

c) How many integer solutions does the equation in Part *a* has if we require that $x_1 \geq 2$, $x_2 \geq 3$, and $x_3 \geq 5$.

Solution. Writing $x_1 = y_1 + 1$, $x_2 = y_2 + 2$, and $x_3 = y_3 + 4$, the equation under the given condition the number of solutions of the given equation is the same as the number of positive integer solutions of the equation

$$(y_1 + 1) + (y_2 + 2) + (y_3 + 4) = 16,$$

i.e., of the equations

$$y_1 + y_2 + y_3 = 9.$$

According to the quoted discussion in the notes, the number of positive solutions of this equation is $\binom{n-1}{r-1}$ with $n = 9$ and $r = 3$, that is

$$\binom{8}{2} = \frac{8 \cdot 7}{1 \cdot 2} = 7 \cdot 7 = 28.$$

2. *a*) From an urn containing 8 red balls and 6 green balls, six balls are taken without replacement. Determine the probability that 4 of the balls are red and 2 of them are green.

Solution. The number of ways one can pick 4 balls out of 8 red balls without replacement is $\binom{8}{4}$. The number of ways one can pick 2 balls out of 6 green balls without replacement is $\binom{6}{2}$. The number of ways one can pick 6 balls out of a total of 14 balls without replacement is $\binom{14}{6}$. Hence, the probability is

$$\binom{8}{4}\binom{6}{2}\bigg/\binom{14}{6} = \frac{\frac{8\cdot7\cdot6\cdot5}{1\cdot2\cdot3\cdot4} \cdot \frac{6\cdot5}{1\cdot2}}{\frac{14\cdot13\cdot12\cdot11\cdot10\cdot9}{1\cdot2\cdot3\cdot4\cdot5\cdot6}} = \frac{7 \cdot 6 \cdot 5 \ \cdot\ 5}{7 \cdot 13 \cdot 11 \cdot 3} = \frac{2 \cdot 5 \cdot 5}{13 \cdot 11} = \frac{50}{143} \approx .34965.$$

b) Give the probability if the same experiment is performed with replacement, and the same outcome is obtained.

Solution. The probability of picking four red balls with replacement $(8/14)^4 = (4/7)^4$. The probability of picking two green balls with replacement is $(6/14)^2 = (3/7)^2$. The probability of first picking four read balls with replacement and then two green balls of replacement is the product of these, that is $(4/7)^4 \cdot (3/7)^2$.

The number of ways four red balls and two balls can be arranged in a sequence (the order in which they are being picked) is

$$\binom{6}{4} = \binom{6}{2} = \frac{6 \cdot 5}{1 \cdot 2} = 15.$$

Hence the probability of picking two red balls and three green balls (in any order) is

$$\binom{6}{2} \cdot \left(\frac{4}{7}\right)^4 \cdot \left(\frac{3}{7}\right)^2 = 15 \cdot \frac{2304}{117649} = \frac{34560}{117649} \approx .29376.$$

3. *a*) On a certain night soon after sunset in March, the probability that an observer sees a meteor (falling star) in any given second is 6/3600 (note that an hour is 3600 seconds; one mentions seconds here, since one second is a very short time under the circumstances – so 6 is the hourly rate when one measures this rate in very short time intervals). What is the probability that this observer will see exactly k meteors in a given hour ($k \geq 0$)?

Solution. If X indicates the number of meteors seen by the given observer, then X follows the Po(6) distribution; that is,

$$\mathrm{P}(X = k) = \frac{6^k}{k!} e^{-6}.$$

b) What is the probability that the same observer will see 4 meteors in a given hour.

Solution. According to the formula given in Part *a*), we have

$$\mathrm{P}(X = 4) = \frac{6^4}{4!} e^{-6} \approx .1338526175399833.$$

Note. The question arises as to how justified one is to consider one second a "very short time" in the problem. In other words, if one sees a meteor with probability approaching $\lambda \Delta t$ in a time interval of length Δt as $\Delta t \to 0$, what value of λ would give rise to a sighting of at least one meteor with probability 6/3600 in any given second. Here λ is given for a time unit of one hour; the corresponding value of λ for a time unit of one second would be $\lambda/3600$. That is, if Y is the random variable denoting the number of meteors seen in a given time interval one second length, then Y follows the Poisson distribution Po($\lambda/3600$); i.e.,

$$\mathrm{P}(Y = k) = \frac{(\lambda/3600)^k}{k!} e^{-\lambda/3600}.$$

Hence

$$\mathrm{P}(Y \geq 1) = 1 - \mathrm{P}(Y = 0) = 1 - e^{-\lambda/3600}.$$

Thus the equation $\mathrm{P}(Y \geq 1) = 6/3600$ gives

$$\frac{6}{3600} = 1 - e^{-\lambda/3600}.$$

From here we obtain

$$\lambda = -3600 \ln\left(1 - \frac{6}{3600}\right) \approx 6.005005562509411.$$

What is going on here is the following. Recall that the expectation of a Po(λ) is λ. We have

$$\frac{\lambda}{3600} \approx .001668057100697058.$$

This is the expected number of meteors seen in any given second. On the other hand, the probability that at least one meteor is seen in any given second is

$$\frac{6}{3600} = \frac{1}{600} \approx .001666666666666666$$

(a repeating decimal); the former number is somewhat larger, since one might see more than one meteor in a given second.